

# Permanence Properties of Quasi-F-Singularities over Arbitrary Fields

BY

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DISSERTATION

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*For Brennan, Robin, and our little family.*

## **Accessibility Statement**

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## Contribution of Authors Statement

Some of the material in chapter 2 section 4 in this thesis is reproduced from a paper coauthored by Jack J Garzella and the author [GJ25]. Jack J Garzella has graciously consented to the reproduction of this material in the author's thesis.

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## Notation and Conventions

- Rings will always be commutative, with identity 1, and of dimension  $d$ .
- $p$  will always denote a positive prime number.
- A ring  $R$  is local if it has a unique maximal ideal; we will not include any Noetherianity assumptions in our definition of local. For any local ring  $R$ ,
  - $\mathfrak{m}$  will denote its maximal ideal and  $k = R/\mathfrak{m}$  will denote its residue field. If  $R$  admits a coefficient field, we will refer to this as  $k \subseteq R$  in an abuse of notation.
  - $\widehat{R}$  will denote the completion with respect to  $\mathfrak{m}$ .
  - $E_R := E_R(k)$  will denote the injective hull of the residue field; when unambiguous we will simply denote this as  $E$ . Let  $(-)^{\vee} := \text{Hom}_R(-, E)$  denote taking the Matlis dual of an  $R$ -Module.
  - For  $M$  an  $R$ -Module, let  $\text{Soc}(M) := \text{Hom}_R(k, M)$  denote the socle of  $M$ .
- For any prime ideal  $P \in \text{Spec}(R)$ ,  $R_P$  will denote the localization  $(R \setminus P)^{-1}R$ , and  $\kappa(P) = R_P/PR_P = \text{Frac}(R_P)$  will denote the residue field at a prime  $P$ .
- All schemes  $X$  will be quasi-compact and separated, with structure sheaf  $\mathcal{O}_X$  and stalks  $\mathcal{O}_{X,x}$  for every  $x \in X$ .
- For any ideal  $I = (f_1, \dots, f_m)$  of a ring  $R$ , let  $I^{[q]} := (f_1^q, \dots, f_m^q)$ .
- For any ring map  $f : R \rightarrow S$ , we will say that  $f$  is finite if  $S$  is finitely generated as an  $R$ -module (e.g. module-finite) and say that  $f$  is of finite type if  $S$  is finitely generated as an  $R$ -algebra.
- If it exists for a ring  $R$ ,  $\omega_R^{\bullet}$  will denote a dualizing complex of  $R$  and  $\omega_R := H^{-d}(\omega_R^{\bullet})$  will be its canonical module. Similarly define  $\omega_X^{\bullet}$  and  $\omega_X$  for a scheme  $X$ . We let  $K_R$  (or  $K_X$ ) denote the associated canonical divisor. We note that when  $R$  is Cohen-Macaulay, the above notion of canonical module coincides with “standard” canonical module, i.e. the module  $M$  for which  $(M_P)^{\vee} = H_P^d(R_P)$  at every localization of  $R$ , or equivalently a maximal Cohen-Macaulay module of type 1 [BH98].
- For any map of  $R$ -modules  $\varphi : M \rightarrow N$ , we let  $L \otimes_R \varphi$  denote functorial image of  $\varphi$  under  $L \otimes_R -$ , namely the map  $L \otimes_R M \rightarrow L \otimes_R N$  sending  $\ell \otimes m \mapsto \ell \otimes \varphi(m)$ .
- A divisor (resp.  $\mathbb{Q}$ -divisor) will always refer to a Weil divisor (resp Weil  $\mathbb{Q}$ -divisor). If not in a setting where Weil divisors and Cartier divisors are equivalent, we will specify when a divisor is a Cartier divisor or  $\mathbb{Q}$ -Cartier divisor (resp. a  $\mathbb{Q}$ -Cartier divisor or a  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor).
- For any  $\mathbb{Q}$ -divisor  $D$ , let  $\lfloor D \rfloor$  denote the round-down of the divisor, i.e. round all coefficients to the nearest integer to get a standard divisor. Similarly let  $\lceil D \rceil$  denote the round-up and  $\{D\}$  denote the fractional part of  $D$ , or the unique divisor for which  $\lfloor D \rfloor = D - \{D\}$ .
- In this thesis we will take a variety to be an irreducible, separated, normal, integral scheme of essentially finite type over a field  $k$ . We will not assume that this field is algebraically closed.

## Summary

A variety  $X$  over a field  $k$  can have singular points, like cusps or crossings, which naturally arise even in the study of smooth varieties. For example, in the minimal model program (MMP), smooth varieties often deform into ones with mild singularities [Mor88]. Quantifying these singularities is central to this program and a key focus of algebraic geometry.

In characteristic 0, singularities of  $X$  can be resolved via a proper birational morphism  $\pi : Y \rightarrow X$  from a smooth variety  $Y$  [Hir64]. The complexity of  $\pi$  reflects how singular  $X$  is, and singularities arising from the MMP, such as KLT or log canonical singularities, can be read off the exceptional divisor of  $\pi$ . In contrast, in positive characteristic  $p > 0$  resolutions are only known to exist up to dimension three. Instead, singularities are studied via the Frobenius morphism. A classical theorem of Kunz [Kun69] states that a Noetherian ring  $R$  is regular if and only if its Frobenius is flat as a map of  $R$ -Modules. Thus, by analyzing Frobenius on the local rings of functions  $\mathcal{O}_{X,x}$ , one can measure singularities at points  $x \in X$ , while studying the global Frobenius  $F : \mathcal{O}_X \rightarrow F_*\mathcal{O}_X$  captures the singularity of  $X$  itself.

Relaxing flatness of Frobenius yields milder singularity conditions: such invariants as (strong)  $F$ -regularity,  $F$ -rationality,  $F$ -splitting/ $F$ -purity, and  $F$ -injectivity are called  $F$ -singularities. These  $F$ -singularities can detect arithmetic data: for K3 surfaces and Abelian varieties, being globally  $F$ -split is equivalent to being ordinary [Yob23]. These connections have led to the recent development of so-called **quasi-F-singularities**; weaker variants of classical  $F$ -singularities that satisfy similar properties yet include a much wider class of examples.

Quasi- $F$ -split varieties were first introduced by [Yob19] as a weakening of  $F$ -splitting that, in a sense, allowed for mild arithmetic supersingularity. Rather than require  $F : \mathcal{O}_X \rightarrow F_* \mathcal{O}_X$  split, quasi- $F$ -split varieties have a certain thickening, or lift, of Frobenius  $\mathcal{O}_X \rightarrow F_* W_n \mathcal{O}_X / p$ , constructed using Witt vectors, that splits in the category of  $\mathcal{O}_X$ -Modules. Work of Kawakami, Takamatsu, Tanaka, Witaszek, Yobuko, and Yoshikawa [KTT<sup>+</sup>25, KTT<sup>+</sup>24a, KTT<sup>+</sup>24b, TWY24] extended this to develop weaker variants of strong  $F$ -regularity and  $F$ -rationality, dubbed quasi- $F$ -regularity and quasi- $F$ -rationality, and applied them to extend results in positive characteristic birational geometry.

The local theory of quasi- $F$ -singularities was advanced in parallel to these developments. Kawakami, Takamatsu, and Yoshikawa [KTY22, KTY25] developed a “quasi” version of Fedder’s criterion [Fed83] for detecting quasi- $F$ -purity, with Yoshikawa developing an analogue of Glassbrenner’s criterion [Gla96] to test for quasi- $F$ -regularity [Yos25] and a “quasi” theory of tight closure to characterize quasi- $F$ -rationality [Yos24]. Work of the author and Garzella showed that quasi- $F$ -pure hypersurfaces have maximal  $F$ -pure threshold [GJ25].

All of these results, however, were only known to hold for  $F$ -finite rings and varieties, or rings and varieties for which the Frobenius  $F : R \rightarrow F_* R$  (or  $F : \mathcal{O}_X \rightarrow F_* \mathcal{O}_X$ ) is a finite map of  $R$ -modules (or  $\mathcal{O}_X$ -modules). This parallels the initial development of  $F$ -singularities; results and definitions were first developed in a more restrictive setting (say, for finite type algebras over  $F$ -finite fields) and later generalized to arbitrary noetherian positive characteristic rings.

The goal of this thesis is to develop a program for the systematic study of quasi- $F$ -singularities in the non- $F$ -finite setting. To do so would require answering the following questions:

QUESTION 0.1. Are the definitions of quasi- $F$ -singularities (quasi- $F$ -splitting, quasi- $F^e$ -splitting, quasi- $F^\infty$ -splitting, quasi- $F$ -regularity, quasi- $F$ -rationality) well defined and a good measure of singularity outside of the  $F$ -finite setting?

We show that, much like classical  $F$ -singularities, detecting (lifts of) Frobenius splitting is not an adequate measure of singularity when moving outside of the  $F$ -finite setting. We show that the class of excellent Henselian DVRs studied in [DM23] can fail to be quasi- $F$ -split (theorem 2.28).

Following the classical setting, we define analogues of quasi- $F$ -singularities introduced in previous work [Yob19, TWY24, KTT<sup>+</sup>24b] by detecting purity of (lifts of) Frobenius instead of splitting (see definitions 2.32, 3.19, 3.58, 3.72). We show that all regular rings satisfy these conditions and that they are equivalent to their  $F$ -finite analogues over  $F$ -finite rings.

QUESTION 0.2. Do these new definitions still satisfy key properties of quasi- $F$ -singularities arising in the  $F$ -finite setting? What are their permanence properties?

We show that our new general definitions are local properties and preserved under completion:

THEOREM A. *Let  $(R, \mathfrak{m})$  be a Noetherian local ring of characteristic  $p > 0$ . Then for any  $n \in \mathbb{N}$  the following hold:*

- (1) *If  $R$  is a  $G$ -ring (e.g. if  $R$  is excellent) then the following are equivalent (corollary 2.39.1)*
  - (a)  *$R$  is  $n$ -quasi- $F$ -pure.*
  - (b)  *$\widehat{R}$  is  $n$ -quasi- $F$ -pure.*
  - (c)  *$R_P$  is  $n$ -quasi- $F$ -pure for all  $P \in \text{Spec}(R)$ .*
- (2) *If  $R$  is normal domain that completes to a normal domain (e.g. an excellent normal domain) and  $\Delta$  is a  $\mathbb{Q}$ -divisor on  $R$  such that  $[\Delta] = 0$ , then the following are equivalent for any  $e \in \mathbb{N} \cup \{\infty\}$  (corollary 3.27.1):*

- (a)  $(R, \Delta)$  is  $n$ -quasi- $F^e$ -pure.
- (b)  $(\widehat{R}, \Delta^\wedge)$  is  $n$ -quasi- $F^e$ -pure.
- (c)  $(R_P, \Delta|_{R_P})$  is  $n$ -quasi- $F^e$ -pure for all  $P \in \text{Spec}(R)$ .

We note that quasi- $F$ -regularity and quasi- $F$ -rationality are also local and stable under completion, but this will be deduced as a direct consequence of theorem C. We also show that these properties are stable under finite étale extension:

**THEOREM B.** *If  $(R, \mathfrak{m})$  is a local G-ring and  $(R, \mathfrak{m}) \rightarrow S$  is a finite étale extension, then  $R$  is  $n$ -quasi- $F$ -pure if and only if  $S$  is (theorem 2.44). Now we further assume that  $(R, \mathfrak{m})$  is an excellent normal domain admitting a dualizing complex  $\omega_R^\bullet$  with associated canonical divisor  $K_R$ , and  $\Delta$  is a  $\mathbb{Q}$ -divisor on  $R$  such that  $[\Delta] = 0$ . Also assume that  $S$  is a domain and let  $\Delta_S = \Delta \otimes_R S$ . Then the following hold:*

- (1) *For any  $e \in \mathbb{N}$ ,  $(R, \Delta)$  is  $n$ -quasi- $F^e$ -pure if and only if  $(S, \Delta_S)$  is. (Theorem 3.40)*
- (2)  *$(R, \Delta)$  is  $n$ -quasi- $F^\infty$ -pure if and only if  $(S, \Delta_S)$  is. (corollary 3.40.1)*
- (3)  *$(R, \Delta)$  is uniformly quasi- $F^\infty$ -pure if and only if  $(S, \Delta_S)$  is. (Corollary 3.40.1)*
- (4) *If  $K_R + \Delta$  is  $\mathbb{Q}$ -Cartier,  $(R, \Delta)$  is  $n$ -quasi- $F$ -regular if and only if  $(S, \Delta_S)$  is. (Theorem 3.70)*
- (5) *If  $\Delta$  is  $\mathbb{Q}$ -Cartier, then  $(R, \Delta)$  is  $n$ -quasi- $F$ -rational if and only if  $(S, \Delta_S)$  is. (Theorem 3.80)*

Due to the local nature of these quasi- $F$ -singularity notions, Theorem B can be described in the language of algebraic geometry. In particular, one can replace the finite étale map  $(R, \Delta) \rightarrow (S, \Delta_S)$  with a dominant finite étale map of schemes  $(Y, \Delta_Y) \rightarrow (X, \Delta)$  (see Theorem 2.48 and Theorem 3.42) and deduce analogous results. We caution that these notions of singularity are only exhibiting local control of the scheme; the theorem above is actually false for global quasi- $F$ -splittings, see example 2.24. This suggests that local quasi- $F$ -singularities are uniquely controllable under taking étale covers.

We also develop the theory of quasi-test ideals and quasi-test submodules, generalizations of the test ideal and parameter test submodules, which test for quasi- $F$ -pure-regularity and quasi- $F$ -rationality. These are generalizations of counterparts initially defined in [KTT<sup>+</sup>24b] which are compatible with the non- $F$ -finite setting. We show that these commute with localization and completion and are also stable under étale extension:

**THEOREM C.** *Let  $(R, \mathfrak{m})$  be an excellent normal local domain admitting a dualizing complex  $\omega_R^\bullet$  and  $\Delta$  a  $\mathbb{Q}$ -divisor on  $R$  such that  $[\Delta] = 0$ . Let  $\tau_n(R, \Delta)$  and  $\tau^q(R, \Delta)$  denote the  $n$ -quasi test ideal and quasi test ideal respectively (definition 3.60). If  $K_R + \Delta$  is  $\mathbb{Q}$ -Cartier, then the following hold:*

- (1)  $\tau_n(R, \Delta) = R$  if and only if  $(R, \Delta)$  is  $n$ -quasi- $F$ -pure regular. (theorem 3.65)
- (2)  $\tau^q(R, \Delta) = R$  if and only if  $(R, \Delta)$  is quasi- $F$ -pure regular. (theorem 3.65)
- (3)  $\tau_n(R, \Delta) \otimes \widehat{R} = \tau_n(\widehat{R}, \Delta^\wedge)$  and  $\tau^q(R, \Delta) \otimes \widehat{R} = \tau^q(\widehat{R}, \Delta^\wedge)$ . (lemma 3.63)
- (4) For any  $P \in \text{Spec}(R)$ ,  $\tau_n(R, \Delta) \otimes R_P = \tau_n(R_P, \Delta_P)$  and  $\tau^q(R, \Delta) \otimes R_P = \tau^q(R_P, \Delta_P)$ . (lemma 3.63)
- (5) Let  $R \rightarrow S$  be a local étale extension to a local domain  $S$ , and let  $\Delta_S = \Delta \otimes_R S$ . Then  $\tau_n(R, \Delta)S = \tau_n(S, \Delta_S)$  and  $\tau^q(R, \Delta)S = \tau^q(S, \Delta_S)$ . (Theorem 3.69)

Let  $\tau_n(\omega_R, \Delta)$  and  $\tau^q(\omega_R, \Delta)$  denote the  $n$ -quasi-test submodule and quasi-test submodule respectively (definition 3.77) If  $\Delta$  is  $\mathbb{Q}$ -Cartier, then the following hold:

- (1)  $\tau_n(\omega_R, \Delta) = \omega_R$  if and only if  $(R, \Delta)$  is  $n$ -quasi- $F$ -rational. (proposition 3.78)
- (2)  $\tau^q(\omega_R, \Delta) = \omega_R$  if and only if  $(R, \Delta)$  is quasi- $F$ -rational. (proposition 3.78)
- (3)  $\tau_n(\omega_R, \Delta) \otimes \widehat{R} = \tau_n(\omega_{\widehat{R}}, \Delta^\wedge)$  and  $\tau^q(\omega_R, \Delta) \otimes \widehat{R} = \tau^q(\omega_{\widehat{R}}, \Delta^\wedge)$ . (proposition 3.79)
- (4) For any  $P \in \text{Spec}(R)$ ,  $\tau_n(\omega_R, \Delta) \otimes R_P = \tau_n(\omega_{R_P}, \Delta_P)$  and  $\tau^q(\omega_R, \Delta) \otimes R_P = \tau^q(\omega_{R_P}, \Delta_P)$ . (proposition 3.79)
- (5) Let  $R \rightarrow S$  be a local étale extension to a local domain  $S$ , and let  $\Delta_S = \Delta \otimes_R S$ . Then  $\tau_n(\omega_R, \Delta)S = \tau_n(\omega_S, \Delta_S)$  and  $\tau^q(\omega_R, \Delta)S = \tau^q(\omega_S, \Delta_S)$ . (theorem 3.80)

In the process of proving these results, we show that, under the above hypotheses, rings admit strong test elements for quasi- $F$ -pure regularity and quasi- $F$ -rationality (theorem 3.65)

and theorem 3.74). As a consequence of theorem C, quasi- $F$ -pure regularity and quasi- $F$ -rationality all complete, localize, and are stable under local étale extension.

## Outline of Thesis

This thesis is divided into 3 chapters. Chapter 1 broadly contains background and preliminaries: section 1 provides an overview of the classical theory of  $F$ -singularities. In this section we develop both the local and global picture for  $F$ -regularity,  $F$ -splitting, and  $F$ -rationality with a particular focus on the  $F$ -finite setting. In section 2 we discuss various ways these singularity conditions can be generalized outside of the  $F$ -finite setting. Along the way we introduce excellent rings and discuss why excellence is a baseline assumption in the above results and develop the  $\Gamma$ -construction; a powerful technique introduced by Hochster and Huneke [HH94a] used to associate to any complete local ring  $R$  an  $F$ -finite, faithfully flat extension  $R^\Gamma$  which inherits many properties of  $R$ .

In chapter 1 section 3 we introduce Witt Vectors, a tool for assigning a  $p$ -nilpotent thickening to any ring of positive characteristic that is heavily utilized in the definitions of quasi- $F$ -singularities. Finally in section 4 we develop the theory of étale extensions; the analogue of local diffeomorphisms in algebraic geometry, and discuss their interactions with both Witt vectors and Frobenius.

In chapter 2 we review the definition of quasi- $F$ -splittings and provide historical motivation for its construction in section 1. In section 2 we review the construction of [DM23] and show that quasi- $F$ -splitting, in its initially devised form, is not a good notion of singularity outside of the  $F$ -finite setting as it can fail to hold for even the nicest class of rings. This motivates the development of quasi- $F$ -purity, coined by the author in [Jag26], in section 3. We show that quasi- $F$ -purity is equivalent to quasi- $F$ -splitting in the  $F$ -finite setting and complete local setting, mirroring the  $F$ -splitting and  $F$ -purity comparison, and show it is stable under limits, localization, completion, and étale extension. In section 4 we reproduce

work of Garzella and the author [GJ25] where we compute the  $F$ -pure threshold of an arbitrary ( $F$ -finite) quasi- $F$ -pure hypersurface; we note that Garzella has graciously consented to the reproduction of this material in the author's thesis.

Chapter 3 extends results of chapter 2 to quasi- $F^e$ -purity, quasi- $F^\infty$ -purity, quasi- $F$ -pure regularity, and quasi- $F$ -rationality and to their variants defined for divisor pairs  $(R, \Delta)$  and  $(X, \Delta)$ . This chapter is certainly the most technical of this thesis due to the complexity inherent in the definitions of these invariants, even in the  $F$ -finite setting. Section 1 develops the theory of twists of sheaves of Witt vectors by divisors on the base scheme in the sense of [Tan22] and uses these constructions to define quasi- $F^e$ -splittings. In section 2 we define a local analogue of this, dubbed quasi- $F^e$ -purity, that is amenable to the non- $F$ -finite setting. We show that Quasi- $F^e$ -purity and Quasi- $F^\infty$ -purity localize, complete and are stable under étale extension.

Chapter 3 section 3 is dedicated to the development of quasi- $F$ -pure regularity, quasi- $F$ -rationality, and their corresponding quasi-test ideals and quasi-test submodules. In this section we prove theorem C and use this to prove that quasi- $F$ -pure regularity and quasi- $F$ -rationality localize, complete, and are stable under étale extension. Section 4 details applications of these quasi- $F$ -singularity notions to birational geometry, though these applications are all in the  $F$ -finite setting.

Chapter 3 section 5 details progress on the author's conjecture posed in [Jag26], more specifically on whether the  $\Gamma$ -construction discussed in chapter 1 section 2.3 can be used to systematically reduce the study of quasi- $F$ -singularities on excellent rings to  $F$ -finite rings. Though the author's thesis provides a way to reduce to the complete local setting, where the  $\Gamma$ -construction can be applied, it is still open as to whether or not the  $\Gamma$ -construction can preserve quasi- $F$ -singularity invariants. This section will be dedicated to exploring this

conjecture, alongside the consequences of it being true.

One consequence of moving to the non- $F$ -finite setting is that rings of Witt vectors  $W_n(R)$  for  $n > 1$  necessarily fail to be Noetherian (lemma 2.31). In Appendix A we show that local cohomology over non-Noetherian rings, if taken to be supported at a finitely generated ideal, still satisfy key properties of local cohomology over Noetherian rings. We use these results heavily in chapter 3. In Appendix B we show how the plethora of quasi- $F$ -singularity invariants discussed in this thesis relate to each other in a diagram.

## CHAPTER 1

**Preliminaries**

Any ring of prime characteristic  $p > 0$  comes equipped with a **Frobenius morphism**  $F : R \rightarrow F_*R$ , assigning  $x \mapsto x^p$ .<sup>1</sup> On any scheme  $X$  of characteristic  $p > 0$  (e.g. schemes for which  $p\mathcal{O}_X = 0$ ), one can glue the local Frobenii to define an **Absolute** Frobenius morphism of schemes  $F : X \rightarrow X$ . On the structure sheaf,  $F : \mathcal{O}_X \rightarrow F_*\mathcal{O}_X$  is an  $\mathcal{O}_X$ -Module homomorphism that, on any open set  $U$ , sends a section  $s$  to its  $p$ th power  $s^p$ . One can similarly define their iterates  $F^e : R \rightarrow F_*^e R$  and  $F^e : \mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X$ .

By studying the behavior and properties of the (absolute) Frobenius morphism on  $R$  (on  $X$ ), one can recover a startling number of properties about the objects themselves. In section 1, we will discuss how the Frobenius morphism can recover key singularity properties of a ring  $R$  or a scheme  $X$  and recover properties of rings and schemes that hold in characteristic 0 but can fail in characteristic  $p > 0$ . We will do this, in part, by reviewing notions of singularity defined via the Frobenius that have come to prominence over the past 50 years.

In section 2, we will remove a simplifying assumption heavily used in section 1 (that the (absolute) Frobenius is a *finite* morphism; rings or schemes that satisfy this condition are called **F-Finite**) and show how otherwise well-formed definitions need to be changed to accommodate this relaxed assumption. We will pay specific attention to the  $\Gamma$ -construction of Hochster and Huneke [**HH94a**], which provides a method to systematically reduce the study of complete local (but possibly not  $F$ -finite) rings to  $F$ -finite rings. In section 3 we will review the construction and historical motivation for Witt Vectors, a tool crucial in

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<sup>1</sup> $F : R \rightarrow R$  is a ring morphism, but by applying  $F_*$ , or the restriction of scalars via Frobenius, on the target,  $F$  turns into a morphism of  $R$ -Modules. It also keeps track of the  $R$ -Module structure on  $F_*R$ :  $r \cdot F_*s := F_*(r^p s)$ .

developing the theory of quasi- $F$ -singularities in later chapters. Finally, in section 4 we will discuss étale extensions and their interactions with Witt vectors and Frobenius.

**SETTING 1.1.** *All rings and schemes in sections 1 and 2 will be Noetherian and of characteristic  $p > 0$  unless otherwise specified. We caution that this will not be implicitly assumed outside of these sections going forward.*

## 1. $F$ -Singularities

The study of singularities via Frobenius begins with a classical theorem of Ernst Kunz from 1969:

**THEOREM 1.2** (Kunz's Theorem [**Kun69**]). *Let  $R$  be a Noetherian ring of characteristic  $p > 0$ . Then  $R$  is regular if and only if the Frobenius morphism  $F^e : R \rightarrow F_*^e R$  is a flat morphism of  $R$ -modules for some (equivalently all)  $e > 0$ .*

For varieties over an  $F$ -finite field, this says that  $X$  is regular if and only if  $F_* \mathcal{O}_X$  is a locally free sheaf. Since the publication of this groundbreaking result, many simpler proofs have been given. We review a proof given in [**MP25**] below:

**PROOF.** [**MP25**] We first assume that  $R$  is regular and prove that  $F_*^e R$  is a flat  $R$ -Module. Frobenius interacts well with localization<sup>2</sup>, and as flatness and regularity are local properties, we can assume that  $R$  is a local ring. As  $R$  is Noetherian, completing  $R \rightarrow \widehat{R}$  is faithfully flat, as is the map  $F_*^e R \rightarrow F_*^e \widehat{R}$ . It is easy to see then that  $R \rightarrow F_*^e R$  is flat if and only if  $\widehat{R} \rightarrow F_*^e \widehat{R}$  is, and hence, we can assume that  $R$  is a complete local ring.

By the Cohen Structure Theorem [**Coh46**],  $R$  is isomorphic to the power series ring  $k[[x_1, \dots, x_d]]$ . There is a natural faithfully flat map  $k[[x_1, \dots, x_d]] \rightarrow \overline{k}[[x_1, \dots, x_d]]$ , where  $\overline{k}$  denotes the algebraic closure of  $k$ . By similar logic to the reduction to the complete case,  $F^e$  is a faithfully flat morphism on  $\overline{k}[[x_1, \dots, x_d]]$  if and only if it is on  $k[[x_1, \dots, x_d]]$ ; hence we can assume that  $k$  is algebraically closed. In this case,  $F_*^e R = F_*^e \overline{k}[[x_1, \dots, x_d]]$  is a free

<sup>2</sup>In fact, Frobenius is an isomorphism on  $\text{Spec}(R)$ .

$\overline{k}[[x_1, \dots, x_d]]$ -module with basis

$$\{F_*^e(x_1^{i_1} \dots x_d^{i_d}) \mid 0 \leq i_j < p^e \quad \forall j \leq d\}$$

These are clearly independent, so it is sufficient to show that it can generate any monomial in  $k[[x_1, \dots, x_d]]$ . If all of the powers of the  $x_i$  in the monomial are less than  $p^e$ , we are immediately done. If not (say,  $i_1 = cp^e + m_1$  for  $c \in \mathbb{N}$  and  $m_1 < p^e$ ), then one can write

$$F_*^e(x_1^{cp^e+m_1} \dots x_d^{i_d}) = x_1^c \cdot F_*^e(x_1^{m_1} \dots x_d^{i_d})$$

And similarly for any other variable; thus we have the forward direction. For the converse we will assume Frobenius is flat and show  $R$  is regular by Serre's criterion. If we assume that  $F_*^e R$  is flat, then  $F_*^{me} R$  is flat for any  $m \in \mathbb{N}$ . Thus, we know that  $F_*^e R$  for infinitely many  $e > 0$ . Regularity and flatness are both local conditions, so we can again assume that  $R$  is local, say of depth  $g$ . Taking a regular sequence in  $\mathfrak{m}$  of maximal length  $x_1, \dots, x_g$ , we have that  $R/(x_1, \dots, x_g)$  is depth 0 and hence has nonzero socle  $N := \text{Soc}(R/(x_1, \dots, x_g))$ . Thus, there is some  $n > 0$  for which  $N \not\subset \mathfrak{m}^n(R/(x_1, \dots, x_g))$ .

Suppose that  $R$  were not regular; then by Serre's criterion  $k = R/\mathfrak{m}$  would have infinite projective dimension over  $R$ . As the Frobenius is flat, we can tensor a minimal free resolution of  $k$  with  $F_*^e R$  and obtain a minimal free resolution of  $R/\mathfrak{m}^{[p^e]}$ . Each morphism of free module in the resolution can be represented by a matrix; by construction every entry must lie in  $\mathfrak{m}^{[p^e]}$ . We will use this to show a contradiction, as this  $e$  can be chosen as large as we'd like.

We claim that for any  $R$ -Module  $M$  of infinite projective dimension,  $\varphi_{g+2}$  as in the minimal free resolution

$$\dots R^{n_{g+2}} \xrightarrow{\varphi_{g+2}} R^{n_{g+1}} \xrightarrow{\varphi_{g+1}} \dots \rightarrow R^{n_1} \rightarrow R^{n_0} \rightarrow M \rightarrow 0$$

has matrix entries that necessarily do not land inside of  $\mathfrak{m}^n$ . If this is true, it would contradict the conclusion of the above paragraph, and hence,  $k$  would have finite projective dimension

and  $R$  must be regular. We now prove the claim.

Since  $R/(x_1, \dots, x_g)$  has projective dimension  $g$  by construction,  $\mathrm{Tor}_R^{g+1}(M, R/(x_1, \dots, x_g))$  vanishes. Thus, tensoring the free resolution above  $R/(x_1, \dots, x_g)$  yields a sequence

$$(R/(x_1, \dots, x_g))^{n_{g+2}} \xrightarrow{\varphi_{g+2}} (R/(x_1, \dots, x_g))^{n_{g+1}} \xrightarrow{\varphi_{g+1}} (R/(x_1, \dots, x_g))^{n_g}$$

That is exact in the middle. By minimality of the resolution,  $N^{n_{g+1}} \subseteq (R/(x_1, \dots, x_g))^{n_{g+1}}$  is contained in  $\mathrm{im}(\varphi_{g+2}) = \ker(\varphi_{g+1})$ . If all entries of  $\varphi_{g+2}$  were indeed contained in  $\mathfrak{m}^n$ , Then  $N^{n_{g+1}} \subseteq \mathfrak{m}^n(R/(x_1, \dots, x_g))^{n_{g+1}}$ , and hence,  $N \subseteq \mathfrak{m}^n(R/(x_1, \dots, x_g))$ , violating the conclusion of an earlier paragraph.  $\square$

Kunz's Theorem [**Kun69**] was proven 5 years after Hironaka's seminal work proving the existence of resolutions of singularities in characteristic 0 [**Hir64**]. Hironaka's work provided tools to measure the severity of singularities in characteristic 0: the more complicated resolution a singular variety admits, the worse the singularities.

The existence of resolutions of singularities in characteristic  $p$  remains an open question to this day in dimensions larger than 3, yet Kunz's theorem nonetheless provides a barometer for measuring singularities of a characteristic  $p$  ring  $R$ . Simply put, one can measure singularity of  $R$  by measuring the failure of (faithful) flatness of  $F$ .

**LEMMA 1.3.** *For any local ring  $R$  and local ring map  $R \rightarrow S$ ,  $S$  is faithfully flat if and only if  $S$  is flat. In particular, for a local ring  $R$ ,  $F_*^e R$  is faithfully flat for all (equivalently for one)  $e > 0$  if and only if  $R$  is regular.*

**PROOF.** faithful flatness is equivalent to being flat and surjective on  $\mathrm{MaxSpec}$ . As  $R \rightarrow S$  is a flat local morphism of local rings, the image of the maximal ideal of  $S$  is just the maximal ideal of  $R$ , and hence the map is faithful. The 'in particular' part is a direct consequence of Kunz's Theorem.  $\square$

Singularity invariants have arose since the inception of Kunz’s theorem that are defined by imposing properties on the Frobenius; the study of these invariants and their properties is the study of **F-Singularities**. In the following subsections we will highlight the most ubiquitous of these invariants, focusing on those that have been generalized to so-called “Quasi- $F$ -Singularities” in recent years.

We caution that this is only a brief summary of a subset of the theory of  $F$ -singularities. For the interested reader, we recommend [MP25] for a local study of  $F$ -singularities and [SS25] for a more comprehensive study of  $F$ -singularities, both local and global. We will be citing results from both these texts heavily in this chapter on preliminaries.

**1.1.  $F$ -Finite Rings.** For the remainder of section 1 we will focus on studying the singularity properties of  $F$ -finite rings. In this subsection, we review the definition of  $F$ -finiteness and its consequences.

DEFINITION 1.4. A ring  $R$  of characteristic  $p > 0$  is called **F-Finite** if  $F : R \rightarrow F_*R$  is a finite map of  $R$ -modules (that is,  $F_*R$  is finitely generated as an  $R$ -module).

$F$ -Finiteness is a well behaved property commutative algebraically.

PROPOSITION 1.5 ([SS25], Proposition 1.21). *If  $R$  is a Noetherian  $F$ -finite ring, then:*

- *The homomorphic image of  $R$  is  $F$ -finite.*
- *Any finite extension of  $R$  is  $F$ -finite.*
- *Any localization of  $R$  is  $F$ -finite.*
- *The completion of  $R$  at any ideal is  $F$ -finite.*

In particular, any essentially of finite type algebra over an  $F$ -finite ring is automatically  $F$ -finite.  $F$ -finiteness may seem an innocuous condition at first glance, but  $F$ -finiteness is a surprisingly strong condition that carries with it a number of powerful consequences.

THEOREM 1.6 ([Gab04]). *A Noetherian  $F$ -finite ring  $R$  is the homomorphic image of an  $F$ -finite regular ring.*

In particular, this implies that any  $F$ -finite ring admits a dualizing complex. Via another theorem of Kunz [Kun76], all  $F$ -finite rings are excellent, and further, any excellent local ring is  $F$ -finite if and only if its residue field is  $F$ -finite. Excellent rings are a very broad class of rings with useful properties; any complete local ring and any field is excellent, as is any essentially of finite type algebra over an excellent ring. We will review the definition of excellence in section 2.1.

As hinted at in Kunz's result,  $F$ -finiteness can fail for many rings that otherwise seem well behaved, even fields. The field  $\mathfrak{K} = \mathbb{F}_p(t_1, t_2, \dots)$ , for instance, is not  $F$ -finite, as  $F_*\mathfrak{K}$  has an infinite basis  $\{\prod_{i \in \mathbb{N}} t_i^{n_i} \mid 0 \leq n_i < p\}$ . Any essentially of finite type algebra over this field is excellent yet it need not be  $F$ -finite, forming a broad class of interesting non- $F$ -finite rings. We will study such rings in section 2 and in later chapters.

**1.2. Frobenius Splittings and Purity.** For an  $F$ -finite regular local ring  $(R, \mathfrak{m})$ , Kunz's theorem implies not only that  $F_*^e R$  is a flat module, but a free module. From the argument given in the proof of Kunz's theorem, one sees that

$$F_*^e R \cong R^{\oplus p^{ed}}$$

provided that  $R$  has perfect residue field (otherwise the free rank would be  $ed \cdot \log_p([F_* k : k])$ ). Rather than requiring  $F_*^e R$  decompose entirely into a direct sum of copies of  $R$ , what if  $F_*^e R$  admitted at least one copy of  $R$  as a free summand? This is equivalent to the  $F^e : R \rightarrow F_*^e R$  splitting as a map of  $R$ -modules, which is equivalent to  $F : R \rightarrow F_* R$  splitting as a map of  $R$ -modules. Rings that admit a splitting of Frobenius are called **F-split**.

**DEFINITION 1.7.** A ring  $R$  is **F-split** if the Frobenius map  $F : R \rightarrow F_* R$  splits as a map of  $R$ -modules.

The local theory of Frobenius splittings was first investigated by Hochster, Huneke and Roberts [Hoc73, HR76, HH94a] to better understand integral extensions of positive characteristic rings.  $F$ -splittings have since found much broader relevance: For instance, via mod

$p$  reduction techniques, varieties that have  $F$ -pure singularities have log canonical singularities [MTW05, HW02], though the converse is a major open question.

From the above discussion. It is clear that any regular  $F$ -finite ring is automatically  $F$ -split. As it turns out,  $F$ -finiteness is a necessary assumption here; we will discuss a broad class of regular, excellent rings that are not  $F$ -split [DM23] in chapter 2 section 2.

REMARK 1.8. *Counting the number of  $R$  direct summands in  $F_*^e R$ , especially in the limit as  $e \rightarrow \infty$ , is of independent interest. Define  $\alpha_e \in \mathbb{N}$  to be the number such that  $F_*^e R \cong R^{\oplus \alpha_e} \oplus M$ , where  $M$  does not admit  $R$  as a direct summand. Then  $s(R)$ , the **F-signature** of  $R$ , is defined as follows:*

$$s(R) = \lim_{e \rightarrow \infty} \frac{\alpha_e}{p^{ed}}$$

First defined by Huneke and Leuschke [HL02], this limit was not known to exist outside of special cases until Tucker [Tuc12] proved the existence of  $F$ -signature in full generality.  $s(R) = 1$  if and only if  $R$  is regular, and  $s(R) > 0$  if and only if  $R$  is strongly- $F$ -regular [AL03], a strengthening of  $F$ -splitting to be defined later in this section.

LEMMA 1.9. *If  $R$  is  $F$ -split, then  $W^{-1}R$  is  $F$ -split for all multiplicative sets  $W \subseteq R$ .*

PROOF. As  $R$  is  $F$ -split, let  $s_F : F_* R \rightarrow R$  denote a splitting of Frobenius.  $W^{-1}s_F : W^{-1}F_* R = F_*(W^{-1}R) \rightarrow W^{-1}R$  is thus a Frobenius splitting for any localization.  $\square$

It is worth noting that the converse of the above lemma, though true for  $F$ -finite rings, is false in general; while one can localize a splitting to get a splitting on a localization, it is not guaranteed that you can “glue” local splittings to get a global splitting on  $R$ . By pre or post composing a Frobenius splitting, one can show that  $F$ -splitting descends along split maps.

LEMMA 1.10. *Let  $\varphi : R \rightarrow S$  be a ring map that splits as a map of  $R$ -modules. If  $S$  is  $F$ -split, then so is  $R$ .*

PROOF. Let  $s_\varphi : S \rightarrow R$  be a splitting of  $\varphi$  and  $s_{F_S} : F_*S \rightarrow S$  denote a splitting of the Frobenius on  $S$ .  $s_{F_S}$  is an  $S$ -module homomorphism and restricts to an  $R$ -module homomorphism as well. Thus we can construct a Frobenius splitting on  $R$  as follows:

$$F_*R \xrightarrow{F_*\varphi} F_*S \xrightarrow{s_{F_S}} S \xrightarrow{s_\varphi} R$$

□

To check additional properties of  $F$ -splitting and provide examples, we first introduce two properties of split maps  $R \rightarrow M$  for any  $R$ -module  $M$ , and show that they characterize splitting in special cases.

PROPOSITION 1.11. *Let  $R$  ring and  $M$  an  $R$ -module with associated map  $\varphi : R \rightarrow M$ . Consider the following statements:*

- (1)  $R \rightarrow M$  is split.
- (2)  $R \rightarrow M$  is pure.
- (3)  $H_{\mathfrak{m}}^d(R) \rightarrow H_{\mathfrak{m}}^d(M)$  is injective at each maximal ideal  $\mathfrak{m}$  of  $R$ .

We always have (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3). If  $R$  is complete local or  $M$  is finitely generated, then (2)  $\Rightarrow$  (1). If  $R$  is Gorenstein, then (3)  $\Rightarrow$  (2).

Before proving the proposition, we review the definition of purity. A map of  $R$ -modules  $M \rightarrow N$  is said to be **pure** if  $L \otimes_R M \rightarrow L \otimes_R N$  is injective for any  $R$ -module  $L$ . As injectivity is locally determined, any map  $M \rightarrow N$  is pure if and only if  $M_P \rightarrow N_P$  is pure for every  $P \in \text{Spec}(R)$ , along us to verify purity locally. If  $R$  is local, then purity of  $R \rightarrow M$  can be checked by tensoring by only 1 specific module.

LEMMA 1.12. *Let  $(R, \mathfrak{m})$  be a local ring and  $M$  be an  $R$ -module. Then  $R \rightarrow M$  is pure if and only if  $E \rightarrow E \otimes_R M$  is injective, where  $E = E_R(R/\mathfrak{m})$  is the injective hull of the residue field  $R/\mathfrak{m}$ .*

PROOF. If  $R \rightarrow M$  is pure, then by definition  $E \rightarrow E \otimes_R M$  is injective, so we prove the converse. Suppose  $L \rightarrow L \otimes_R M$  was not injective for some  $R$ -module  $L$ ; we need to

show that  $E \rightarrow E \otimes_R M$  is not injective. Without loss of generality we can assume that  $L$  is finitely generated, as any element of  $\ker(L \rightarrow L \otimes_R M)$  lies inside some finitely generated submodule  $L' \subseteq L$ , which maps to  $L' \otimes_R M$ . Now choose  $t \in \ker(L \rightarrow L \otimes_R M)$  and  $n$  such that  $t \notin \mathfrak{m}^n L$ .

Looking at the map of quotients  $L/\mathfrak{m}^n L \rightarrow L/\mathfrak{m}^n L \otimes_R M$ , we see that the image of  $t \in L/\mathfrak{m}^n L$  is nonzero by construction, but still  $t \in \ker(L/\mathfrak{m}^n L \rightarrow L/\mathfrak{m}^n L \otimes_R M)$ .  $L/\mathfrak{m}^n L$  has finite length and hence  $L/\mathfrak{m}^n L \hookrightarrow E^{\oplus N}$  for some  $N \in \mathbb{N}$ , defining a map  $L/\mathfrak{m}^n L \otimes_R M \rightarrow E^{\oplus N} \otimes_R M$  it follows that the image of  $t \in E^{\oplus N}$  is nonzero, yet again maps to zero in  $E^{\oplus N} \otimes_R M$ . It follows that  $0 \neq t \in \ker(E^{\oplus N} \rightarrow E^{\oplus N} \otimes_R M)$ , and hence by the distributive property,  $E \rightarrow E \otimes_R M$  cannot be injective.  $\square$

Using this lemma, we are ready to prove the proposition.

PROOF OF PROPOSITION 1.11. If  $\varphi$  splits, then  $M \cong R \oplus M'$  for some other  $R$ -module  $M'$ . Thus for any  $R$ -module  $L$ ,  $L \otimes_R \varphi : L \rightarrow L \oplus (L \otimes_R M')$ , which is very clearly injective. This proves that (1) implies (2). Purity is a local property, so it suffices to assume that  $R$  is a local ring with unique maximal ideal  $\mathfrak{m}$ . If  $\varphi$  is pure then  $H_{\mathfrak{m}}^d(R) \otimes_R \varphi$  is injective. This is precisely the map

$$H_{\mathfrak{m}}^d(R) \rightarrow H_{\mathfrak{m}}^d(R) \otimes_R M = H_{\mathfrak{m}}^d(M)$$

And hence (2) implies (3). For the converse statements, suppose that  $R$  is complete local and  $R \rightarrow M$  was pure. We will show that  $R \rightarrow M$  splits by showing the evaluation map  $\mathrm{Hom}_R(M, R) \rightarrow R$  is surjective; the preimage of 1 under the evaluation map is precisely our splitting.

If  $R \rightarrow M$  is pure,  $E \rightarrow E \otimes_R M$  is injective. By Matlis duality, we get a surjection  $\mathrm{Hom}_R(E \otimes_R M, E) \rightarrow \mathrm{Hom}_R(E, E) \cong R$ . By tensor hom adjunction,

$$\mathrm{Hom}_R(E \otimes_R M, E) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(E, E)) \cong \mathrm{Hom}_R(M, R)$$

One should check that after applying Matlis duality that this recovers the evaluation map for complete local rings, but this is easy to check. Hence this argument shows that (2) implies (1) in the complete local case. If  $M$  was finitely generated and  $R \rightarrow M$  is pure, we show it is split by again checking that  $e_\varphi : \text{Hom}_R(M, R) \twoheadrightarrow R$ . This can be checked locally, and further, as  $M$  is finitely generated and  $\widehat{R}$  is faithfully flat, we get that

$$\widehat{R} \otimes_R \text{Hom}_R(M, R) \cong \text{Hom}_{\widehat{R}}(\widehat{M}, \widehat{R})$$

which surjects onto  $\widehat{R} \otimes_R R = \widehat{R}$  by right exactness of tensor product. Thus we can reduce to the complete local case and conclude (2) implies (1) under either hypotheses.

Showing (3) implies (2) is an application of the previous lemma and the fact that  $H_{\mathfrak{m}}^d(R) = E$  when  $R$  is Gorenstein.  $\square$

When  $M$  as above is specifically  $F_*R$ , this gives rise to weaker variants of the  $F$ -splitting singularity condition.

**DEFINITION 1.13.** If  $R \rightarrow F_*R$  is pure, we say that  $R$  is **F-Pure**, and if  $H_{\mathfrak{m}}^d(R) \rightarrow H_{\mathfrak{m}}^i(F_*R) = F_*H_{\mathfrak{m}}^i(R)$  is injective for every maximal ideal  $\mathfrak{m}$  of  $R$  and  $i \leq d$ , we say that  $R$  is **F-Injective**<sup>3</sup>.

**REMARK 1.14.**  $F$ -purity and  $F$ -injectivity are analogous to singularity classes arising from characteristic 0.  $F$ -pure singularities are related to (semi-)log canonical singularities [HW02], whereas  $F$ -injective singularities are related to Du Bois singularities [Sch09].

As a corollary of the proposition and the preceding discussion, we have the following:

**COROLLARY 1.14.1.** *Let  $R$  be a ring.*

- *If  $R$  is  $F$ -split, then  $R$  is  $F$ -pure.*
- *If  $R$  is  $F$ -pure, then  $R$  is  $F$ -injective.*

<sup>3</sup> $F$ -injectivity is most often studied with  $R$  is Cohen-Macaulay, and hence, when  $H_{\mathfrak{m}}^i(R) = 0$  for  $i < d$ . In this setting,  $F$ -injectivity is characterized by injectivity of a single map  $H_{\mathfrak{m}}^d(R) \rightarrow F_*H_{\mathfrak{m}}^d(R)$ . A similar argument to proposition 1.11 shows that  $F$ -purity still implies  $F$ -injectivity, even in the more general context.

- If  $R$  is  $F$ -finite or a complete local ring, then  $F$ -splitting and  $F$ -purity are equivalent.
- If  $R$  is Gorenstein, then  $F$ -purity and  $F$ -injectivity are equivalent.
- $R$  is  $F$ -pure (resp.  $F$ -injective) if and only if  $R_P$  is  $F$ -pure (resp.  $F$ -injective) for all  $P \in \text{Spec}(R)$ , if and only if  $R_{\mathfrak{m}}$  is  $F$ -pure (resp.  $F$ -injective) for all maximal ideals  $\mathfrak{m}$  of  $R$ .

Via faithful flatness of completion and the fact that completion plays well with Frobenius and local cohomology, we also have the following lemmas.

LEMMA 1.15. *Any regular ring is  $F$ -pure.*

PROOF. As regularity and  $F$ -purity are local notions, assume that  $R$  is local. A regular local ring has a faithfully flat Frobenius map  $R \rightarrow F_*R$  (see lemma 1.3) and faithfully flat maps are pure. Hence  $R$  has pure Frobenius and is  $F$ -pure.  $\square$

LEMMA 1.16. *Let  $(R, \mathfrak{m})$  be a local ring. Then  $R$  is  $F$ -pure if and only if  $\widehat{R}$  is  $F$ -pure.*

PROOF. This follows from faithful flatness of the completion map  $R \rightarrow \widehat{R}$  and the fact that  $\widehat{E}_R \cong E_{\widehat{R}}$ .  $\square$

LEMMA 1.17. *Let  $R \rightarrow S$  be a map of  $R$ -modules. If  $M \rightarrow N$  is a pure map of  $S$ -modules, then it is a pure map of  $R$ -modules.*

PROOF. Let  $L$  be any  $R$ -module. Then  $L \otimes_R S$  is an  $S$ -module, and the map  $(L \otimes_R S) \otimes_S M \rightarrow (L \otimes_R S) \otimes_S N$  is injective by  $S$ -purity. By associativity of tensor product,  $L \otimes_R S \otimes_S M \cong L \otimes_R M$ , and similarly for  $N$ , so  $M \rightarrow N$  must be  $R$ -pure.  $\square$

LEMMA 1.18. *If  $\varphi : R \rightarrow S$  is a pure map of  $R$ -modules and  $S$  is  $F$ -pure, then so is  $R$ .*

PROOF. If  $\varphi : R \rightarrow S$  is pure as a map of  $R$ -modules, and  $S \rightarrow F_*S$  is pure as a map of  $S$ -modules and hence as  $R$ -modules, then the composition  $R \rightarrow S \rightarrow F_*S$  is pure as a map of  $R$ -modules by the preceding lemma. This is the same map as the composition  $R \rightarrow F_*R \xrightarrow{F_*\varphi} F_*S$ , which immediately implies that  $R \rightarrow F_*R$  is pure.  $\square$

We note that none of these lemmas require  $R$  to be  $F$ -finite; this makes  $F$ -purity a better notion of singularity to consider in the non- $F$ -finite setting, and is precisely the notion of singularity we generalize in later chapters. We caution that the above lemmas do not always hold if you replace “ $F$ -pure” with “ $F$ -split”, unless you are in a setting when they are equivalent.

A combination of corollary 1.14.1 and lemma 1.16 allows us to reduce checking  $F$ -purity of a ring  $R$  to the complete local setting, where  $R \cong A/I$  for  $A$  a regular complete local ring by Cohen’s Structure Theorem. In this setting we have access to a powerful criterion to check for  $F$ -purity called **Fedder’s Criterion**, named after its creator Richard Fedder.

**THEOREM 1.19 ([Fed83]).** *Let  $(A, \mathfrak{m})$  be a regular local ring,  $I \subseteq A$  an ideal, and  $R \cong A/I$ . Then  $R$  is  $F$ -pure if and only if  $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$ .*

*In particular, for any  $f \in A$ ,  $A/(f)$  is  $F$ -pure if and only if  $f^{p-1} \in \mathfrak{m}^{[p]}$ .*

There also exists a graded analogue of Fedder’s criterion, replacing  $A$ , a regular local ring, with a polynomial ring  $k[x_1, \dots, x_d]$ . Fedder’s result does not require that  $R$  be  $F$ -finite; his proof required a reduction to the  $F$ -finite setting. He did this by showing that for  $R$  an  $F$ -pure algebra over  $\mathfrak{K}$  a field of characteristic  $p > 0$ . Then for any extension  $L/\mathfrak{K}$ ,  $L \otimes_{\mathfrak{K}} R$  is  $F$ -pure. We will prove this in section 2 (proposition 1.90).

After completing (cf. lemma 1.16), this allowed him to base change the coefficient field of  $R$  to its perfect closure, reducing to the  $F$ -finite setting. We warn that  $F$ -singularity properties typically need not be preserved by arbitrary field extension, typically only separable ones. Examples of where this fails will be discussed later this section, as well as in later chapters. We will discuss a work-around to this problem, called the  $\Gamma$ -construction [HH94a], and related results in section 2.3.

Fedder's Criterion provides a remarkably simple way to verify  $F$ -purity. We use it to construct example of  $F$ -pure rings (which, in the  $F$ -finite setting, are  $F$ -split) below. These examples are from [MP25, Example 2.10], though the proofs are standard knowledge.

EXAMPLE 1.20. *Any Stanley-Riesner ring is  $F$ -pure.*

PROOF. Recall that a ring  $k[x_1, \dots, x_d]/I$  is a Stanley-Riesner ring if  $I$  is generated by square-free monomials. Let  $g = x_{i_1} \cdots x_{i_k}$  be any such square-free monomial. Observe that

$$(x_1 \cdots x_d)^{p-1} \cdot g = \prod_{j \in \{i_1, \dots, i_k\}} x_j^p \cdot \prod_{\ell \notin \{i_1, \dots, i_k\}} x_\ell^{p-1} = g^p \cdot \prod_{\ell \notin \{i_1, \dots, i_k\}} x_\ell^{p-1} \in (g^p)$$

As this holds for any square free monomial, for any ideal generated by square free monomials  $I$ ,  $(x_1 \cdots x_d)^{p-1} \in (I^{[p]} : I)$ . Thus,  $(I^{[p]} : I) \not\subseteq \mathfrak{m}^{[p]}$ .  $\square$

EXAMPLE 1.21. *Let  $A = k[x, y, z]$  and  $f = x^3 + y^3 + z^3 \in A$ . Then  $A/(f)$  is  $F$ -pure if and only if  $\text{char}(k) \equiv 1 \pmod{3}$ .*

PROOF. Let  $p = \text{char}(k)$ . If  $p = 3$  then  $A$  is not reduced, and hence cannot be  $F$ -pure. If  $p \equiv 1 \pmod{3}$ , then  $\frac{p-1}{3} \in \mathbb{Z}$ , and hence

$$c \cdot (x^3)^{\frac{p-1}{3}} \cdot (y^3)^{\frac{p-1}{3}} \cdot (z^3)^{\frac{p-1}{3}} = c(xyz)^{p-1}$$

is a monomial of  $(x^3 + y^3 + z^3)^{p-1}$ . As  $(xyz)^{p-1} \notin \mathfrak{m}^{[p]}$ ,  $(x^3 + y^3 + z^3)^{p-1} \notin \mathfrak{m}^{[p]}$  so  $A/(f)$  is  $F$ -pure by Fedder's criterion. If  $p \equiv 2 \pmod{3}$ , then  $\frac{p-1}{3} \notin \mathbb{Z}$ , and in particular, we claim that every nonzero monomial

$$M_{i,j,\ell} := \binom{p-1}{i,j,\ell} x^{3i} y^{3j} z^{3\ell}$$

of  $(x^3 + y^3 + z^3)^{p-1}$ , where  $i + j + \ell = p - 1$  must satisfy least one of  $3i > p - 1$ ,  $3j > p - 1$ , or  $3\ell > p - 1$ . This would imply that  $M_{i,j,\ell} \in \mathfrak{m}^{[p]}$  for any  $i, j, \ell$ ; therefore  $(x^3 + y^3 + z^3)^{p-1} \in \mathfrak{m}^{[p]}$  and  $A/(f)$  is not  $F$ -pure by Fedder's criterion. To prove the claim, suppose that none of the inequalities held. Then  $i, j, \ell \leq \frac{p-1}{3}$ . As  $i, j, \ell \in \mathbb{N}$ ,  $i, j, \ell \leq \lfloor \frac{p-1}{3} \rfloor = \frac{p-2}{3}$ . It would follow then that  $i + j + \ell \leq p - 2$ , which cannot hold as  $i + j + \ell = p - 1$ .  $\square$

We will return to example 1.21 in section 1.4 to compute the  $F$ -pure threshold, or measure of failure of  $F$ -purity, when  $p \equiv 2 \pmod{3}$  (proposition 1.34). Viewed in another context,  $x^3 + y^3 + z^3$  cuts out an elliptic curve which is ordinary if and only if  $p \equiv 1 \pmod{3}$ ; otherwise it is supersingular. It turns out that the computation above is an example of a more general trend; an elliptic curve is ordinary if and only if its cone point is  $F$ -pure (see proposition 1.29). Supersingular elliptic curves, while not  $F$ -pure, turn out to be the simplest example of a so-called “quasi- $F$ -pure” variety; we will return to study these in chapter 2.

**1.3. Global Frobenius Splittings.**  $F$ -purity is a local notion; thus for a scheme  $X$ ,  $F : \mathcal{O}_X \rightarrow F_* \mathcal{O}_X$  is a pure map of  $\mathcal{O}_X$ -modules if and only if the map is pure at all stalks. If  $\mathcal{O}_X \rightarrow F_* \mathcal{O}_X$  splits in the category of  $\mathcal{O}_X$ -modules, this localizes to being  $X$  being  $F$ -split at every stalk (see lemma 1.9). However, the converse is false; admitting a global splitting is a much stronger condition that imposes significant geometric control of your variety.

**DEFINITION 1.22.** Let  $X$  be a scheme of characteristic  $p > 0$ . Then  $X$  is **(globally) F-split** if  $F : \mathcal{O}_X \rightarrow F_* \mathcal{O}_X$  splits in the category of  $\mathcal{O}_X$ -modules.  $X$  is **Locally F-split** if every stalk  $\mathcal{O}_{X,x}$  is an  $F$ -split local ring.  $X$  is **F-pure** if  $F : \mathcal{O}_X \rightarrow F_* \mathcal{O}_X$  is a pure map of  $\mathcal{O}_X$ -modules (or equivalently,  $\mathcal{O}_{X,x}$  is  $F$ -pure for all  $x \in X$ ).

Equivalently,  $X$  is (globally)  $F$  split if there exists a surjection

$$H^0(X, \mathcal{H}om_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X)) \rightarrow H^0(X, \mathcal{O}_X)$$

It is also easy to check that  $X$  is  $F$ -split if and only if  $\mathcal{O}_X \xrightarrow{F^e} F_*^e \mathcal{O}_X$  splits for some (equivalently, all)  $e > 0$ . For an  $F$ -finite scheme, local  $F$ -splitting and  $F$ -purity are equivalent. For an affine  $F$ -finite scheme, all three definitions are equivalent by corollary 1.14.1.

**LEMMA 1.23.** *Let  $X$  be a scheme of finite type over a field  $k$  of positive characteristic. Then for any extension  $L/k$ , if  $L \times_k X$  is (locally or globally)  $F$ -split, then so is  $X$ .*

**PROOF.** Recall that any field is  $F$ -split, as  $F_* k$  is always a  $k$  vector space. Thus the local argument is exactly that of lemma 1.10. The global argument follows much the same.

Taking the projection map  $\pi : L \times_k X \rightarrow X$ , this descends to a split map  $\mathcal{O}_X \rightarrow \pi^* \mathcal{O}_X$ . There exists a splitting of Frobenius on  $\pi_* \mathcal{O}_X$ , and hence we can construct a splitting on  $F : \mathcal{O}_X \rightarrow F_* \mathcal{O}_X$  with the following composition:

$$F_* \mathcal{O}_X \rightarrow \pi_* F_* \mathcal{O}_X \rightarrow \pi_* \mathcal{O}_X \rightarrow \mathcal{O}_X$$

□

We caution that the converse of this is false, even in simple cases; examples that specialize to  $F$ -splittings will be discussed in chapter 2.

LEMMA 1.24 ([SS25], Example 3.1.4). *Over any field  $k$  of positive characteristic,  $\mathbb{P}^n$  is globally  $F$ -split.*

PROOF. Citing the prior lemma, we can assume that  $k$  is perfect. Fix homogeneous coordinates  $x_0, \dots, x_n$  of  $\mathbb{P}^n$ .  $\mathbb{P}^n$  is covered by affine charts

$$D_*(x_i) := \text{Spec} \left( k \left[ \frac{x_0}{x_i}, \dots, \frac{x_n}{x_i} \right] \right)$$

Which are  $F$ -split via the splitting

$$F_* \left( \lambda \left( \frac{x_0}{x_i} \right)^{t_0} \dots \lambda \left( \frac{x_n}{x_i} \right)^{t_n} \right) \mapsto \begin{cases} \lambda^{1/p} \left( \frac{x_0}{x_i} \right)^{t_0/p} \dots \lambda^{1/p} \left( \frac{x_n}{x_i} \right)^{t_n/p} & t_j/p \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

These local splittings can be glued together as they are equivalently defined on intersections of the affine patches  $D_+(x_i) \cap D_+(x_j) = D_+(x_i x_j)$ . □

It is again worth stressing that you cannot 'glue' local splittings in general. This really only works because  $\mathbb{P}^n$  is  $\text{Proj}(k[x_0, \dots, x_n])$  and the polynomial ring is  $F$ -split. This correspondence holds in far more generality:

THEOREM 1.25 ([SS25], Theorem 3.4.1). *Let  $R$  be an  $\mathbb{N}$ -graded  $F$ -finite ring of characteristic  $p > 0$  that is finitely generated over its degree 0 subring. If  $R$  is  $F$ -split then  $\text{Proj}(R)$  is globally  $F$ -split.*

We also can prove a global analogue of lemma 1.12 utilizing Serre duality:

**THEOREM 1.26** ([**SS25**], Theorem 3.1.27). *Let  $X$  be a smooth projective variety of dimension  $d$ . Then  $X$  is globally  $F$ -split if and only if*

$$H^d(X, \omega_X) \xrightarrow{F} H^d(X, \omega_X^p)$$

*is injective.*

**PROOF.** Frobenius splitting is equivalent to constructing a surjection

$$H^0(X, \mathcal{H}om_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X)) \twoheadrightarrow H^0(X, \mathcal{O}_X)$$

Applying Serre duality, this is equivalent to the injectivity of

$$\text{Ext}^d(\mathcal{O}_X, \omega_X) \rightarrow \text{Ext}^d(\mathcal{H}om_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X), \omega_X)$$

But  $X$  is smooth, so by Kunz's theorem  $F_* \mathcal{O}_X$  is a locally free sheaf and hence

$$\text{Ext}^d(\mathcal{H}om_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X), \omega_X) \cong \text{Ext}^d(\mathcal{O}_X, F_* \mathcal{O}_X \otimes \omega_X) \cong H^d(X, F_* \mathcal{O}_X \otimes \omega_X) \cong H^d(X, \omega_X^p)$$

$\text{Ext}^d(\mathcal{O}_X, \omega_X) \cong H^d(X, \omega_X)$ , and hence  $H^0(X, \mathcal{H}om_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X)) \twoheadrightarrow H^0(X, \mathcal{O}_X)$  is surjective if and only if  $H^d(X, \omega_X) \xrightarrow{F} H^d(X, \omega_X^p)$  is injective, completing the proof.  $\square$

**REMARK 1.27.** *Both smoothness and projectivity can be relaxed; this theorem holds for any proper normal variety  $X$  [**SS25**, theorem 3.2.16].*

Parallel to the development of the local study of  $F$ -purity and  $F$ -splittings, global Frobenius splittings were introduced by Mehta, Ramanan and Ramanathan in order to study Grassmannians and Schubert varieties in positive characteristic [**MR85**, **RR85**, **Ram87**]. Their crucial insight, which has since expanded in scope, was that Frobenius-split varieties recover key properties of characteristic 0 varieties that are known not to extend to characteristic  $p$ . For instance, though a smooth projective variety in characteristic  $p$  need not satisfy Kodaira Vanishing [**Ray78**],  $F$ -split varieties do.

**THEOREM 1.28** (Kodaira Vanishing for  $F$ -split Varieties). *Let  $X$  be a Cohen Macaulay projective scheme of characteristic  $p > 0$  and dimension  $d$ , and  $L$  an ample line bundle. If  $X$  is globally  $F$ -split, then  $X$  satisfies Kodaira vanishing. That is,*

$$H^i(X, \omega_X \otimes L) = 0$$

for all  $i > 0$ .

**PROOF.** Fix  $i > 0$ . By Serre duality, it is sufficient to show that  $H^{d-i}(X, L^{-1}) = 0$ . Tensoring the iterated Frobenius  $F^e : \mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X$  with  $L^{-1}$  yields a map  $L \rightarrow F_*^e L^{-p^e}$ . As  $F^e$  splits, the corresponding map on cohomology  $H^{d-i}(X, L^{-1}) \hookrightarrow H^{d-i}(X, L^{-p^e})$  is injective, so it is sufficient to check that  $H^{d-i}(X, L^{-p^e}) = 0$  for  $e \gg 0$ . This is Serre dual to  $H^i(X, \omega_X \otimes L^{p^e})$ ; for  $e$  large  $L^{p^e}$  is very ample, and hence  $H^i(X, \omega_X \otimes L^{p^e})$  vanishes due to Serre vanishing, a characteristic independent result.  $\square$

The Cohen-Macaulay hypothesis can be weakened slightly; see [SS25, Theorem 3.2.32]. As an application of these results, we can classify the Frobenius splitting of curves.

**PROPOSITION 1.29.** *Let  $X$  be a smooth projective curve over an algebraically closed field. Then  $X$  is always locally  $F$ -split. Further, global  $F$ -splitting is completely determined by the genus  $g(X) = g$  of the curve:*

- (1) *If  $g = 0$ ,  $X$  is  $F$ -split.*
- (2) *If  $g = 1$ , then  $X$  is an elliptic curve that is  $F$ -split if and only if it is ordinary.*
- (3) *If  $g > 1$ , then  $X$  is never  $F$ -split.*

**PROOF.** The genus 0 case follows immediately from lemma 1.24. For genus 1, recall that an elliptic curve is ordinary if Frobenius acts bijectively on  $H^1(X, \mathcal{O}_X)$ . As  $\mathcal{O}_X = \omega_X$  for elliptic curves, we conclude by the theorem 1.26. If  $g > 1$ , then  $\omega_X$  is ample; it follows from a proof similar to that of Kodaira vanishing that if  $X$  were  $F$ -split,  $H^1(X, \omega_X) = 0$ . However, this is Serre dual to  $H^0(X, \mathcal{O}_X)$  which cannot be 0.  $\square$

Through more involved arguments, various stronger vanishing theorems are known to hold for smooth projective  $F$ -split varieties, such as Kawamata-Viehweg Vanishing. This can be deduced from the fact that smooth  $F$ -split varieties lift modulo  $p^2$ , that is,  $X$  lifts to  $W_2(k)$  [Zda18]. Satisfying key vanishing theorems, in part, makes testing for global  $F$ -splitting quite useful in birational geometry and in the Minimal Model Program in positive characteristic. As quasi- $F$ -split varieties are also  $W_2(k)$ -liftable, they also satisfy these vanishing theorems; thus we will discuss the implications of (quasi-) $F$ -split singularities in birational geometry [KTT<sup>+</sup>25, KTT<sup>+</sup>24a, KTT<sup>+</sup>24b] in chapter 3 section 4.

Through work of Deligne and Illusie [DI87], varieties of characteristic  $p$  that are  $W_2(k)$  liftable, if less than dimension  $p$ , have a degenerate Hodge-de-Rham spectral sequence, implying that all  $F$ -split varieties of low enough dimension admit a degenerate Hodge-de-Rham spectral sequence. This dimension requirement was recently removed by Petrov [Pet25], even for quasi- $F$ -split varieties.

$F$ -splitting also detects arithmetic data: For K3 surfaces and Abelian varieties, being  $F$ -split is equivalent to being ordinary<sup>4</sup> motivating connections to the weak ordinarity conjecture [BST17]. Applications in these directions form the basis for the study of quasi- $F$ -splittings; these will be discussed in chapter 2 section 1.

**1.4.  $F$ -Pure Thresholds.** In this subsection we will briefly review the  $F$ -pure threshold, an asymptotic invariant of Takagi and Watanabe [TW04] used to measure the failure of  $F$ -purity. This will be a (roughly) self contained exposition of  $F$ -pure thresholds for principal quotients of regular rings; thus we will take the simplified definition of the  $F$ -pure threshold introduced in [MTW05] to be our definition.

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<sup>4</sup>as these are Calabi-Yau varieties,  $\omega_X = \mathcal{O}_X$ ; one can then deduce that Frobenius acts bijectively on  $H^{\dim X}(X, \mathcal{O}_X)$  from theorem 1.26. This is called being “weakly ordinary”, which is equivalent to ordinarity for Abelian varieties and K3 surfaces. See chapter 2 section 1.4 for a detailed discussion on ordinarity in general.

$F$ -pure thresholds are the lowest  $F$ -threshold, a more general set of invariants with ties to the study of test ideals in positive characteristic.  $F$ -purity, the  $F$ -pure threshold, and the test ideal are characteristic  $p$  analogues of log canonicity, the log canonical threshold [MTW05], and the multiplier ideal in characteristic 0 [BST15]. Finally, we will reference this section to compute the  $F$ -pure threshold of any quasi- $F$ -pure hypersurface in chapter 2 section 4.

Let  $A = k[x_1, \dots, x_n]$ . For a polynomial  $f \in \mathfrak{m}$  and any  $e \geq 1$ , we define

$$\nu_f(p^e) := \max \{N \mid f^N \notin \mathfrak{m}^{[p^e]}\}$$

Via Fedder's criterion (Theorem 1.19), if  $R := A/(f)$  is  $F$ -pure, it can be readily seen that  $\nu_f(p) = p - 1$ . More generally, for  $e \gg 0$ ,  $\nu_f(p^e) \sim p^e$ , i.e. their ratio is 1 in the limit. If  $R$  was not  $F$ -pure,  $\nu_f(p^e)$  would be smaller than  $p^e$  as  $e$  grows. The ratio between these quantities in the limit is precisely the  $F$ -pure threshold.

**DEFINITION 1.30.** Let  $A = k[x_1, \dots, x_n]$ . For a polynomial  $f \in \mathfrak{m}$  and any  $e \geq 1$ , we define the **F-pure threshold** as follows:

$$\text{fpt}(f) := \lim_{e \rightarrow \infty} \frac{\nu_f(p^e)}{p^e} \in (0, 1]$$

$\text{fpt}(f)$  is always a rational number [BMS08] contained in  $(0, 1]$  and  $\text{fpt}(f) = 1$  if and only if the corresponding quotient singularity  $k[x_1, \dots, x_n]/(f)$  is  $F$ -pure. The lower the  $F$ -pure threshold is, the further away the hypersurface is from being  $F$ -pure. We note that  $\left\{ \frac{\nu_f(p^e)}{p^e} \right\}_e$  forms a non-decreasing sequence, so for any  $e \geq 1$ ,

$$\text{fpt}(f) \geq \frac{\nu_f(p^e)}{p^e} > 0$$

Just as  $F$ -pure singularities are related to log canonical singularities in characteristic 0, there exists an analogous relationship between the  $F$ -pure threshold and log canonical threshold:

**THEOREM 1.31** ([**MTW05**] Theorem 3.3,3.4). *Let  $f \in \mathbb{C}[x_1, \dots, x_n]$  be a polynomial with integer coefficients. For a given prime  $p$ , let  $f_p \in \mathbb{F}_p[x_1, \dots, x_n]$  denote  $f$  with coefficients reduced modulo  $p$ . Then for all primes  $p$ ,  $\text{fpt}(f_p) \leq \text{lct}(f)$ , and further,*

$$\lim_{p \rightarrow \infty} \text{fpt}(f_p) = \text{lct}(f)$$

The  $F$ -pure threshold has been computed explicitly for wide classes of examples [**MTW05**, **BMS06**, **HMTW08**, **BMS09**, **Her11**, **Her14**, **Her15**, **DSNnB18**, **BS15**, **HNnBWZ16**, **KKP<sup>+</sup>22**] and specific examples can be easily computed via the `FrobeniusThresholds` package [**HSTW21**] in `Macaulay2`.

We highlight two such results below that we will apply in chapter 2 section 4, in forms general enough for our needs.

**THEOREM 1.32** ([**HNnBWZ16**], theorem 3.5). *Fix an  $\mathbb{N}$ -grading on  $k[x_1, \dots, x_n]$ , for  $k$  a field of characteristic  $p$ . Let  $f \in k[x_1, \dots, x_n]$  be a homogeneous isolated singularity such that  $\sum \deg(x_i) \geq \deg(f)$ . Then*

$$\text{fpt}(f) = 1 - \frac{E}{p^L}$$

for  $E, L \in \mathbb{N}$ , where  $L \geq 1$  and  $0 \leq E \leq n - 2$ . If  $p > n - 2$ , then  $L = 1$ .

This result shows that, for characteristic large enough,  $\text{fpt}(f)$  is of the form  $h/p$  for some  $1 \leq h \leq p$ . We caution that this is a much less general form of the theorem given in [**HNnBWZ16**], but is all we will need for our purposes.

We recall that the Fermat cubic  $x^3 + y^3 + z^3$  cuts out an  $F$ -pure hypersurface if and only if  $p$ , the characteristic of  $k$ , is congruent to 1 mod 3; it follows in this case that  $\text{fpt}(x^3 + y^3 + z^3) = 1$ . If  $p \equiv 2 \pmod{3}$  we know that  $\text{fpt}(x^3 + y^3 + z^3) < 1$ , but the exact value was not known until work of Bhatt and Singh whom computed the  $F$ -pure threshold of not only elliptic curves, but any Calabi-Yau hypersurface (see also [**Bha12**]).

**THEOREM 1.33 ([BS15]).** *Suppose  $R = k[x_1, \dots, x_n]$  is a polynomial ring over a field of characteristic  $p > n + 2$  and  $f$  is a homogeneous polynomial such that  $X = \text{Proj}(R/(f))$  is a smooth Calabi-Yau hypersurface. Then*

$$\text{fpt}(f) = 1 - \frac{h}{p}$$

for  $0 \leq h \leq \dim(X)$ . If  $p^2 > n^2 + n - 1$ , then  $h$  is the order of vanishing of the Hasse Invariant on the versal deformation space of  $X \subseteq \mathbb{P}^n$ . If  $X$  admits a lift modulo  $p^2$ ,  $h$  can be taken to be the  $a$ -number of  $X$ .

$a = a(X)$  is the  $a$ -number of a variety in positive characteristic, a generalization of the  $a$ -number of an abelian variety, defined in full generality by van der Geer and Katsura [vdGK13]. This invariant will be further discussed in chapter 2, particularly its relation with the so-called Artin-Mazur height of a Calabi-Yau variety [AM77] in section 1. For curves, it follows from the dimension bound on  $h$  that  $h$  can be exactly 0 or 1. This, coupled with the fact that supersingular elliptic curves are not  $F$ -pure and hence  $\text{fpt} \neq 1$ , imply the following proposition:

**PROPOSITION 1.34.** *Let  $f \in k[x, y, z]$  be the defining equation of an elliptic curve over a field of characteristic  $p > 11$ . Let  $R = k[x, y, z]/(f)$ . Then,*

$$\text{fpt}(f) =^{(*)} \begin{cases} 1 & \text{Proj}(R) \text{ is ordinary} \\ 1 - \frac{1}{p} & \text{Proj}(R) \text{ is supersingular} \end{cases}$$

In particular,

$$\text{fpt}(x^3 + y^3 + z^3) = \begin{cases} 1 & p \equiv 1 \pmod{3} \\ 1 - \frac{1}{p} & p \equiv 2 \pmod{3} \end{cases}$$

In chapter 2 section 4, we will show  $(*)$  holds not only for supersingular elliptic curves, but *any* quasi- $F$ -pure but not  $F$ -pure singularity.

**1.5. Strong  $F$ -Regularity.** In this subsection we recall the definition of a strengthening of Frobenius splitting, called strong  $F$ -regularity, in both the local and global setting. We will review this invariants' fundamental properties, consequences, and applications.

DEFINITION 1.35. Let  $R$  be an  $F$ -finite<sup>5</sup> ring of characteristic  $p$ .  $R$  is **strongly- $F$ -Regular** if for all nonzero divisors  $c \in R$ , there exists an  $e \in \mathbb{N}$  for which the composition  $R \xrightarrow{F^e} F_*^e R \xrightarrow{\cdot F_*^e c} F_*^e R$  assigning  $r \mapsto F_*^e r^{p^e} \mapsto F_*^e r^{p^e} c$  splits as a map of  $R$ -modules.

If  $R \xrightarrow{F^e} F_*^e R \xrightarrow{\cdot F_*^e c} F_*^e R$  splits for a specific  $c$ , we will say that  $R$  is  $F$ -split along  $c$ , and refer to this splitting  $F_*^e R \rightarrow R$  as a splitting along  $c$ . Another way of defining strong  $F$ -regularity is that  $R$  is  $F$ -split along every nonzero divisor.

LEMMA 1.36. *If  $R$  is strongly- $F$ -regular then it is  $F$ -split.*

PROOF. Take the splitting along  $c = 1$ . □

In the original definition of strong  $F$ -regularity [HH89], Hochster and Huneke required strongly  $F$ -regular rings to be split along all elements not contained in a minimal prime. In view of the prior lemma, all strongly  $F$ -regular rings are  $F$ -split and hence reduced. In this case, the set of non-zero divisors is exactly the union of the minimal primes.

We also note that, if  $R \rightarrow F_*^e R$  assigning  $1 \mapsto F_*^e c$  splits, then so will  $R \rightarrow F_*^E R$  for any  $E \geq e$ , but  $R \rightarrow F_*^{e'} R$  need not split for any  $e' < e$ . These are all equivalent notions for  $c = 1$  (hence,  $F$ -splittings are defined by a single iterate of Frobenius splitting) but splitting along an element may require a “sufficiently” high iterate (cf. [SS25][lemma 4.7, caution 4.8]).

REMARK 1.37. *One reason for the “strongly” adverb is due to this definition being an a priori strengthening of earlier definitions of  $F$ -regularity defined via tight closure [HH89,*

<sup>5</sup>As we will see in section 2 and in chapter 2 section 2, Frobenius splitting (even along elements) is not a good notion of singularity outside of the  $F$ -finite setting. We will discuss non- $F$ -finite generalizations of this definition in section 2.

**HH90, HH91**]. A ring is weakly- $F$ -regular if all of its ideals are so-called tightly closed and a ring is  $F$ -regular if its localizations are weakly- $F$ -regular (tight closure famously does not localize!).

It is known that strongly  $F$ -regular implied  $F$ -regular implies weakly  $F$ -regular [**HH94a**], though the equivalence is a major open question in commutative algebra, verified in specific cases such as graded rings [**LS99**]. In this thesis we will not cover the machinery of tight closure, but it is worth stressing the mathematical importance of the “weak implies strong” conjecture.

Strong- $F$ -regularity localizes and forces the ring to be a Cohen-Macaulay normal domain, imposing a significant amount of control on the ring for a condition defined solely through Frobenius.

**PROPOSITION 1.38** ([**HH89, HH94a**]). *All strongly  $F$ -regular rings are Cohen-Macaulay normal domains. Further, strong  $F$ -regularity is a local property. That is, for any  $F$ -finite ring  $R$ ,  $R$  is strongly  $F$ -regular if and only if  $R_P$  is strongly  $F$ -regular for every  $P \in \text{Spec}(R)$ .*

All  $F$ -finite regular rings are strongly- $F$ -regular, so strong- $F$ -regularity can be viewed as a mild singularity condition.

**LEMMA 1.39.** *Let  $R$  be an  $F$ -finite regular ring. Then  $R$  is strongly- $F$ -regular.*

**PROOF.** We follow the proof given in [**MP25**][theorem 3.4]. In view of the prior lemma, we can assume that  $(R, \mathfrak{m})$  is an  $F$ -finite regular local ring, in which case  $F_*^e R$  is a finitely generated free  $R$ -module due to Kunz’s theorem. For any nonzero  $c \in R$ , we claim that there exists an  $e > 0$  for which  $F_*^e c$  is an element of a minimal free basis of  $F_*^e R$ . If true, then it immediately follows that  $1 \mapsto F_*^e c$  splits as a morphism of  $R$ -modules, proving the lemma.

If the claim were false, then that  $F_*^e c \in \mathfrak{m} \cdot F_*^e R = F_*^e(\mathfrak{m}^{[p^e]})$  for all  $e$ , implying that  $c \in \bigcap_e \mathfrak{m}^{[p^e]} = 0$ , a contradiction.  $\square$

We recall that a ring  $R$  is called a **splinter** if any finite extension  $R \rightarrow S$  splits as a morphism of  $R$ -modules. Splinters form the basis for the **direct summand conjecture**,

one of the famous homological conjectures posed in the 1960s, asking whether all regular rings are splinters. In characteristic 0, there is an elementary proof that a ring is a splinter if and only if it is normal (you reduce to the case where  $\text{Frac}(S)/\text{Frac}(R)$  is a normal extension, then construct a splitting by averaging over the Galois group.). In characteristic  $p$  this was proven by Hochster [**Hoc73**], though the case in mixed characteristic remained open for over 50 years. Recently, this conjecture was proven in full generality by André [**And18**], who proved the mixed characteristic case via Scholze’s theory of perfectoid algebras [**Sch12**].

Yet, the question of both a necessary and sufficient condition for being a splinter in positive (and mixed) characteristic remains. One candidate for this condition in positive characteristic is strong  $F$ -regularity.

**THEOREM 1.40.** *Every strongly  $F$ -regular ring is a splinter.*

**PROOF.** We follow the proof of [**MP25**][theorem 3.5]. Via proposition 1.38 we can reduce to the case where  $R$  is a local strongly  $F$ -regular domain. Let  $S$  be a finite extension of  $R$ : after quotienting by a minimal prime, we can assume that  $S$  is a domain as well. As  $S$  is now a torsion-free  $R$ -module, one can construct a section  $S \rightarrow R$  assigning  $1 \mapsto c \neq 0$ . Using the strong  $F$ -regularity of  $R$ , one can find an  $e$  for which map  $R \rightarrow F_*^e R$  sending  $1 \mapsto F_*^e c$  splits. This means we can construct a splitting  $S \rightarrow R$  as follows:

$$S \xrightarrow{F^e} F_*^e S \xrightarrow{F_*^e 1 \mapsto F_*^e c} F_*^e R \xrightarrow{F_*^e c \mapsto 1} R$$

□

It is currently a major open question as to whether the converse of theorem 1.40 holds; that is, all splinters are strongly- $F$ -regular. The equivalence is known to hold for  $\mathbb{Q}$ -Gorenstien rings [**HH94b**, **Sin99**] yet is very much open in general. Weakly  $F$ -regular rings are splinters, so resolving this conjecture will also resolve the weak implies strong conjecture discussed in remark 1.37.

LEMMA 1.41. *If  $R \rightarrow S$  is a finite or faithfully flat extension and  $S$  is strongly  $F$ -regular, then so is  $R$ .*

PROOF. Theorem 1.40 implies that if  $R \rightarrow S$  is finite,  $R \rightarrow S$  splits. The proof of this follows similarly to that of lemma 1.10. Now suppose  $R \rightarrow S$  is faithfully flat and pick a non-zerodivisor  $c \in R$ . By strong  $F$ -regularity of  $S$ , there is an  $e > 0$  for which  $S \rightarrow F_*^e S$  assigning  $1 \mapsto F_*^e c$  splits, viewing  $c$  as the image of  $c \in R$  in  $S$ . As  $R \rightarrow S$  is faithfully flat, the map  $R \rightarrow F_*^e R$  assigning  $1 \mapsto F_*^e c$  splits if and only if  $S \rightarrow F_*^e R \otimes_R S$  splits, i.e the map splits after applying  $- \otimes_R S$ . We can construct a splitting of this as follows, using the splitting on  $S$  along  $c$ :

$$F_*^e R \otimes_R S \rightarrow F_*^e S \rightarrow S$$

□

Much like  $F$ -purity, strong- $F$ -regularity of quotients of regular rings can be tested for explicitly by Glassbrenner's Criterion:

THEOREM 1.42 (Glassbrenner's Criterion [Gla96]). *Let  $(S, \mathfrak{m})$  be a  $F$ -finite regular local ring,  $I \subseteq S$  an ideal, and  $R = S/I$ . Then,  $R$  is strongly  $F$ -regular if and only if for every  $c \in S$  not in any minimal prime of  $I$ , there exists an  $e > 0$  such that  $c(I^{[p^e]} : I) \not\subseteq \mathfrak{m}^{[p^e]}$ .*

One issue in verifying strong  $F$ -regularity, either directly or via Glassbrenner's criterion, is verifying the given conditional for any possible  $c$ . If  $R$  is strongly  $F$ -regular if and only if, for a fixed  $c$ , there is an  $e > 0$  for which  $R \rightarrow F_*^e R$  assigning  $1 \mapsto F_*^e c$  splits, we call this  $c$  a **test element**.

THEOREM 1.43 ([HH89], theorem 3.4). *Let  $R$  be an  $F$ -finite ring. Then for any  $c$  such that  $R_c$  is strongly  $F$ -regular, there exists a  $m$  for which  $c^m$  is a test element.*

REMARK 1.44. *Such  $c$ , notably, exist and are somewhat easy to find. For any excellent reduced ring, the ideal  $I$  cutting out the non-regular locus of  $\text{Spec}(R)$  (closed by excellence) is not contained in any minimal prime. Hence, by prime avoidance one can always find a  $c \in I$  outside of any minimal prime. Localizing  $R$  at  $c$  yields a regular ring, which in*

particular is strongly  $F$ -regular. The existence of test elements is very useful for checking various permanence properties of strong  $F$ -regularity, such as invariance under completion.

LEMMA 1.45. *Let  $(R, \mathfrak{m})$  be an  $F$ -finite local ring. then  $R$  is strongly  $F$ -regular if and only if  $\widehat{R}$  is strongly  $F$ -regular.*

PROOF. Working in either direction, we can assume that  $R$  is a domain (proposition 1.38). As  $R$  is  $F$ -finite it is also excellent, and hence there exists  $c \in R$  for which  $R_c$  is regular, and hence, strongly  $F$ -regular. Thus, it is sufficient to show that  $R \rightarrow F_*^e R$  assigning  $1 \mapsto F_*^e c$  splits if and only if the analogous map  $\widehat{R} \rightarrow F_*^e \widehat{R}$  splits. By  $F$ -finiteness, the fact that  $E_R \cong E_{\widehat{R}}$ , and proposition 1.11, this is equivalent to showing that  $E \rightarrow E \otimes_R F_*^e R$  is injective if and only if  $E \rightarrow E \otimes_{\widehat{R}} F_*^e \widehat{R}$  is injective. This equivalence follows from the considering the composition  $E \rightarrow E \otimes_R F_*^e R \rightarrow E \otimes_{\widehat{R}} F_*^e \widehat{R}$  and faithful flatness, and hence purity, of completion.  $\square$

We caution that the  $F$ -finite hypothesis is necessary; there exist DVRs that are not  $F$ -finite but have  $F$ -finite completion [DS18]. Similarly to  $F$ -splittings, strong  $F$ -regularity can be extended to a global invariants for schemes of positive characteristic. The definition of strong  $F$ -regularity is motivated by the following fact about strongly  $F$ -regular rings.

LEMMA 1.46. *Let  $R$  be strongly  $F$ -regular. then for any effective divisor  $D$ , there exists an  $e > 0$  for which  $R \rightarrow F_*^e R \rightarrow F_*^e R(D)$  splits.*

PROOF. Choosing  $f$  such that  $D \leq \text{div}(f)$ , it is sufficient to check that  $R \rightarrow F_*^e R \rightarrow F_*^e R(\text{div}(f)) = F_*^e R[1/f]$  splits for some  $e > 0$ . Clearing denominators yields an equivalence

$$\text{im} \left( R \xrightarrow{F^e} F_*^e R \rightarrow F_*^e R[1/f] \right) = \text{im} \left( R \xrightarrow{F^e} F_*^e R \xrightarrow{\cdot F_*^e f} F_*^e R \right)$$

and hence, we can find an  $e$  for which the composition splits from the definition of strong  $F$ -regularity.  $\square$

Eventually splitting along every divisor  $D$  is precisely the definition of global  $F$ -regularity, introduced by Smith in [Smi00]. We recommend [SS10, SS25] for further exposition and

proofs of the following results in this section, though the ideas are quite similar to the theory of global Frobenius splittings discussed in subsection 1.3. Global  $F$ -regularity is closely connected to the KLT (or “Kawamata-Log-Terminal”) singularity condition arising from birational geometry, see [HW02].

DEFINITION 1.47. An  $F$ -finite Noetherian scheme  $X$  is **Globally F-regular** if for all effective divisors  $D$  there exists an  $e > 0$  for which

$$\mathcal{O}_X \xrightarrow{F^e} F_*^e \mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X(D)$$

splits as a morphism of  $\mathcal{O}_X$  modules.  $X$  is **Locally F-regular** if  $\mathcal{O}_{X,x}$  is strongly  $F$ -regular for all  $x \in X$ .

As strong  $F$ -regularity localizes (proposition 1.38) it is easy to see that these notions are both equivalent for  $X = \text{Spec}(R)$  to strong  $F$ -regularity of  $R$ . From a similar argument to lemma 1.36, one sees that globally  $F$ -regular varieties are automatically globally  $F$ -split.

Much like the affine case, global  $F$ -regularity can be checked along a single effective divisor:

THEOREM 1.48 ([SS25], Theorem 3.2.14). *Suppose  $X$  is an  $F$ -finite scheme and  $B$  an effective divisor for which  $X$  is globally  $F$ -regular outside of the support of  $B$ . Then  $X$  is globally  $F$ -regular if and only if there exists an  $e$  for which  $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X(B)$  splits.*

Similarly to global  $F$ -splitting, splitting along a divisor  $D$  can be verified at the top level of cohomology.

THEOREM 1.49. *Let  $X$  be a normal proper variety of dimension  $d$ . For an effective divisor  $D$  on  $X$ ,  $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X(B)$  splits if and only if the map induced by Frobenius*

$$H^d(X, \omega_X) \rightarrow H^d(X, \omega_X \otimes_{\mathcal{O}_X} F_*^e \mathcal{O}_X(D))$$

*is injective.*

PROOF. A full proof can be found in [SS25, Theorem 3.2.16], but it is similar in style to Theorem 1.26.  $\square$

COROLLARY 1.49.1.  *$X$  as above is globally  $F$ -regular if and only if*

$$H^d(X, \omega_X) \hookrightarrow H^d(X, \omega_X \otimes_{\mathcal{O}_X} F_*^e \mathcal{O}_X(B))$$

For  $B$  chosen as in theorem 1.48.

A particularly important application of this corollary is that Calabi-Yau varieties, or any varieties for which  $K_X = 0$  (or equivalently,  $\omega_X = \mathcal{O}_X$ ), can never be globally  $F$ -regular. If such a variety were, then by the preceding corollary one can find an  $e$  for which

$$k \cong H^d(X, \omega_X) \hookrightarrow H^d(X, \omega_X \otimes_{\mathcal{O}_X} F_*^e \mathcal{O}_X(B))$$

This forces  $H^d(X, \omega_X \otimes_{\mathcal{O}_X} F_*^e \mathcal{O}_X(B)) \neq 0$ , and hence  $H^d(X, \mathcal{O}_X(B)) \neq 0$ . But this is dual to  $H^0(X, \mathcal{O}_X(-B))$ , which must be 0 due to  $B$  being effective, deriving a contradiction. Following the curves classification of  $F$ -splitting in theorem 1.29, it follows that curves of positive genus can never be globally  $F$ -regular. More generally, globally  $F$ -regular varieties must be log-Fano [SS10]. Global  $F$ -regularity, much like Global Frobenius splitting, can be checked at the ring level.

THEOREM 1.50 ([SS25], Theorem 3.4.1). *Let  $R$  be an  $\mathbb{N}$ -graded  $F$ -finite ring of characteristic  $p > 0$  that is finitely generated over its degree 0 subring. If  $R$  is strongly  $F$ -regular then  $\text{Proj}(R)$  is globally  $F$ -regular.*

This theorem implies, for instance, that  $\mathbb{P}_k^n$ , for  $k$  an  $F$ -finite field, is globally  $F$ -regular. The converse of this theorem, however, need not hold, see [SS25][example 4.4] The main roadblock to this example is that it is not projectively normal (or even normal!). Projective normality implies the coordinate ring is the same as the section ring  $S(X, L) := \bigoplus_{n \in \mathbb{N}} H^0(X, L^n)$ , for  $L$  defining the embedding of the variety into  $\mathbb{P}^n$ . It turns out that passing to the section ring gives both a necessary and sufficient condition for testing for global  $F$ -regularity, and even global  $F$ -splitting.

THEOREM 1.51 ([SS25], Theorem 3.4.12). *Let  $X$  be a projective variety and  $L$  any ample line bundle on  $X$ . Then  $X$  is globally  $F$ -regular (resp. globally  $F$ -split) if and only if  $S(X, L)$  is strongly  $F$ -regular (resp.  $F$ -split).*

One reason this correspondence is particularly strong is that global  $F$ -splitting or  $F$ -regularity can be verified by considering the section ring of any ample line bundle, not just the line bundle defining the embedding of the projective variety into  $\mathbb{P}^n$ . It also does not require the variety to be smooth.

One can interpret this result as saying the global singularities of  $X = \text{Proj}(R)$  are completely determined by the local properties of the cone point on  $\text{Spec}(R)$ . This allows for the study of global singularities to reduce to the local study in quite a systematic way.

**1.6.  $F$ -Rationality.** Much as  $F$ -injectivity can be viewed as a weakening of  $F$ -purity/ $F$ -splitting, requiring only an injective map on local cohomology rather than purity of (or the existence of a splitting of) Frobenius directly, one can define similarly define a weakening of strong  $F$ -regularity.

DEFINITION 1.52. [HH94a, HH94b] (cf. [MP25, section 4]) A local ring  $(R, \mathfrak{m})$  is  **$F$ -Rational** if  $R$  is Cohen-Macaulay and for every  $c$  outside of a minimal prime of  $R$ , there exists an  $e > 0$  such that the composition

$$H_{\mathfrak{m}}^d(R) \xrightarrow{F} H_{\mathfrak{m}}^d(F_*^e R) \xrightarrow{\cdot F_*^e [c]} H_{\mathfrak{m}}^d(F_*^e R)$$

is injective. Any ring  $R$  is  $F$ -rational if  $R_{\mathfrak{m}}$  is  $F$ -rational for all maximal ideals  $\mathfrak{m}$  of  $R$ .

REMARK 1.53. *The original definition of  $F$ -rationality was defined via tight closure; a local ring was said to be  $F$ -rational every ideal generated by a system of parameters is tightly closed. This definition is equivalent to the initial definition for excellent rings (and hence for  $F$ -finite rings), which is often the most general setting we will work in in this thesis anyway.*

$F$ -rationality localizes and can be verified at the completion of a local ring. The converse of this completion statement only holds when  $R$  is excellent, and proving it requires a much more intricate argument.

LEMMA 1.54.  *$R$  is  $F$ -rational if and only if  $R_P$  is  $F$ -rational for all  $P \in \text{Spec}(R)$ . If  $(R, \mathfrak{m})$  is a local ring, then  $\widehat{R}$  being  $F$ -rational implies that  $R$  is  $F$ -rational. If  $(R, \mathfrak{m})$  is a Cohen Macaulay excellent local ring, then  $R$  is  $F$ -rational if and only if  $\widehat{R}$  is  $F$ -rational.*

The requirement of being excellent here is subtle, but necessary [LR01]. Along these lines, we note that this definition of  $F$ -rationality does not require that  $R$  is  $F$ -finite.

LEMMA 1.55 ([HH94a], Theorem 4.2(b)). *Any  $F$ -rational ring is normal.*

As strongly  $F$ -regular rings are Cohen Macaulay (proposition 1.38), the following can be seen as via a similar argument used to prove proposition 1.11 after reducing to the complete local case:

PROPOSITION 1.56. *If  $R$  is strongly  $F$ -regular then it is  $F$ -rational, with equivalence if  $R$  is  $F$ -finite and Gorenstein. If  $R$  is  $F$ -rational, then it is  $F$ -injective.*

We note that, as all  $F$ -rational rings are Cohen-Macaulay by definition,  $H_{\mathfrak{m}}^i(R) = 0$  for all  $i \neq d$ . Thus, the final implication is immediate by picking the splitting along  $c = 1$ . Finally we provide a useful characterization of  $F$ -rationality in terms of  $F$ -stable submodules. Recall that an  $R$ -module  $M$  with a Frobenius action (such that  $F(rm) = r^p F(m)$ ) is **F-Stable** if  $F(M) \subseteq M$ .

PROPOSITION 1.57 ([Smi97]). *Let  $(R, \mathfrak{m})$  be a local ring. Then  $\widehat{R}$  is  $F$ -rational if and only if  $R$  is Cohen-Macaulay and the only  $F$ -stable submodules of  $H_{\mathfrak{m}}^d(R)$  are 0 and  $H_{\mathfrak{m}}^d(R)$ .*

This was used by Smith in her thesis to show that, if a characteristic 0 variety had  $F$ -rational singularities modulo infinitely many  $p$ , then the variety has rational singularities. This is in fact an equivalence proven by Mehta and Srinivas [MS97], remarkably tying

together two invariants: one from characteristic 0 defined via resolutions of singularities, and one from characteristic  $p$  defined via Frobenius.

REMARK 1.58. *This characterization of  $F$ -rationality can be checked at a specific submodule; the tight closure of 0 in  $H_m^d(R)$ :*

$$0_{H_m^d(R)}^* := \left\{ [x] \in H_m^d(R) \mid cF^e[x] = 0 \text{ for } c \text{ not in any minimal prime of } R \text{ and } e \gg 0 \right\}$$

$R$  is  $F$ -rational if and only if  $R$  is Cohen-Macaulay and  $0_{H_m^d(R)}^* = 0$ . In chapter 3 we will introduce a “quasi” version of this ideal that is used to test for quasi- $F$ -regularity and quasi- $F$ -rationality.

LEMMA 1.59. *Any regular ring of characteristic  $p > 0$  is  $F$ -rational.*

PROOF. This follows from regular rings being weakly  $F$ -regular [HH94a, Theorem 3.4] and that weakly  $F$ -regular rings are  $F$ -rational [HH94a, Theorem 4.2].  $\square$

We note that the lemma above does not assume that  $R$  is  $F$ -finite; in the  $F$ -finite setting this can be deduced as a direct consequence of lemma 1.39 and proposition 1.56.

**1.7. Test Ideals.** Test ideals are a powerful tool to detect strong  $F$ -regularity and  $F$ -rationality in positive characteristic rings and schemes, with close ties to applications in birational geometry and the theory of multiplier ideals from characteristic 0 [BST15]. We will briefly touch on the test ideal construction, linking it to the study of  $F$ -pure thresholds (as discussed in subsection 1.4) and to the eventual construction of “quasi-test ideals” in chapter 3.

We recall that a test element in a Noetherian  $F$ -finite ring  $R$  is a non-zero-divisor  $c \in R$  such that the map  $1 \mapsto F_*^e c$  splits if and only if  $R$  is strongly  $F$ -regular. Such elements exist and are somewhat ubiquitous. To canonical construct an ideal of such test elements, however, we will need to strengthen our definition. We say that  $c \in R$  is a **Strong Test**

**Element** if for all non-zero-divisors  $f$ , there exists an  $e_0 > 0$  such that

$$c \in \text{im} \left( \text{Hom}_R(F_*^e R, R) \xrightarrow{\text{ev.}} R \right)$$

where  $\text{Hom}_R(F_*^e R, R) \rightarrow R$  is the evaluation map sending  $\varphi \mapsto \varphi(F_*^e f)$ .  $R$  is strongly  $F$ -regular if and only if this map is surjective for all non-zero-divisors  $f$ : thus one can say a ring is strongly  $F$ -regular if and only if 1 is a strong test element. It is easy to check that the set of strong test elements forms an ideal of  $R$ . This is precisely the test ideal

**DEFINITION 1.60.** The **Test Ideal**  $\tau(R) \subseteq R$  of  $R$  is the set of all strong test elements of  $R$ , as defined above.

From our discussion above,  $R$  is strongly  $F$ -regular if and only if  $1 \in \tau(R)$ , i.e.  $\tau(R) = R$ , and  $\tau(R)$  cuts out the non-strongly- $F$ -regular locus of points in  $R$ . This construction localizes and completes:

**LEMMA 1.61.** *Let  $R$  be an  $F$ -finite ring. Then for all  $P \in \text{Spec}(R)$ ,*

$$\tau(R) \otimes_R R_P \cong \tau(R_P)$$

*As  $R_P$ -modules. If  $R$  is local, then*

$$\tau(R) \otimes_R \widehat{R} \cong \tau(\widehat{R})$$

*As  $\widehat{R}$ -modules.*

Providing an independent argument for strong  $F$ -regularity localizing and completing. As test ideal localize, for an  $F$ -finite scheme  $X$ , one can define the **Global Test Ideal**  $\tau(X)$  as the sheaf of ideals cutting out the (closed) non-strongly- $F$ -regular locus of points in  $X$ .

It is not immediately apparent that strong test elements will always exist. Fortunately, they are just as ubiquitous as the standard test elements.

THEOREM 1.62 ([HH89], theorem 3.4). *Let  $R$  be an  $F$ -finite ring. Then for any  $c$  such that  $R_c$  is strongly  $F$ -regular, there exists a  $m$  for which  $c^m$  is a strong test element.*

Such elements  $c$  exist and can be chosen to be non-zerodivisors through an argument similar to remark 1.44.

REMARK 1.63. *Test elements were first defined in the language of tight closure, which were then used to define the “finitistic” test ideal, which tests for weak- $F$ -regularity. The ideal presented above is the “non-finitistic” test ideal, which in turn tests for strong  $F$ -regularity (with the global invariant testing for global  $F$ -regularity). These ideals would of course be equivalent assuming the weak implies strong conjecture discussed in remark 1.37.*

One can also define the test ideal utilizing the language of uniform compatibility. In practice, this description, while not as concrete, is helpful for proving permanence statements about the test ideal such as lemma 1.61. Given an ideal  $I \subseteq R$  and a map  $\varphi \in \text{Hom}_R(F_*^e R, R)$ , we say that  $I$  and  $\varphi$  are **Compatible** if  $\varphi(F_*^e I) \subseteq I$ . It is easy to check that compatibility can be verified locally and at the level of completion. We say that an ideal  $I$  is **Uniformly Compatible** if it is compatible with all  $\varphi \in \text{Hom}_R(F_*^e R, R)$ , for all  $e > 0$ .

LEMMA 1.64. *If  $R$  is strongly- $F$ -regular, then  $0$  and  $R$  are the only uniformly compatible ideals.*

PROOF. This is mostly an exercise in working through definitions. Suppose  $I \neq 0$  is a universally compatible ideal; our goal is to show that  $I = R$ . For any  $c \in I$  non-zero, there exists some  $e > 0$  and  $\varphi \in \text{Hom}_R(F_*^e R, R)$  such that  $\varphi(F_*^e c) = 1$  by the definition of strong  $F$ -regularity.  $I$  and  $\varphi$  are compatible, and hence,  $1 = \varphi(F_*^e c) \in \varphi(F_*^e I) \subseteq I$ . Therefore  $I = R$ . □

THEOREM 1.65. *Let  $R$  be an  $F$ -finite reduced ring. Then  $\tau(R)$  is the smallest non-zero uniformly compatible ideal that contains a non-zero divisor. In particular,*

$$\tau(R) = \sum_{e \geq 0} \sum_{\varphi \in \text{Hom}_R(F_*^e R, R)} \varphi(c)$$

for any non-zero divisor  $c \in \tau(R)$ .

PROOF. We follow the proof of [SS25, Theorem 6.15]. To show that the test ideal is uniformly compatible it is sufficient to check that for any  $c \in \tau(R)$ ,  $i > 0$ , and any  $\varphi \in \text{Hom}_R(F_*^i R, R)$ ,  $\varphi(F_*^i c) \in \tau(R)$ . Well,  $c$  is a strong test element so there is an  $e_0 > 0$  such that for all  $e \geq e_0$ , there is some  $\psi \in \text{Hom}_R(F_*^e R, R)$  for which  $c = \psi(F_*^e f)$ . After applying  $\varphi$  we get that  $\varphi(F_*^e c) = (\varphi \circ \psi)(F_*^{e+i}(f))$ , viewing  $\varphi \circ \psi \in \text{Hom}_R(F_*^{e+i} R, R)$  in a bit of an abuse of notation. This precisely says that for any  $f \geq e + i$ ,

$$\varphi(F_*^i c) \in \text{im} \left( \text{Hom}_R(F_*^f R, R) \xrightarrow{\text{ev. at } d} R \right)$$

To show  $\tau(R)$  is the smallest non-zero uniformly compatible ideal, take another such ideal  $I$  containing a non-zero divisor.  $c \in \tau(R)$  has some  $\psi \in \text{Hom}_R(F_*^e R, R)$  and  $d$  for which  $\psi(F_*^e d) = c$ . Since  $I$  is compatible with  $\psi$ ,  $c = \psi(F_*^e d) \in I$ . For the final claim, one can see that  $\sum_{e \geq 0} \sum_{\varphi \in \text{Hom}_R(F_*^e R, R)} \varphi(c)$  is contained in  $\tau(R)$  by uniform compatibility, and equality follows from minimality.  $\square$

One can of course extend the notions of (uniform) compatibility from ideals of rings to submodules of modules much the same way. Explicitly, let  $N \subset M$  both be  $R$ -modules.  $N$  is **compatible** with  $\varphi \in \text{Hom}_R(F_*^e M, M)$  if  $\varphi(F_*^e N) \subseteq N$ , and  $N$  is **uniformly compatible** if it is compatible with all such  $\varphi \in \text{Hom}_R(F_*^e M, M)$ , for all  $e > 0$ . These notions are again compatible with localization and completion.

To construct an analogous object to the test ideal to test for  $F$ -rationality, we consider this more general notion when  $M = \omega_R$ , the canonical module of  $R$ . To forge the connection, note that the definition of  $F$ -rationality for a local ring is Matlis dual to  $R$  being

Cohen-Macaulay and  $\mathrm{Tr}^e(F_*^e c \omega_R) = \omega_R$  for every non-zero-divisor  $c$ , for  $\mathrm{Tr}^e : F_*^e \omega_R \rightarrow \omega_R$  the (iterated) Frobenius trace, or the Matlis dual of the Frobenius on top local cohomology  $F : H_m^d(R) \rightarrow H_m^d(F_* R) = F_* H_m^d(R)$ .

As it turns out, this trace map  $\mathrm{Tr}$  encodes uniform compatibility of submodules of  $\omega_R$ ; that is,  $N \subseteq \omega_R$  is uniformly compatible if and only if it is compatible with  $\mathrm{Tr}$ . This provides the basis for constructing the so-called parameter test ideal:

**THEOREM 1.66** ([Smi95]). *Let  $R$  be an  $F$ -finite reduced ring with canonical module  $\omega_R$ .*

*Then*

$$\tau(\omega_R) := \sum_{e>0} \mathrm{Tr}^e(F_*^e c \cdot \omega_R) \subseteq \omega_R$$

*Where  $c$  is any strong test element. This is called the **parameter test ideal** of  $R$ .  $R$  is  $F$ -rational if and only if  $R$  is Cohen Macaulay and  $\tau(\omega_R) = \omega_R$ .*

In [Smi95] she also (equivalently) characterizes  $\tau(\omega_R)$  as the smallest submodule of  $\omega_R$  that is supported on all of  $\mathrm{Spec}(R)$  and compatible with  $\mathrm{Tr}$  (and hence, uniformly compatible as a submodule of  $\omega_R$ ).  $\tau(\omega_R)$  is compatible with localization and completion; hence for an  $F$ -finite integral, equidimensional scheme  $X$  one can define  $\tau(\omega_X)$  to be the coherent subsheaf of  $\omega_X$  that agrees with  $\tau(\omega_R)$  in affine charts.

**1.8.  $F$ -Singularities of pairs.** We will conclude our section on  $F$ -singularities by introducing analogous singularity conditions on pairs  $(X, \Delta)$ , where  $X$  is a noetherian scheme of characteristic  $p$  and  $\Delta$  is an effective  $\mathbb{Q}$ -divisor. Singularities of  $(X, \Delta)$  should be thought of as singularities on  $X$  along  $\Delta$ . In later chapters we will introduce many new generalizations of these  $F$ -singularity notions to the “quasi” setting, all for pairs, and all motivated by the definitions for classical  $F$ -singularities in this section. Much of this theory was developed by Smith and Schwede in [SS10].

DEFINITION 1.67. Let  $(X, \Delta)$  be a pair, where  $X$  is a normal irreducible<sup>6</sup>  $F$ -finite scheme of characteristic  $p$ , and  $\Delta$  is an effective  $\mathbb{Q}$ -divisor on  $X$ .

- $(X, \Delta)$  is **Globally F-Regular** if for every effective divisor  $D$ , there exists an  $e > 0$  such that  $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + D)$  splits as a morphism of  $\mathcal{O}_X$  modules.
- $(X, \Delta)$  is **Globally Sharply F-Split** if there exists an  $e > 0$  such that  $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil)$  splits as a morphism of  $\mathcal{O}_X$  modules.
- $(X, \Delta)$  is **Globally F-Split** if there exists an  $e > 0$  such that  $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X(\lfloor (p^e - 1)\Delta \rfloor)$  splits as a morphism of  $\mathcal{O}_X$  modules.
- $(X, \Delta)$  is **Globally Keenly F-Split** if there exists an  $e > 0$  such that  $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X(\lfloor p^e \Delta \rfloor)$  splits as a morphism of  $\mathcal{O}_X$  modules.
- $(X, \Delta)$  is **Globally Naively Keenly F-Split** if  $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X(\lfloor p \Delta \rfloor)$  splits as a morphism of  $\mathcal{O}_X$  modules.

If  $(X, \Delta)$  can be covered by open affines such that  $(U, \Delta|_U)$  is globally  $F$ -Regular, then  $(X, \Delta)$  is **Locally F-Regular**. If  $(X, \Delta)$  can be covered by open affines such that  $(U, \Delta|_U)$  is globally  $F$ -split, then  $(X, \Delta)$  is **Locally (sharply) F-Split** (defining local analogues of the other  $F$ -splitting variants similarly).

REMARK 1.68. *These definitions subsume our definitions for global  $F$ -splitting and global  $F$ -regularity when  $\Delta = 0$ , as well as other earlier notions of  $F$ -splitting and  $F$ -regularity of pairs, see [SS10, remark 3.2]. In addition, local and global  $F$ -splitting and  $F$ -regularity for pairs are equivalent for affine schemes.*

We recall that if  $D \leq E$  as  $\mathbb{Q}$ -divisors, then there is a natural inclusion  $\mathcal{O}_X(D) \hookrightarrow \mathcal{O}_X(E)$ . This will be useful in justifying the following lemma.

LEMMA 1.69 ([SS10], lemma 3.5). *If  $(X, \Delta)$  is globally  $F$ -regular (resp. globally  $F$ -split) then so is  $(X, \Delta')$  for any  $\Delta' \leq \Delta$ .*

<sup>6</sup>As noted in [SS10], normal schemes are a disjoint union of their irreducible components, so nothing is lost with this requirement.

PROOF. For any effective divisor  $D$ , if the composition

$$\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X (\lceil (p^e - 1)\Delta' \rceil + D) \hookrightarrow_*^e \mathcal{O}_X (\lceil (p^e - 1)\Delta \rceil + D)$$

splits, so does  $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X (\lceil (p^e - 1)\Delta' \rceil + D)$ . Similarly, if the composition

$$\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X (\lceil (p^e - 1)\Delta' \rceil) \hookrightarrow F_*^e \mathcal{O}_X (\lceil (p^e - 1)\Delta \rceil)$$

splits, so does  $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X (\lceil (p^e - 1)\Delta' \rceil)$ .  $\square$

Global  $F$ -regularity for pairs, much like boundary free case, can be checked along a single divisor:

PROPOSITION 1.70 ([SS10], Theorem 3.9).  *$(X, \Delta)$  is globally  $F$ -regular if and only if there exists an effective divisor  $C$  such that  $(X \setminus C, \Delta_{X \setminus C})$  is globally  $F$ -regular and there exists an  $e > 0$  such that  $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X (\lceil (p^e - 1)\Delta \rceil + C)$  splits as a morphism of  $\mathcal{O}_X$  modules.*

this yields various equivalent definitions for global  $F$ -regularity of pairs:

PROPOSITION 1.71 ([SS10]). *Let  $(X, \Delta)$  be a pair, where  $X$  is a normal irreducible  $F$ -finite scheme of characteristic  $p$ , and  $\Delta$  is an effective  $\mathbb{Q}$ -divisor on  $X$ . Then the following are equivalent:*

- $(X, \Delta)$  is globally  $F$ -regular.
- For every effective divisor  $D$  and all  $e \gg 0$ ,  $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X (\lceil (p^e - 1)\Delta \rceil + D)$  splits as a morphism of  $\mathcal{O}_X$  modules.
- There exists an effective divisor  $C$  such that  $(X \setminus C, \Delta_{X \setminus C})$  is globally  $F$ -regular and  $(X, \Delta + \varepsilon C)$  is globally  $F$ -split for sufficiently small  $\varepsilon > 0$ .

This last condition can be seen as being globally  $F$ -split up to small perturbation by a divisor, a correspondence analogous to that of strong  $F$ -regularity and  $F$ -splitting. It also yields the following corollary:

COROLLARY 1.71.1. *If  $(X, \Delta)$  is globally  $F$ -regular, then for any effective divisor  $D$  there exists an  $\varepsilon > 0$  such that  $(X, \Delta + \varepsilon D)$  is globally  $F$ -regular.*

As discussed earlier in subsection 1.5, global  $F$ -regularity is linked to the notion of being KLT via “reduction mod  $p$ ” techniques. Yet, the KLT condition (as well as other singularity conditions arising in the MMP) can actually be framed in a way that does not require the existence of a (log) resolution of singularities, and hence, can be a well defined singularity notion in any characteristic.

DEFINITION 1.72 ([KM98], Definition 2.34). Let  $(X, \Delta)$  be a pair where  $X$  is a normal variety and  $\Delta$  is a  $\mathbb{Q}$ -divisor such that  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier. For any proper birational map  $Y \rightarrow X$ , fix  $E_Y$  such that  $\pi_*(K_X + \Delta) = K_Y + E_Y$ , and let  $E_Y = \sum a_{Y,i} E_{Y,i}$  be its decomposition into irreducible divisors. Then  $(X, \Delta)$  is KLT (resp. log canonical) if for all  $Y$ ,  $a_{Y,i} > -1$  and  $\lfloor \Delta \rfloor \leq 0$  (resp.  $a_{Y,i} \geq -1$ ). If  $X$  is of characteristic 0, it is sufficient to verify this condition on just one proper birational map; the log resolution of singularities  $Y \rightarrow X$ .

This lets us compare singularity conditions directly for positive characteristic varieties. Unfortunately, KLT varieties, even surfaces, need not be globally  $F$ -regular, or even globally  $F$ -split [CTW18] in any characteristic. But what about local singularities? KLT surfaces are necessarily locally  $F$ -regular in characteristic  $p > 5$  [CGS16, Har98], yet there known counter examples, a subset of DuVal singularities, in characteristic  $p = 2, 3, 5$  which even fail to be locally  $F$ -split<sup>7</sup>. Even worse, KLT threefolds in any characteristic need not be locally  $F$ -split [CTW18]. It turns out many of these counterexamples, though not locally  $F$ -regular, are locally “quasi”- $F$ -regular; these examples and related results will be discussed in chapter 3.

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<sup>7</sup>In literature, being locally  $F$ -split is sometimes referred as being  $F$ -pure.  $F$ -purity localizes completely, and as the authors were working in the  $F$ -finite setting, global  $F$ -purity is equivalent to local  $F$ -purity, which is equivalent to local  $F$ -splitting, see corollary 1.14.1. As we will be focusing on the non- $F$ -finite setting later in this thesis, where local  $F$ -splitting and  $F$ -purity no longer coincide, we will refer to such singularities as locally  $F$ -split.

## 2. $F$ -Singularities outside of the $F$ -Finite Setting

In this section we investigate the  $F$ -singularity notions discussed in the previous section, but without assuming that we are necessarily working over  $F$ -finite rings or fields. As discussed in subsection 1.1,  $F$ -finite rings are automatically excellent and are homomorphic images of regular rings, and hence admit dualizing complexes [Kun76, Gab04]. We will still generally work over excellent rings and, when a definition requires it, also assume that our rings admit a dualizing complex, even a canonical module.

The first hurdle when working in the non- $F$ -finite setting is that  $F$ -purity and  $F$ -splitting are no longer equivalent (see corollary 1.14.1). Indeed, regular rings are still always  $F$ -pure, but it does not immediately follow from the definitions that any regular ring is  $F$ -split. It was open for decades whether there exist regular rings that were not  $F$ -split; this was answered by Datta and Smith [DS16] who proved the following equivalence for DVRs with  $F$ -finite fraction field:

**THEOREM 1.73 ([DS16]).** *For a discrete valuation ring  $V$  with  $F$ -finite fraction field, the following are equivalent:*

- (a)  $V$  is  $F$ -split.
- (b)  $V$  is  $F$ -finite.
- (c)  $V$  is excellent.

In conjunction with this, they constructed an example of non- $F$ -finite DVR:

**EXAMPLE 1.74 ([DS16], Example 4.5.1).** *Let  $\mathbb{F}_p((t))$  be the fraction field of the DVR  $\mathbb{F}_p[[t]]$ . Since the field of rational functions  $\mathbb{F}_p(t)$  is countable,  $\mathbb{F}_p((t))$  can not be algebraic over  $\mathbb{F}_p(t)$ . Choose a power series  $f = \sum_{n=1}^{\infty} a_n t^n$  in  $\mathbb{F}_p[[t]]$  that is transcendental over  $\mathbb{F}_p(t)$ . This defines an injective ring map*

$$\mathbb{F}_p[[x, y]] \hookrightarrow \mathbb{F}_p[[t]]$$

assigning  $x \mapsto t, y \mapsto f(t)$ . Restricting the  $t$ -adic valuation on  $\mathbb{F}_p((t))$  to the subfield  $\mathbb{F}_p(x, y)$  produces a discrete valuation on  $\mathbb{F}_p(x, y)$  with valuation ring  $V$ .  $V$  is not  $F$ -finite as a consequence of [DS16, theorem 4.3.1].

Via the preceding theorem, this DVR (a regular local ring) fails to be  $F$ -split. This means that, outside of the  $F$ -finite setting,  $F$ -splitting is no longer a good notion of singularity, as the commutative algebraic analogue of a smooth object can fail to satisfy this condition. This example, also as a result of the preceding theorem, is not excellent. Unfortunately even excellence could not save the  $F$ -split condition as more recently, Datta and Murayama constructed a class of excellent Henselian DVRs that fail to  $F$ -split [DM23]. We will discuss this example in chapter 2 section 2 and show the excellent Henselian DVRs they discussed even fail to be *quasi- $F$ -split*.

Though we cannot no longer consider Frobenius splitting, and thus its global variant, as a particularly useful notion of singularity in the  $F$ -finite setting, we have a viable, well studied, and similar replacement in  $F$ -purity. The drawback of this is that we will primarily concern ourselves with the study of local singularities going forward, as global  $F$ -splitting and global  $F$ -regularity fail to be a good singularity conditions for varieties defined over non- $F$ -finite fields for similar reasons.

$F$ -injectivity and  $F$ -rationality, fortunately, are already well defined in the non- $F$ -finite setting, though some of the key properties of  $F$ -rationality were initially proved with  $F$ -finite added as a hypothesis; we will develop a tool of Hochster and Huneke called the  $\Gamma$ -construction [HH94a] in subsection 2.3 which provides a systematic way to reduce the study of  $F$ -singularity conditions of complete local rings to  $F$ -finite rings, allowing us to extend results from the  $F$ -finite setting to the general setting.

All that remains then is devising an alternative definition for strong  $F$ -regularity. There are multiple proposed extensions of this definition appearing in literature that, fortunately,

all turn out to be equivalent for algebras of finite type over an excellent local ring. We will discuss these definitions and their equivalence in subsection 2.2. We will now introduce our new baseline assumption: excellence.

**2.1. Excellent Rings.** Excellent rings were introduced by Grothendieck [Gro67] as a very general class of well-behaved rings. An arbitrary noetherian ring  $R$  can be spectacularly poorly behaved, but excellent rings satisfy many basic properties and form a good base of study. They are also the broadest class of rings for which resolutions of singularities are expected to hold, which is fortunate as the vast majority of rings naturally occurring in algebraic geometry are (quasi-) excellent. In this subsection, we will define excellent rings and highlight some of their most useful properties.

DEFINITION 1.75. A  $k$ -algebra  $R$  is **Geometrically Regular** if for any finite extension  $L/k$  (or equivalently, any finite purely inseparable extension  $L$ )  $L \otimes_k R$  is regular. More generally, a ring morphism  $R \rightarrow S$  is **Geometrically Regular** if it is flat and for all  $P \in \text{Spec}(R)$ ,  $\kappa(P) \otimes_R S$  is a geometrically regular  $\kappa(P)$ -algebra.

If  $k$  is perfect then a  $k$ -algebra  $R$  is geometrically regular if and only if it is regular. Geometrically regular maps are ubiquitous in commutative algebra; any étale map or smooth map is automatically geometrically regular. In fact, a filtered colimit of smooth  $R$ -algebras is still geometrically regular. In his celebrated theorem, Popescu proved that the converse also holds.

THEOREM 1.76 (Néron-Popescu Approximation [Pop86]). *A morphism of (arbitrary characteristic) noetherian rings  $R \rightarrow S$  is geometrically regular if and only if  $S = \varinjlim S_i$ , a filtered colimit of smooth  $R$ -algebras  $S_i$ .*

We will use this result in chapter 2 section 3 to show that quasi- $F$ -purity completes.

DEFINITION 1.77. A ring  $R$  is called a **G-ring** (or Grothendieck Ring) if for all  $P \in \text{Spec}(R)$ ,  $R_P \rightarrow \widehat{R}_P$  is a geometrically regular morphism of rings.

We are now ready to define (quasi-) excellence.

DEFINITION 1.78 ([Gro67], 7.8.2). A ring  $R$  is called **Quasi-Excellent** if it is a G-ring and any finite type  $R$ -algebra  $S$  has an open regular locus.  $R$  is **Excellent** if it is quasi-Excellent and universally catenary. A scheme  $X$  is **(Quasi)-Excellent** if it admits an affine open cover, where each affine open is Spec of an excellent ring.

The universally catenary requirement amounts to equi-dimensionality, a mild requirement commonplace in algebraic geometry used to rule out pathological examples. A classical example of a noetherian ring with infinite krull dimension due to Nagata [Nag62] would not satisfy this condition, for instance.

THEOREM 1.79 ([Sta24], 07PX).  $\mathbb{Z}$ , any complete local ring, and any field is excellent. Any finite type algebra over an excellent ring is excellent. Excellence is preserved under localization.

REMARK 1.80. We caution that, for a Noetherian ring  $R$ ,  $R_P$  being excellent for all prime ideals  $P \in \text{Spec}(R)$  does **not** guarantee that  $R$  is itself excellent. This is because the openness of regular loci is not a local property. We call such a ring **locally excellent**. Similarly, a scheme  $X$  is **locally excellent** if  $\mathcal{O}_{X,x}$  is an excellent local ring for all  $x \in X$ .

Any essentially of finite type algebra over a field is excellent, and hence all varieties are (locally) excellent. Geometric regularity of completion enforces that basic ring theoretic properties are preserved under completion.

LEMMA 1.81 ([Sta24], 0C23). Let  $R$  be a quasi-excellent local ring. If  $R$  is reduced (resp. normal), then  $\widehat{R}$  is reduced (resp. normal).

This means that for excellent local rings, being normal implies being analytically normal and being reduced implies being analytically reduced. Excellent rings are also closely tied to resolutions of singularities. In [Gro67], Grothendieck deduced that if one can resolve singularities of any complete local ring, then one can resolve the singularities of any excellent local ring. Following Hironaka's proof of the former in characteristic 0, we have the following:

**THEOREM 1.82** ([Gro67, Hir64]). *Let  $X/k$  be an excellent scheme over a field  $k$  of characteristic 0. Then  $X$  admits a proper birational map  $\tilde{X} \rightarrow X$  for which  $\tilde{X}$  is smooth (i.e.  $X$  admits a resolution of singularities). Further, if any reduced locally Noetherian scheme  $X$  (of any characteristic) were to admit a resolution of singularities, then it must be quasi-excellent.*

Quasi-excellence can be upgraded to excellence in a geometric context, where equidimensionality is often assumed. Thus, excellence is a good baseline assumption when studying singularities in any characteristic. We recall the following theorem of Kunz for local rings of positive characteristic:

**THEOREM 1.83** ([Kun76]). *Let  $(R, \mathfrak{m})$  be a noetherian local ring of characteristic  $p$ . Then  $R$  is  $F$ -finite if and only if  $R$  is excellent and  $R/\mathfrak{m}$  is  $F$ -finite.*

Many results in the theory of  $F$ -singularities assume  $F$ -finiteness, and hence, assume excellence by default. As a consequence of this theorem, any excellent ring that is not  $F$ -finite only fails to be  $F$ -finite at the level of the residue field. Thus, if we want to extend results outside of the  $F$ -finite setting, this theorem provides a systematic way to do so.

**COROLLARY 1.83.1.** *Let  $\mathcal{A}$  denote a certain class of local rings closed under completion and field extension and  $\mathcal{P}$  a property of local rings. If*

- (a) *Any ring  $R$  in  $\mathcal{A}$  has property  $\mathcal{P}$  if and only if  $\hat{R}$  has property  $\mathcal{P}$ .*
- (b) *Any algebra  $R$  over a field  $k$  in  $\mathcal{A}$  has property  $\mathcal{P}$  if and only if  $L \otimes_k R$  has property  $\mathcal{P}$  for an algebraic  $F$ -finite extension  $L/k$ .*

*Then property  $\mathcal{P}$  holds for any local ring in  $\mathcal{A}$  if and only if it holds for any  $F$ -finite local ring in  $\mathcal{A}$ .*

**PROOF.** Let  $R$  be a ring in  $\mathcal{A}$ . Passing to the completion of  $R$  yields the isomorphism

$$\hat{R} \cong \frac{k[[x_1, \dots, x_n]]}{I}$$

by Cohen’s structure theorem. This is a complete local, hence excellent ring. If  $k$  is  $F$ -finite, then we are done by the preceding theorem. If not, base change by an  $F$ -finite extension  $L$  then complete again. This yields

$$(L \otimes_k \widehat{R})^\wedge \cong \frac{L[[x_1, \dots, x_n]]}{LI}$$

This is an excellent ring with  $F$ -finite residue field  $L$ , and hence is  $F$ -finite. Since property  $\mathcal{P}$  is invariant under completion and such a field extension, the result follows.  $\square$

Thus, to remove the  $F$ -finiteness hypothesis one can (naively) hope to verify (a) and (b) for a given property  $\mathcal{P}$ . For (a) to hold excellence is often required; we will show that, assuming excellence, (a) holds for all of our extended definitions of quasi- $F$ -singularities in later chapters. Condition (b) is much more subtle, and is known not to hold for many properties  $\mathcal{P}$ , as  $L$  is necessarily not inseparable, and  $F$ -singularities need not be preserved under arbitrary inseparable extensions. Resolving (b) was the central goal of Hochster and Huneke in developing the  $\Gamma$ -Construction, which in some sense constructs the “correct”  $F$ -finite extension for a given property  $\mathcal{P}$  such that (b) will hold. We will discuss this further in subsection 2.3.

**2.2.  $F$ -Pure Regularity.** When  $R$  is an  $F$ -finite noetherian ring of characteristic  $p$ , we defined strong  $F$ -regularity in subsection 1.5 to be that, for any non-zerodivisor  $c$ , there exists an  $e$  for which the map  $R \rightarrow F_*^e R$  assigning  $1 \mapsto F_*^e c$  splits. Outside of the  $F$ -finite setting, splitting is not an ideal condition to impose on Frobenius [DS16, DM23]. Instead one can replace “splits” with “is pure” and get a workable definition in the more general setting.

**DEFINITION 1.84 ([DS16, Has10]).** A (not necessarily noetherian) ring  $R$  is **F-pure regular** if for every non-zerodivisor  $c \in R$ , the map  $R \rightarrow F_*^e R$  sending  $1 \mapsto F_*^e c$  is a pure map of  $R$ -modules.

**REMARK 1.85.** *This definition was originally introduced for noetherian rings in [Has10] and dubbed “very strong  $F$ -regularity”, as it was seen as a strengthening of earlier definition*

extensions of Smith [Smi93] and Hochster that defined strong  $F$ -regularity via tight closure. Datta and Smith, concerned with the study of valuation rings, extended this definition to non-Noetherian rings and dubbed it  $F$ -pure regularity. As all split maps are pure, this definition of “very strong  $F$ -regularity” is a priori **weaker** than the definition of strong  $F$ -regularity given in subsection 1.5. Therefore we stick with Datta and Smith’s naming convention to avoid confusion and to emphasize the fact that purity is involved in the definition. The author recommends [Mur21, Appendix B] and [HY23] for additional overview of these historical definitions.

When showing that  $F$ -finite regular rings are strongly  $F$ -regular (lemma 1.39). The  $F$ -finite hypotheses was crucially used to, after reducing to the local setting, conclude that  $F_*R$  is a free  $R$ -module. If  $F_*R$  is not finitely generated, though it is flat by Kunz’s Theorem, may fail to be a free module and hence the same proof may not hold. It is then pertinent to ask whether any regular ring is necessarily  $F$ -pure regular. Unfortunately, the answer is no.

EXAMPLE 1.86 ([HY23], example 6.4, 6.8). *There exists Noetherian regular domains  $R$  of dimension 2 that are not  $F$ -pure regular.*

They also proved that such an example is not excellent, and that any such counterexample is necessarily non-excellent.

PROPOSITION 1.87 ([HY23], corollary 2.18). *Any (locally) excellent regular ring is  $F$ -pure regular.*

1.11 In fact, for excellent rings  $F$ -pure regularity is equivalent to the earlier extensions of strong- $F$ -regularity to the non- $F$ -finite setting introduced by Smith and Hochster and discussed in remark 1.85.  $F$ -rationality, still well defined for non- $F$ -finite rings, only completes in the excellent setting (see lemma 1.54 and [LR01]). This further motivates excellence as an initial assumption in the non- $F$ -finite setting.

Much like strong  $F$ -regularity,  $F$ -pure regularity is a local property, completes, implies  $F$ -regularity (and is hence normal), and descends along pure morphisms.

PROPOSITION 1.88 ([HY23], Proposition 2.10, Theorem 2.11, theorem 2.23). *Let  $R$  be a Noetherian ring of characteristic  $p$ . Then  $R$  is  $F$ -pure regular if and only if  $R_P$  is  $F$ -pure regular for all  $P \in \operatorname{Spec}(R)$ , if and only if  $R_{\mathfrak{m}}$  is  $F$ -pure regular for every maximal ideal  $\mathfrak{m}$  in  $R$ . If  $(R, \mathfrak{m})$  is an excellent local ring, then  $R$  is  $F$ -pure regular if and only if  $\widehat{R}$  is  $F$ -pure regular.*

*$F$ -pure regular rings are  $F$ -regular, and hence, normal. Further, if  $R \rightarrow S$  is a pure map of  $R$ -modules, then if  $S$  is  $F$ -pure regular, so is  $R$ .*

We note that these results, in the generality above, can be proven similarly to the proofs of the corresponding statements for strong  $F$ -regularity, taking care to only utilize purity, rather than splitting, of the morphisms in the proof. The cited results are not expressed in full generality however, and we encourage the reader to read their paper for the most general statements where more technical proofs are provided.

As the second part of proposition 1.88 holds for any noetherian ring, not just an excellent ring, example 1.86 is the lowest possible dimension example of a  $F$ -pure regular ring that is not regular; dimension 1  $F$ -pure regular rings are normal and hence regular. Further, Hochster and Yao proved the existence of test elements for  $F$ -pure regularity, recovering many of strong  $F$ -regularity's key properties in this non- $F$ -finite variant.

THEOREM 1.89 ([HY23], Theorem 2.24). *Let  $R$  be a locally excellent ring and  $c \in R$  for which  $R_c$  is  $F$ -pure regular. Then  $R$  is  $F$ -pure regular if and only if there exists an  $e > 0$  for which the map  $R \rightarrow F_*^e R$  sending  $1 \mapsto F_*^e c$  is pure.*

**2.3. The Gamma Construction.** We recall the remarkable result of Fedder [Fed83] that a  $k$ -algebra  $R$  is  $F$ -pure if and only if  $L \otimes_k R$  is  $F$ -pure for every (or equivalently, any) algebraic extension  $L/k$ . The statement for separable extensions is quite easy to show; any separable extension  $L/k$  is a filtered colimit of finite separable extensions  $L_i/k$ , which are étale. As we will see in section 4,  $F$ -singularities are well preserved under étale extensions,

and so the separable case follows from such results.

Any algebraic extension  $L/k$  can be decomposed as a separable extension and purely inseparable extension; thus it is sufficient to check this claim in the purely inseparable case. This can further be reduced to adjoining a single  $p$ th root:

**PROPOSITION 1.90** ([Fed83], proposition 1.11). *Let  $R$  be a  $k$ -algebra, for  $k$  a non-perfect field. Choose  $\lambda \in k$  so that  $\lambda^{1/p} \notin k$ . Then  $R$  is  $F$ -pure if and only if  $R[\lambda^{1/p}]$  is  $F$ -pure.*

To prove this lemma, we introduce an equivalent notion of  $F$ -purity:

**LEMMA 1.91** ([Fed83], lemma 1.8). *Let  $(R, \mathfrak{m})$  be an excellent reduced local ring. Then  $R$  is  $F$ -pure if and only if there exists a sequence of ideals  $q_n$ , cofinal with powers of  $\mathfrak{m}$ , such that  $R/q_n$  is 0-dimensional Gorenstein and each  $q_n$  is contracted with respect to the Frobenius map.*

**PROOF.** We recall that  $R$  is  $F$ -pure if and only if  $E \rightarrow E \otimes_R F_* R$  is injective 1.11. Excellent reduced rings are so-called “approximately Gorenstein” by work of Hochster [Hoc77], meaning that there exists a sequence  $q_n$  as described above.  $E = \varinjlim R/q_n$ , and since purity is preserved under colimit,  $R$  is  $F$ -pure if and only if  $R/q_n \rightarrow R/q_n \otimes_R F_* R$  is injective for each  $n$ . This occurs precisely when  $q_n$  is contracted with respect to the Frobenius map.  $\square$

We will use this to prove proposition 1.90.

**PROOF OF PROPOSITION.** As  $F$ -purity localizes and completes, we can assume that  $(R, \mathfrak{m})$  is a complete local  $k$ -algebra. Further, if  $R$  is non-reduced then it can never be  $F$ -pure, so we further assume that  $R$  is reduced. It is easy to check that  $R[\lambda^{1/p}]$  remains complete, local, and reduced with maximal ideal  $\mathfrak{m} \cdot R[\lambda^{1/p}]$ .  $R \rightarrow R[\lambda^{1/p}]$  is a pure map of  $R$ -modules, and hence we immediately conclude the  $\Leftarrow$  direction by lemma 1.18.

For the converse, choose  $\{q_n\}$  as in lemma 1.91. It follows that for each  $n$ ,  $R[\lambda^{1/p}]/q_n[\lambda^{1/p}]$  is a free  $R/q_n$ -module and that  $\{q_n R[\lambda^{1/p}]\}$  is a sequence of ideals in  $R[\lambda^{1/p}]$  for which

$R[\lambda^{1/p}]/q_n R[\lambda^{1/p}]$  is 0-dimension Gorenstein and powers are cofinal with  $\mathfrak{m}R[\lambda^{1/p}]$ . It is thus sufficient to show that  $\{q_n R[\lambda^{1/p}]\}$  are contracted along Frobenius to conclude  $F$ -purity via lemma 1.91. Well, suppose there exists an  $n$  for which  $q_n R[\lambda^{1/p}]$  is not contracted via Frobenius. then there exists  $y \notin q_n R[\lambda^{1/p}]$  satisfying  $y^p \in (q_n R[\lambda^{1/p}])^{[p]}$ . Let  $x_n$  generate the socle of  $R/q_n$  and hence  $R[\lambda^{1/p}]/q_n R[\lambda^{1/p}]$ ; dimension 0 Gorenstein rings; as  $y \notin q_n R[\lambda^{1/p}]$  there exists  $s \in R[\lambda^{1/p}], q \in q_n R[\lambda^{1/p}]$  for which  $x_n = sy + q$ . Hence.  $x_n^p = s^p y^p + q^p \in (q_n R[\lambda^{1/p}])^{[p]} = q_n^{[p]} R$ .  $q_n$  contracting under Frobenius implies that  $x_n \in q_n$ , but this contradicts the fact that  $x_n$  generates the socle of  $R/q_n$ .  $\square$

From the discussion preceding the proposition, we have the following corollary:

**COROLLARY 1.91.1.** *A  $k$ -algebra  $R$  is  $F$ -pure if and only if  $L \otimes_k R$  is  $F$ -pure for every (or equivalently, any) algebraic extension  $L/k$ .*

Such arguments do not always extend to other classes of  $F$ -singularities. For instance, take  $R$  to be an excellent (but not complete)  $F$ -pure but not  $F$ -split algebra over a non- $F$ -finite field  $k$ . Taking  $L$  to be the perfect closure of  $k$ ,  $(L \otimes_k R)^\wedge$  is an  $F$ -finite ring that, via the proposition above, is  $F$ -pure and hence  $F$ -split. Inseparable base changes can drastically change the underlying ring; even if  $R$  is an  $F$ -finite complete intersection,  $L \otimes_k R$  can fail to even be reduced (see chapter 3 section 5 for an explicit such example).

This is mostly due to adjoining  $p$ th roots of a certain finite classes of bad elements of the  $p$ -basis of  $k$  (typically those that are used to express the defining equations of  $R$ , in the case where  $R$  is finite type over  $k$ ). This suggests that one can take the base change to an extension  $L/k$  that adjoins all  $p$ th roots of elements *outside* this bad set; if this bad set has  $n$  elements,  $F_* L$  is a  $p^n$ -dimensional vector space over  $L$  and hence an  $F$ -finite field, and thus  $(L \otimes_k R)^\wedge$  will be  $F$ -finite and (hopefully) inherit the corresponding singularity properties of  $R$ . This is the motivating idea behind the  $\Gamma$ -**construction**, a powerful tool used to reduce questions about complete local rings to the  $F$ -finite setting.

DEFINITION 1.92 ([HH94a]). Let  $(R, \mathfrak{m})$  be a complete local ring. We may identify  $R/\mathfrak{m} = k$  as a coefficient field  $k \subseteq R$  by Cohen's structure theorem with  $p$ -basis  $\Lambda$ . Let  $\Gamma \subseteq \Lambda$  be a co-finite subset of  $\Lambda$ , and for every  $e > 0$  define

$$k^{\Gamma, e} = k [\lambda^{1/p^e} \mid \lambda \in \Gamma]$$

These form an ascending chain, and hence we can define

$$R^\Gamma := \varinjlim_e (k^{\Gamma, e} \otimes_k R)^\wedge$$

$R^\Gamma$ , by construction, is an  $F$ -finite ring and a faithfully flat  $R$ -module with local complete intersection fibers [Has10].

REMARK 1.93. Defining  $k^\Gamma := \varinjlim_e k^{\Gamma, e}$ , one can check that  $R^\Gamma \cong (k^\Gamma \otimes_k R)^\wedge$ .

It can be shown that for  $\Gamma$  chosen sufficiently small, any prime  $P \in \text{Spec}(R)$  expands to a prime ideal  $PR^\Gamma \in \text{Spec}(R^\Gamma)$  such that  $R_P$  and  $R_{PR^\Gamma}^\Gamma$  have equal depth. This allows us to identify (loci of)  $\text{Spec}(R)$  and  $\text{Spec}(R^\Gamma)$ .

The  $\Gamma$ -construction was first used by Hochster and Huneke to reduce the study of weak  $F$ -regularity to the  $F$ -finite setting:

LEMMA 1.94 ([HH94a], lemma 6.19). Let  $(R, \mathfrak{m})$  be a complete local ring. If  $R$  is weakly  $F$ -regular and Gorenstein, then there exists a  $\Gamma \subseteq \Lambda$  sufficiently small such that  $R^\Gamma$  is weakly  $F$ -regular.

Following this result, the  $\Gamma$ -construction has found significant application for other ring theoretic properties. We catalog a number of these results below:

THEOREM 1.95 ([HR76, HH94a, Vél95, EH08, Has10, Ma14, Mur21]). Let  $(R, \mathfrak{m})$  be a complete local ring. If  $R$  has property  $\mathcal{P}$ , then there exists a  $\Gamma \subseteq \Lambda$  sufficiently small such that  $R^\Gamma$  has property  $\mathcal{P}$  for any of the following properties  $\mathcal{P}$ :

- (a) Regular

- (b) *Complete Intersection*
- (c) *Gorenstein*
- (d) *Cohen-Macaulay*
- (e)  $(S_n)$
- (f)  $(R_n)$
- (g) *(weakly) normal*
- (h) *Reduced*
- (i) *Strongly  $F$ -regular*
- (j)  *$F$ -pure (or equivalently,  $F$ -split)*
- (k)  *$F$ -rational*
- (l)  *$F$ -injective*

The  $\Gamma$ -construction is also compatible with asymptotic invariants such as the  $F$ -rational signature [Lyu24] and has a global variant preserving  $\mathcal{P}$ -loci due to work of Murayama [Mur21]:

THEOREM 1.96 ([Mur21], Theorem A). *Let  $X$  be a scheme essentially of finite type over  $k$ . Then there exists a purely inseparable  $F$ -finite field extension  $k^\Gamma/k$  such that the projection morphism*

$$X \times_k k^\Gamma \rightarrow X$$

*is a homeomorphism identifying  $\mathcal{P}$  loci, for any  $\mathcal{P}$  as in theorem 1.95.*

In addition, if such a variety  $X$  admits a resolution of singularities, say when  $\dim(X) \leq 3$ , then the  $\Gamma$  construction preserves log canonical and KLT singularities, see [Mur21, remark 3.7].

### 3. Witt Vectors

In this section we will review the Witt Vector construction alongside their properties. Though Witt Vectors can be defined over a ring of arbitrary characteristic, we will restrict our focus to rings (and sheaves) of positive characteristic. Much of this section will be a

reproduction of the author’s prior published work [Jag26], the author recommend this work alongside [Bor11], [LZ04, Appendix], [DK14], and [Len19] for a more thorough general treatment, and Serre’s Local Fields [Ser79] for an excellent and concise treatment of Witt Vectors over characteristic 0 rings.

For any ring  $R$  we define  $W(R) := \{(r_0, r_1, \dots) \mid r_i \in R\}$  to be the ring of **(p-typical) Witt Vectors**. We will typically denote elements of  $W(R)$  as  $\alpha = (\alpha_0, \alpha_1, \dots)$ . For any  $r \in R$  the element  $[r] := (r, 0, \dots) \in W(R)$  denotes the **lift**<sup>8</sup> of  $r$ . Attached to  $W(R)$  are addition and multiplication operations defined via universal Witt Polynomials  $S_\bullet(\alpha, \beta)$  and  $P_\bullet(\alpha, \beta)$  such that

$$\begin{aligned}\alpha + \beta &= (S_0(\alpha_0, \beta_0), S_1(\alpha_0, \alpha_1, \beta_0, \beta_1), \dots, S_n(\alpha_{\leq n}, \beta_{\leq n}), \dots) \\ \alpha \cdot \beta &= (P_0(\alpha_0, \beta_0), P_1(\alpha_0, \alpha_1, \beta_0, \beta_1), \dots, P_n(\alpha_{\leq n}, \beta_{\leq n}), \dots)\end{aligned}$$

The construction of these Witt polynomials, at first glance, seems fairly nonstandard. Consider  $S_i$  and  $P_i$  for  $i = 0$  and 1:

$$\begin{aligned}S_0(\alpha_0, \beta_0) &= \alpha_0 + \beta_0, & S_1(\alpha_0, \alpha_1, \beta_0, \beta_1) &= \alpha_1 + \beta_1 - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} \alpha_0^i \beta_0^{p-i} \\ P_0(\alpha_0, \beta_0) &= \alpha_0 \beta_0, & P_1(\alpha_0, \alpha_1, \beta_0, \beta_1) &= \alpha_1 \beta_0^p + \alpha_0^p \beta_1 + p \alpha_1 \beta_1\end{aligned}$$

As  $\binom{p}{i}$  is  $p$ -divisible for all valid  $i$ , division by  $p$  in formula of  $S_1$  is purely formal. This paper will focus on Witt Vectors over rings of characteristic  $p$ ; in this case the term  $p \alpha_1 \beta_1 = 0$  in  $P_1(\alpha_{\leq 1}, \beta_{\leq 1})$  and similarly for the other  $P_i$ . We also note that  $W(-)$  is functorial – for any ring morphism  $f : R \rightarrow S$  one can define a ring morphism  $W(f) : W(R) \rightarrow W(S)$  assigning  $\alpha \mapsto (f(\alpha_0), f(\alpha_1), \dots)$ . While this clearly a map of sets, some work is needed to show that this is a ring map. This follows from the fact that, in coordinates, addition and multiplication are identified via polynomial relations. As  $f$  is a ring map,  $f(S_n(\alpha_{\leq n}, \beta_{\leq n})) =$

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<sup>8</sup>This is sometimes referred to as a “Teichmüller lift” in literature, though the author dislikes this convention.

$S_n(f(\alpha_{\leq n}), f(\beta_{\leq n}))$ , and similarly for  $P_n$ ; it follows that  $W(f)$  is a map of rings. Rings of Witt Vectors over a ring of characteristic  $p$  have the following associated maps:

- Frobenius:  $F : W(R) \rightarrow W(R)$  is simply the image of  $F$  under the functor  $W(-)$  assigning  $F(\alpha) = (\alpha_0^p, \alpha_1^p, \dots)$ . The author stresses that the Witt Frobenius is not the same as the 'standard' Frobenius  $\alpha \mapsto \alpha^p$ , which is not a ring map on  $W(R)$ .
- Verschiebung:  $V : W(R) \rightarrow W(R)$  assigning  $(\alpha_0, \alpha_1, \dots) \mapsto (0, \alpha_0, \alpha_1, \dots)$ .

We define  $W_n(R) := W(R)/\text{im}(V^n)$  to be the ring of **(n-truncated, p-typical) Witt Vectors**, with similar addition and multiplication operations defined by Witt polynomials  $S_i, P_i : W_i(R) \times W_i(R) \rightarrow R$  for  $i < n$ . Based on the definition of  $S_0$  and  $P_0$ , it is clear that  $W_1(R) \cong R$ . This construction is naturally also functorial, and has maps  $F : W_n(R) \rightarrow W_n(R)$  and  $V : W_n(R) \rightarrow W_{n+1}(R)$  defined similarly to the above, with the additional restriction map  $\mathfrak{R} : W_{n+1}(R) \rightarrow W_n(R)$  assigning  $(\alpha_0, \dots, \alpha_n) \mapsto (\alpha_0, \dots, \alpha_{n-1})$ . It's worth noting that  $F$  and  $\mathfrak{R}$  are ring homomorphisms while  $V$  is additive but not multiplicative.

REMARK 1.97. *In this section, we are no longer assuming that all rings are of characteristic  $p$ . Indeed, for  $R$  of characteristic  $p > 0$ ,  $W(R)$  is a ring of mixed characteristic  $(0, p)$  and  $W_n(R)$  is not characteristic  $p$  for  $n > 1$ .  $W_n(R)$  should be viewed as a  $p$ -nilpotent thickening of  $R$ ; indeed if  $R$  is a  $p$ -torsion object in the category of rings, then  $W_n(R)$  is a  $p^n$ -torsion object and  $\text{Spec}(W_n(R)) = \text{Spec}(R)$ .*

We note  $p \in W_n(R)$  is of the form  $(0, 1, 0, \dots, 0)$  and the map  $p : \alpha \mapsto p \cdot \alpha$  yields the identity  $p = FV = VF$ . Let  $\overline{W}_n(R) := W_n(R)/p$  be the mod  $p$  reduction of  $W_n(R)$ ; when  $R$  is perfect (i.e.  $F$  is an isomorphism on  $R$ ) one sees that  $\text{im}(V) \cong \text{im}(p : \alpha \mapsto p \cdot \alpha)$ . This implies that  $\overline{W}_n(R) \cong R$ , though this isomorphism does not hold in general.

$F_*\overline{W}_n(R)$  is an  $R$ -module via the action  $r \cdot F_*\overline{\alpha} := F_*\overline{[r^p] \cdot \alpha}$ , and is a  $W_n(R)$ -Module in the expected way. This  $R$ -module, often denoted  $Q_{R,n}$  in literature, will be of particular importance when defining quasi- $F$ -splitting and quasi- $F$ -purity in chapter 2.  $R$  is also a

$W_n(R)$  module via the restriction map  $W_n(R) \xrightarrow{\mathfrak{R}^{n-1}} R$ . While there certainly can be others, we will assume this  $W_n(R)$ -module structure on  $R$  unless otherwise stated.

EXAMPLE 1.98. For  $R = \mathbb{F}_p$ ,  $W_n(\mathbb{F}_p) = \mathbb{Z}/p^n\mathbb{Z}$  and  $W(\mathbb{F}_p) = \mathbb{Z}_p$ , the  $p$ -adic integers. This is in some sense the motivating example for the Witt-Vector construction; one can see the latter isomorphism via the assignment

$$\sum_{n \geq 0} a_n p^n \mapsto (a_0, a_1, a_2, \dots)$$

This is an isomorphism set-theoretically, but making this assignment into an isomorphism of rings essentially defines the Witt Vector addition and multiplication rules.

EXAMPLE 1.99. More generally, for any perfect field  $k$  of characteristic  $p$ ,  $W(k)$  is always an (unramified) mixed characteristic DVR with uniformizer  $p$ . The author cautions that, outside of the case where  $R$  is a perfect field,  $W(R)$  and  $W_n(R)$  rarely have such nice descriptions.

In light of remark 1.97, it is easy to show that the Witt vector construction globalizes. That is, for any scheme  $X$  one can define the sheaves  $W \mathcal{O}_X$  and  $W_n \mathcal{O}_X$  as follows:

$$W_n \mathcal{O}_X(U) = W_n(\mathcal{O}_X(U)), \quad W \mathcal{O}_X(U) = W(\mathcal{O}_X(U))$$

In an abuse of notation,  $WX$  and  $W_n X$  will refer to the locally ringed spaces  $(X, W_n \mathcal{O}_X)$  and  $(X, W \mathcal{O}_X)$  respectively; these are in fact schemes that are topologically isomorphic to  $(X, \mathcal{O}_X)$ .

REMARK 1.100. One can also define Witt Vectors entirely categorically. Any ring of Witt vectors  $W(R)$  comes equipped with a natural  $\delta$ -ring structure, making  $W(-)$  a functor from the category of rings to the category of  $\delta$ -rings.  $W(-)$  can equivalently be formulated as the right adjoint of the forgetful functor from the category of  $\delta$ -rings to the category of rings. We will not make use of this characterization much, if at all, in this thesis.

As they can be formulated as a right adjoint, Witt vectors automatically commute with limits. In addition, it can be easily seen that truncated Witt Vectors commute with colimits.

LEMMA 1.101. *Let  $R = \varinjlim R_i$  be a direct limit in the category of rings. Then  $W_n(R) \cong \varinjlim W_n(R_i)$  as rings, i.e.  $W_n(-)$  commutes with direct limits.*

PROOF. As  $W_n(R_i)$  consists of finite length vectors with coefficients in  $R_i$ , it can be easily deduced that  $W_n(\varinjlim R_i)$  satisfies the universal property of colimits with respect to the directed system  $\{W_n(R_\bullet)\}$ .  $\square$

As the following lemmas will show, Witt Vectors also behave well with respect to local rings.

LEMMA 1.102 ([KTY22], Proposition 2.6). *If  $(R, \mathfrak{m})$  is a Noetherian local ring, then  $W_n(R)$  is a local ring.*

PROOF.  $\ker(\mathfrak{A}^{n-1} : W_n(R) \rightarrow R) = \{(0, r_1, \dots, r_{n-1}) \mid r \in R\}$  corresponds to  $\text{im}(V)$ . As  $\text{im}(V^n) = 0$  in  $W_n(R)$ , it follows that  $\ker(\mathfrak{A}^{n-1})$  is nilpotent.  $\mathfrak{A}^{n-1}$  is surjective, and hence  $R \cong \frac{W_n(R)}{\ker(\mathfrak{A}^{n-1})}$ . The inverse image under the restriction map of  $\mathfrak{m}$  yields a maximal ideal of  $W_n(R)$ , and the inverse image of the complement of  $\mathfrak{m}$  (e.g. the units of  $R$ ) lift to units of  $W_n(R)$  by nilpotence of  $\ker(\mathfrak{A}^{n-1})$ . It follows then that all elements outside of  $(\mathfrak{A}^{n-1})^{-1}(\mathfrak{m}) \subseteq W_n(R)$  are unit, and hence  $W_n(R)$  is local.  $\square$

This proof yields a description of of the maximal ideal of  $W_n(R)$ , denoted  $W_n\mathfrak{m}$ :

$$W_n\mathfrak{m} = \{\alpha \in W_n(R) \mid \alpha_0 \in \mathfrak{m}\}$$

LEMMA 1.103 ([Jag26], lemma 2.2). *Let  $(R, \mathfrak{m})$  be a Noetherian local ring. Then  $W_n(\widehat{R}) \cong W_n(R)^\wedge$ . In other words, The  $\mathfrak{m}_{W_n(R)}$ -adic completion of  $W_n(R)$  is ring-isomorphic to the image of the  $\mathfrak{m}$ -adic completion of  $R$  under  $W_n(-)$ .*

PROOF. Consider the ideal

$$W_n(\mathfrak{m}^k) := \ker(W_n(R) \twoheadrightarrow W_n(R/\mathfrak{m}^k)) = \{(a_0, \dots, a_{n-1}) \in W_n(R) \mid a_i \in \mathfrak{m}^k \forall i\}$$

One can see that  $W_n(\widehat{R}) \cong \varprojlim \frac{W_n(R)}{W_n(\mathfrak{m}^k)}$ , so it is sufficient to show that the following two directed systems of  $W_n(R)$ -modules are cofinal:

$$\{W_n(\mathfrak{m}^k)\}_k \quad , \quad \{(W_n \mathfrak{m})^k\}_k$$

Fix  $k$ . We want to find an  $\ell$  such that  $(W_n \mathfrak{m})^\ell \subseteq W_n(\mathfrak{m}^k)$ . Notice that

$$(W_n \mathfrak{m})^2 \subseteq \{(a_0, \dots, a_{n-1}) \in W_n(R) \mid a_0 \in \mathfrak{m}^2, a_1 \in \mathfrak{m}\}$$

And further

$$(W_n \mathfrak{m})^t \subseteq \{(a_0, \dots, a_{n-1}) \in W_n(R) \mid a_0 \in \mathfrak{m}^t, a_1 \in \mathfrak{m}^{t-1}, \dots, a_{n-1} \in \mathfrak{m}^{t-n+1}\}$$

Thus to make sure every term  $a_i \in \mathfrak{m}^k$ , we just need to  $\ell$  such that  $k = \ell - n + 1$ . Thus  $\ell = k + n - 1$  works. Conversely, for a fixed  $k$  we want to find an  $\ell$  such that  $W_n(\mathfrak{m}^\ell) \subseteq (W_n \mathfrak{m})^k$ . By similar logic, all  $(a_0, \dots, a_{n-1}) \in W_n(\mathfrak{m}^\ell)$  satisfy  $a_i \in \mathfrak{m}^\ell$ , and thus are contained in  $(W_n \mathfrak{m})^k$  for sufficiently large  $k$  dependent only on  $\ell$  and  $n$ . The result follows.  $\square$

**REMARK 1.104.** *Witt Vectors also behave well with respect to localizations, allowing us to reduce to the local study easily. Taking  $P \in \text{Spec}(R)$  and  $W_n P \in \text{Spec}(W_n(R))$  the unique image of  $P$  under the isomorphism*

$$(\mathfrak{A}^{n-1})^* : \text{Spec}(R) \xrightarrow{\cong} \text{Spec}(W_n(R))$$

*one can deduce from the universal property of localization as well as the set theoretic properties of  $W_n(R)$  that*

$$W_n(R)_{W_n P} \cong W_n(R_P)$$

#### 4. Étale Extensions

A map of rings  $R \rightarrow S$  is **Étale** if it is flat and unramified, or equivalently, it is smooth and of relative dimension 0 (see [Gro67, 17.6.2, 17.6.3] for many more equivalent definitions). A morphism of schemes  $f : Y \rightarrow X$  is **Étale** if each morphism of local rings  $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,f(x)}$  is étale. Étale morphisms are the scheme theoretic analogue of a local diffeomorphism from

differential geometry and are ubiquitous in algebraic geometry. This section will cover étale morphisms in very light generality, but with specific focus on their permanence properties for positive characteristic singularities.

If one intuitively understands étale maps as local diffeomorphisms, then it stands to reason that they should preserve local ( $F$ -)singularity properties. Such facts follow from a remarkable property about the relative Frobenius associated to an étale map, which we now define in full generality.

DEFINITION 1.105. Let  $\varphi : R \rightarrow S$  be a ring map of rings of characteristic  $p$ . One can realize  $F_*R \otimes_R S$  as the following pushout:

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \downarrow F_R & & \downarrow i \\ F_*R & \xrightarrow{F_*R \otimes \varphi} & F_*R \otimes_R S \end{array}$$

Utilizing the universal property of pushouts, the maps  $F_*R \xrightarrow{F_*\varphi} F_*S$  and  $F_S : S \rightarrow F_*S$  yields the morphism  $F_{S/R}$ : the **Relative Frobenius**.

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \downarrow & & \downarrow \\ F_*R & \longrightarrow & F_*R \otimes_R S \\ & \searrow & \downarrow \\ & & F_*S \end{array}$$

$\xrightarrow{F_{S/R}}$  (dashed arrow from  $F_*R \otimes_R S$  to  $F_*S$ )  
 $\xrightarrow{F_{S/R}}$  (solid arrow from  $F_*R$  to  $F_*S$ )

PROPOSITION 1.106. *If  $\varphi : R \rightarrow S$  is an étale morphism of noetherian rings of characteristic  $p$ , then  $F_{S/R}$  is an isomorphism.*

PROOF. Étale morphisms are locally of finite presentation and unramified; in particular, étale maps are locally finite étale. Thus, without loss of generality we can assume that  $R \rightarrow S$  is finite étale. If  $R \rightarrow S$  is étale then any base change is étale [Sta24, 00U2]. Hence both  $F_*R \rightarrow F_*R \otimes_R S$  and  $F_*R \rightarrow F_*S$  are étale; it follows from the diagram above and [Sta24, 02GW] that  $F_{S/R}$  is also étale. Étale morphisms are locally of finite presentation and unramified; in particular, étale maps are locally finite étale. Thus, we can reduce to the

local picture and assume that  $F_{S/R}$  is a finite local étale map of local rings.

As  $F_{S/R}$  is étale, it has separable fibers (indeed, the only finite étale algebras over a field are finite separable extensions). However,  $F_{S/R}(1 \otimes s) = s^p$ , implying that the map on fibers is purely inseparable. The only purely inseparable extension that is separable is the trivial extension, and hence,  $F_{S/R}$  is an isomorphism on fibers. Well, any flat finite map is free and hence, we can conclude that  $F_*S$  is a free  $F_*R \otimes_R S$  module. As  $F_{S/R}$  is an isomorphism on fibers, this forces  $F_*S$  to be a rank 1 free module. It follows that  $F_{S/R}$  must be an isomorphism.  $\square$

This allows us to base change the Frobenius on  $R$  ( $R \rightarrow F_*R$ ) to the Frobenius on  $S$  ( $S \cong R \otimes_R S \rightarrow F_*R \otimes_R S \cong F_*S$ ). Expressed more succinctly, this proposition shows that  $F_R \otimes_R S = F_S$ . When  $R$  and  $S$  are local and  $\varphi : R \rightarrow S$  is a local map, it is not only flat but faithfully flat (lemma 1.3). Hence, one can closely relate the singularities of  $R$  and  $S$  via faithfully flat descent.

LEMMA 1.107. *Let  $(R, \mathfrak{m})$  be a local ring,  $R \rightarrow M$  a map of  $R$ -modules, and  $R \rightarrow S$  a local étale extension of local Noetherian rings of characteristic  $p$ . Then  $R \rightarrow M$  is pure if and only if  $S \rightarrow S \otimes_R M$  is pure, and  $H_{\mathfrak{m}}^d(R) \rightarrow H_{\mathfrak{m}}^d(M)$  is injective if and only if  $H_{\mathfrak{m}}^d(S) \rightarrow H_{\mathfrak{m}}^d(S \otimes_R M)$  is injective.*

PROOF. As  $R \rightarrow S$  is unramified,  $E_R \otimes_R S \cong E_S$ . It follows that

$R \rightarrow M$  is pure

$\iff E_R \rightarrow E_R \otimes_R M$  is injective (proposition 1.11)

$\iff S \otimes_R E_R \rightarrow S \otimes_R E_R \otimes_R M$  is injective (by faithful flatness of  $S$ )

$\iff E_S \rightarrow E_S \otimes_R M$  is injective

$\iff S \rightarrow S \otimes_R M$  is pure

The for the second claim,  $H_m^d(R) \rightarrow H_m^d(M)$  is injective if and only if  $S \otimes_R H_m^d(R) \rightarrow S \otimes_R H_m^d(M)$  is injective. As we are working over the top local cohomology module,

$$S \otimes_R H_m^d(R) \cong H_m^d(S) \quad \text{and} \quad S \otimes_R H_m^d(M) \cong H_m^d(S \otimes_R M)$$

so the second claim follows.  $\square$

**PROPOSITION 1.108.** *Let  $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$  be an étale local map of local Noetherian rings of characteristic  $p$ . Then  $R$  is  $F$ -pure (resp. strongly  $F$ -pure regular,  $F$ -rational,  $F$ -injective) if and only if  $S$  is  $F$ -pure (resp.  $F$ -pure regular,  $F$ -rational,  $F$ -injective).*

**PROOF.** The statements for  $F$ -purity and  $F$ -injectivity follow immediately from the prior lemma, setting  $M := F_*R$  and  $S \otimes_R M = S \otimes_R F_*R \cong F_*S$ . The corresponding statement for  $F$ -pure regularity and  $F$ -rationality follow from the existence of strong test elements (theorems 1.43, 1.62, 1.89). Choose  $c$  such that  $R_c$  is regular. As  $R \rightarrow S$  is étale, so is  $R_c \rightarrow S \otimes R_c = S_c$ . As  $S_c$  is étale over the regular local ring  $R_c$ , it too is regular, and hence the image of  $c$  is a test element. It follows from the prior lemma that  $R \rightarrow F_*^e R$  assigning  $1 \mapsto F_*^e c$  is a pure map (resp. injective on local cohomology) if and only if  $S \rightarrow F_*^e S$  assigning  $1 \mapsto F_*^e c$  is pure (resp. injective on local cohomology).  $\square$

**REMARK 1.109.** *If  $R$  and  $S$  as above were  $F$ -finite,  $F$ -pure regularity is equivalent to strong  $F$ -regularity, and hence, strong  $F$ -regularity ascends and descends along étale maps.*

**REMARK 1.110.** *The analogous statements for global singularities can fail in somewhat subtle ways. For a morphism of schemes  $Y \rightarrow X$  of characteristic  $p$ , one can define the relative Frobenius  $F_{Y/X}$  similarly to the above. It is true that if  $Y \rightarrow X$  is a finite étale map,  $Y$  is  $F$ -split if and only if  $X$  is  $F$ -split provided that  $K_Y \sim_{\mathbb{Q}} 0$ . If  $X$  is  $F$ -split then  $Y$  is  $F$ -split, but the converse need not be true general. As it turns out, quasi- $F$ -splittings, while locally preserved under étale maps, is not necessarily preserved globally (see example 2.24)*

One goal in the later chapters is to show that étale local maps also preserve quasi- $F$ -singularities. Here we will need another important fact about étale maps:

THEOREM 1.111 ([vdK86] (cf. [Bor11])). *If  $\varphi : R \rightarrow S$  is an étale map of noetherian rings, then  $W_n(\varphi) : W_n(R) \rightarrow W_n(S)$  is an étale map of rings for all  $n \in \mathbb{N}$ .*

This remarkable result allows us to extend proposition 1.106 to the similarly defined relative-Witt Frobenius:

LEMMA 1.112 ([Jag26]). *Let  $R \rightarrow S$  be an étale extension. Then the relative Witt Frobenius*

$$W_n(F)_{S/R} : F_*W_n(R) \otimes_{W_n(R)} W_n(S) \rightarrow F_*W_n(S)$$

*is an isomorphism.*

PROOF. Taking the hypotheses above, Let  $R'$  be any  $R$ -algebra and  $S' := R' \otimes_R S$ . Then for any  $n$ , [LZ04, corollary A.12] states that there exists a canonical isomorphism:

$$W_n(S) \otimes_{W_n(R)} W_n(R') \cong W_n(S')$$

Set  $R' = F_*R$ . Then  $S' := F_*R \otimes_R S \cong F_*S$ , where the isomorphism follows from proposition 1.106. As the pullback along Frobenius commutes with the Witt functor, i.e.  $W_n(F_*R) \cong F_*W_n(R)$  and similarly for  $S$ , we obtain our desired result.  $\square$

This result will be used extensively alongside the methods of lemma 1.107 and proposition 1.108 to show that the corresponding quasi versions are stable under taking étale covers.

## CHAPTER 2

**Quasi- $F$ -Splitting and Purity**

In this chapter we will review the definition of and motivation for the study of quasi- $F$ -split varieties. Recently introduced by Yobuko [Yob19], Quasi- $F$ -split varieties can loosely be thought of a generalization of  $F$ -split varieties that allow for “mild” arithmetic supersingularity. Initially introduced as a global invariant with a view towards Calabi-Yau varieties, quasi- $F$ -splitting has proven to be a useful invariant in arithmetic and birational geometry [NY21, KTT<sup>+</sup>25, KTT<sup>+</sup>24a, KTT<sup>+</sup>24b, TWY24]. We will discuss these developments more so in chapter 3 section 4 after developing the necessary tools to understand the quasi- $F$ -splitting of pairs  $(X, \Delta)$ . We will develop the arithmetic-geometric connections between quasi- $F$ -splitting and the Artin-Mazur height of a Calabi-Yau variety in section 1.

In section 2 we will analyze quasi- $F$ -splitting as an invariant outside of the  $F$ -finite setting and show that there exist excellent Henselian DVRs, specifically Tate algebras studied by Datta and Murayama [DM23], that fail to be quasi- $F$ -split (Theorem 2.28). In some sense a remedy to this phenomenon, the author introduced quasi- $F$ -purity in [Jag26], a local analog of quasi- $F$ -splittings that is generalizable to outside of the  $F$ -finite setting. These developments as well as various permanence results will be discussed in section 3, containing material previously published as Jagathese V. (2026) On Quasi- $F$ -Purity of Excellent Rings, Communications in Algebra 54, 1760-1771 (see [Jag26]). Finally in section 4 we will compute the  $F$ -pure threshold (see chapter 1 section 1.4) of an arbitrary quasi- $F$ -pure hypersurface, showing that it takes on a maximal value. This section contains material previously published as Jagathese V., Garzella J. (2026) On  $F$ -Pure Thresholds and Quasi- $F$ -Pure Hypersurfaces (submitted, not yet in print) <sup>1</sup> (see [GJ25]).

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<sup>1</sup>Jack J Garzella has graciously consented to the reproduction of this material in the author’s thesis.

## 1. Quasi- $F$ -Splitting and Artin-Mazur Heights of Calabi-Yau Varieties

The study of quasi- $F$ -splittings begins with the study of arithmetic complexity and  $W_2(k)$ -liftability of Calabi-Yau varieties. In this section, we will investigate this connection and provide a definition for what it means to be “quasi-”  $F$ -split. We will conclude relating (quasi)- $F$ -splittings to arithmetic properties like being ordinary or Hodge-Witt, then provide numerous examples of quasi- $F$ -split varieties.

**1.1. Lifting modulo  $p^2$  and the  $a$ -Number.** Let  $X$  be a scheme over a field  $k$  of characteristic  $p$ . One can ask whether  $X$  lifts (or in some sense, thickens) to a variety  $\tilde{X}$  over  $W_2(k)$ . Such a variety  $\tilde{X}$  fits into the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(W_2(k)) \end{array}$$

With the absolute Frobenius  $F : \tilde{X} \rightarrow \tilde{X}$  restricting to  $F : X \rightarrow X$  and being compatible with the Witt Frobenius on  $W_2(k)$ . Such varieties are often referred to as “lifting modulo  $p^2$ ” in the sense that, if  $k = \mathbb{F}_p$ ,  $\tilde{X}$  is defined over  $W_2(\mathbb{F}_p) = \mathbb{Z}/p^2$ . Liftability of varieties was first considered in work of Mehta and Srinivas [MS87], where they show ordinary abelian varieties always admit a  $W_2(k)$  lift. More generally, any  $F$ -split variety admits a  $W_2(k)$  lift due to unpublished work of Mehta.

**THEOREM 2.1 ([DI87]).** *Let  $X$  be a smooth projective variety of dimension less than  $p$ , defined over a perfect field of characteristic  $p$ . If  $X$  admits a  $W_2(k)$  lift, then the associated Hodge-to-De-Rham spectral sequence degenerates at the  $E_1$  page.*

This was proven on the way to proving the existence of Hodge Decompositions over arbitrary fields of characteristic 0, but has garnered independent interest. This dimension restriction was removed for (quasi)- $F$ -split varieties in recent work of Petrov [Pet25].

As one consequence of this result, one can recover powerful vanishing theorems for globally  $F$ -split varieties. Yet, it was known to experts at the time that  $F$ -splitting was not

a necessary condition to admit a lift modulo  $p^2$ ; many varieties that were not  $F$ -split yet admitted a  $W_2(k)$  lift (such as, for example, any non- $F$ -split Calabi-Yau surface) existed in literature.

For Calabi-Yau varieties that admit a degenerate Hodge-to-de-Rham spectral sequence (such as liftable varieties above), we will introduce an invariant known as the  $a$ -number, an invariant for Abelian varieties introduced by Oort [Oor66] that has been extended to the aforementioned generality by Van der Geer and Katsura [vdGK02]. Recall that the **DeRham Cohomology** of a variety  $X$  of dimension  $d$  is the hyper-cohomology of the DeRham complex  $H_{\text{dR}}^i(X) := \mathbb{H}^i(X, \Omega_X^\bullet)$ . The top De Rham cohomology group has the associated **Hodge Filtration**:

$$H_{\text{dR}}^d(X) = F_0 \supset F_1 \supset \cdots \supset F_d$$

If the Hodge-to-deRham spectral sequence degenerates, the graded pieces are of the form

$$F_i/F_{i+1} \cong H^{n-i}(X, \Omega_X^i)$$

Each cohomology group  $H^{n-i}(X, \Omega_X^i)$  admits an action by the Frobenius  $F$ ; the  $a$ -number is defined to be

$$a(X) := \max\{i \mid \text{im}(F) \cap F^i \neq 0\}$$

In Oort's original definition for Abelian varieties, the  $a$ -number was defined to be the  $k$ -dimension of  $\text{Hom}(\ker(F : \mathbb{G}_a \rightarrow \mathbb{G}_a), X)$ . This can equivalently be formulated as the  $k$ -dimension of the intersection  $H^0(X, \Omega_X^1) \cap \text{im}(F)$ . For Abelian varieties the above definition recovers this one.

LEMMA 2.2 ([vdGK13], proposition 2.2). *The two definitions of the  $a$ -number coincide for Abelian varieties.*

PROOF. rewriting  $H_{\text{dR}}^d(X) \cong \bigwedge^d H_{\text{dR}}^1(X)$ , we recall that  $H_{\text{dR}}^1(X)$  decomposes as  $H^0(X, \Omega_X^1) \oplus B$ , for  $B$  a complementary subspace. This implies that the Hodge filtration on  $H_{\text{dR}}^d(X)$  is

$$F^r = \sum_{j=r}^d \left( \bigwedge^j H^0(X, \Omega_X^1) \otimes \bigwedge^{d-j} B \right)$$

Commuting wedge with Frobenius yields the equalities

$$F(H^d(X, \mathcal{O}_X)) \cong F\left(\bigwedge^d H^1(X, \mathcal{O}_X)\right) \cong \bigwedge^d F(H^1(X, \mathcal{O}_X))$$

Decomposing

$$F(H^1(X, \mathcal{O}_X)) \cong (F(H^1(X, \mathcal{O}_X)) \cap H^0(X, \Omega_X^1)) \oplus B$$

yields

$$F(H^d(X, \mathcal{O}_X)) \cong \bigwedge^a (F(H^1(X, \mathcal{O}_X)) \cap H^0(X, \Omega_X^1)) \otimes \bigwedge^{d-a} B$$

For  $a = \dim(F(H^1(X, \mathcal{O}_X)) \cap H^0(X, \Omega_X^1))$ . It follows that  $F(H^d(X, \mathcal{O}_X)) \subseteq F^a$  but  $F(H^d(X, \mathcal{O}_X)) \not\subseteq F^{a+1}$ .  $\square$

**1.2. The Artin-Mazur Height.** Though all Calabi-Yau surfaces admit a lifting, there exist many known examples of Calabi-Yau threefolds that necessarily do not [Hir99, HIS07, HIS08]. Many of these were obtained by studying the obstruction class for lifting living in  $\text{Ext}^2(\Omega_X, \mathcal{O}_X)$  (see [Yob19, MS87] for a further discussion on this). All of these examples just so happened to have infinite **Artin-Mazur height**, in invariant of Calabi-Yau varieties introduced in [AM77] that we now define.

DEFINITION 2.3. Let  $X$  be a Calabi-Yau variety over a field of characteristic  $p$  of dimension  $d$ . We define the functor  $\Phi_X$  from the category of Artin local  $k$ -algebras to the category of Abelian groups as follows:

$$\Phi_X(A) := \ker(H_{\text{ét}}^d(X \otimes_k A, \mathbb{G}_m) \rightarrow H_{\text{ét}}^d(X, \mathbb{G}_m))$$

[AM77] showed that this functor is pro-representable by a 1-dimensional formal group. Over characteristic  $p$ , such groups are classified up to isomorphism by height<sup>2</sup>; let  $\text{AM}(X)$  denote the height of this 1-dimensional group, an invariant of  $X$ . This is precisely the **Artin-Mazur Height**.

It is well known that a variety is  $F$ -split if and only if it has Artin-Mazur height 1 [MS87, vdGK03]; this can be seen from [vdGK03] who developed an alternative formulation of the Artin-Mazur height:

**THEOREM 2.4** ([vdGK03]). *For a Calabi-Yau variety  $X$  of dimension  $d$ ,*

$$\text{AM}(X) = \min \{n \geq 1 \mid [F : H^d(X, W_n \mathcal{O}_X) \rightarrow H^d(X, W_n \mathcal{O}_X)] \neq 0\}$$

As  $\mathcal{O}_X = \omega_X$  for Calabi-Yau varieties, if Frobenius has a non-zero action for  $n = 1$  one can deduce that  $X$  is  $F$ -split via theorem 1.26. As we will see in the following subsection, the Artin-Mazur height can also be realized as the minimal Witt-vector thickening which admits a splitting of a lift of Frobenius; this is used by Yobuko [Yob19] to define quasi- $F$ -splittings. The Artin-Mazur height also has the following relationship with the  $a$ -number:

**THEOREM 2.5** ([vdGK13]). *If  $\text{AM}(X) = 1$  (e.g.  $X$  is  $F$ -split) then  $a(X) = 0$ . If  $1 < \text{AM}(X) < \infty$ , then  $a(X) = 1$ .*

For certain subclasses of varieties, the Artin-Mazur height can take on finitely many possible values.

**THEOREM 2.6** ([Art74, AM77]). *If  $X$  is a K3 surface,  $\text{AM}(X) \in \{1, \dots, 10\} \cup \{\infty\}$ .*

K3 surfaces with infinite Artin-Mazur height are called **supersingular K3 surfaces** and are of general importance in number theory. It can be shown that supersingular K3 surfaces necessarily have Picard number 22 [Art74, AM77], whereas K3 surfaces over characteristic 0 have Picard number at most 20 (which can be easily read off the Hodge diamond). K3

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<sup>2</sup>The theory of heights of formal groups is beyond the scope of this thesis, especially as we will soon see that these Artin-Mazur heights can be computed in far simpler ways.

surfaces in characteristic 0 which hit this bound were classically known as **singular K3 surfaces**, hence the characterization of infinite Artin-Mazur height K3 surfaces as “super-singular”. This mimics the naming convention of singular and supersingular elliptic curves.

In general, one can use similar techniques of [Art74, AM77] to bound the maximal possible finite height of a Calabi-Yau variety of a fixed dimension (the bound for 3-folds, for instance, is 102). However, there do exist Calabi-Yau varieties of arbitrarily large but finite Artin Mazur height<sup>3</sup>.

**1.3. Quasi- $F$ -Splittings.** Let  $X$  be a scheme of positive characteristic. We say that  $X$  is  **$n$ -quasi- $F$ -split** [Yob19] if there exists a morphism of  $W_n \mathcal{O}_X$  modules  $F_* W_n \mathcal{O}_X \rightarrow \mathcal{O}_X$  that makes the diagram below commute:

$$\begin{array}{ccc} W_n \mathcal{O}_X & \xrightarrow{F} & F_* W_n \mathcal{O}_X \\ \mathfrak{F}^{n-1} \downarrow & \swarrow \text{dotted} & \\ \mathcal{O}_X & & \end{array}$$

When  $n = 1$ , this dotted map is simply a splitting of Frobenius, and hence, being 1-quasi- $F$ -split is equivalent to being  $F$ -split. This can be framed equivalently as follows. Let  $Q_{X,n}$  be a pushout of the diagram above, and  $\Phi_{X,n} : \mathcal{O}_X \rightarrow Q_{X,n}$  the induced morphism. Then by the universal property of pushouts, there exists a morphism of  $W_n \mathcal{O}_X$  modules  $F_* W_n \mathcal{O}_X \rightarrow \mathcal{O}_X$  as above if and only if  $\Phi_{X,n}$  splits in the category of  $W_n \mathcal{O}_X$ -modules.

$$\begin{array}{ccc} W_n \mathcal{O}_X & \xrightarrow{F} & F_* W_n \mathcal{O}_X \\ \mathfrak{F}^{n-1} \downarrow & \swarrow \text{dotted} & \downarrow \\ \mathcal{O}_X & \xrightarrow{\Phi_{X,n}} & Q_{X,n} \end{array}$$

As it turns out,  $Q_{X,n} \cong \frac{F_* W_n \mathcal{O}_X}{p F_* W_n \mathcal{O}_X} =: F_* \overline{W}_n \mathcal{O}_X$ , and  $Q_{X,n}$  has a natural  $\mathcal{O}_X$ -module structure: for a section  $s \in \mathcal{O}_X$ ,  $s \cdot F_* \overline{\alpha} = F_* [s^p] \cdot \overline{\alpha}$  defines an  $\mathcal{O}_X$ -module structure on  $Q_{X,n}$ . Thus, it is both necessary and sufficient that  $\Phi_{X,n}$  splits as a morphism of  $\mathcal{O}_X$ -modules. From the diagram

<sup>3</sup>It was shown in [Art74] that there exist K3 surfaces of all possible heights, with explicit examples at each height in characteristics  $p = 2, 3, 5$  and  $7$  by [KS16, KTY22, BGP25] and [BGP25] respectively. Explicit examples of Calabi-Yau varieties of every (even) height in characteristic 2 can be found in [KTY22, section 7.5].

$$\begin{array}{ccc}
W_{n+1} \mathcal{O}_X & \xrightarrow{F} & F_* W_{n+1} \mathcal{O}_X \\
\mathfrak{R} \downarrow & & \mathfrak{R} \downarrow \\
W_n \mathcal{O}_X & \xrightarrow{F} & F_* W_n \mathcal{O}_X \\
\mathfrak{R}^{n-1} \downarrow & \dashleftarrow & \downarrow \\
\mathcal{O}_X & \xrightarrow{\Phi_{X,n}} & Q_{X,n}
\end{array}$$

One can see that being  $n$ -Quasi- $F$ -split implies that you are  $(n+1)$ -Quasi- $F$ -split. It follows then that  $n$ -quasi- $F$ -splitting is a weaker condition than (global)  $F$ -splitting that weakens further as  $n$  grows. We say that  $X$  is **quasi-F-split** if it is  $n$ -quasi- $F$ -split for some  $n$ , and define  $\text{ht}_{\text{QFS}}(X)$ , the **quasi-F-split height** of  $X$ , to be the minimal  $n$  for which  $X$  is  $n$ -quasi- $F$ -split, where  $\text{ht}_{\text{QFS}}(X) = \infty$  if no such  $n$  exists. It follows from this that  $X$  is  $F$ -split if  $\text{ht}_{\text{QFS}}(X) = 1$ , quasi- $F$ -split if  $\text{ht}(X)$  is finite, and is otherwise not quasi- $F$ -split.

If  $\Phi_{X,n}$  splits locally at each subset in an open cover we say that  $X$  is **locally  $n$ -quasi-F-split**, and if  $X$  is locally  $n$ -quasi- $F$ -split for some  $n$ , then it is **locally quasi-F-split**<sup>4</sup>. If  $X = \text{Spec}(R)$  is an affine scheme, we denote these objects as  $Q_{R,n}(= F_* W_n(R)/p =: F_* \overline{W}_n(R))$  and  $\Phi_{R,n}$  for simplicity.

**THEOREM 2.7 ([Yob19]).** *Any smooth quasi- $F$ -split scheme over a perfect field  $k$  of characteristic  $p > 0$  admits a lifting to  $W_2(k)$ . Further, for Calabi-Yau varieties over  $k$ , the Artin-Mazur height and the Quasi- $F$ -split height coincide.*

The proof of these theorems involve the theory of (higher) Cartier operators [Ill79] and their connection with both the Artin-Mazur height and the module  $Q_{X,n}$ . We won't develop this theory here, but we encourage the interested reader to read the proofs in [Yob19].

$\Phi_{X,n}$  splits if and only if the evaluation map  $\mathcal{H}om(Q_{X,n}, \mathcal{O}_X) \rightarrow \mathcal{O}_X$  is surjective. This can be used to develop a criterion for quasi- $F$ -splitting much like Frobenius splitting (Theorem 1.26):

<sup>4</sup>Much like Frobenius splittings, these are often referred to as quasi- $F$ -pure singularities in literature. locally quasi- $F$ -split and quasi- $F$ -pure are equivalent in the  $F$ -finite setting (theorem 2.34) but due to our focus on the non- $F$ -finite setting, we will refer to these local conditions as above.

PROPOSITION 2.8 ([**KTT<sup>+</sup>25**], lemma 2.10). *Let  $X$  be an integral noetherian scheme over a perfect field  $k$  of characteristic  $p > 0$ . Then  $X$  is quasi- $F$ -split if and only if for some  $n > 0$ ,*

$$H^d(X, \omega_X) \rightarrow H^d(X, Q_{X,n} \otimes_{\mathcal{O}_X} \omega_X)$$

*is injective.*

Quasi- $F$ -splittings, much like  $F$ -splittings, also satisfy key vanishing theorems:

PROPOSITION 2.9. *Let  $X$  be a projective quasi- $F$ -split scheme over a perfect field  $k$  of characteristic  $p > 0$  and let  $L$  be an ample line bundle on  $X$ . Then for any  $i > 0$ ,*

- (a)  $H^i(X, \omega_X \otimes L) = 0$  (*Kodaira Vanishing*).
- (b)  $H^i(X, L) = 0$ .

PROOF. Kodaira vanishing follows from the fact that quasi- $F$ -split varieties have degenerate Hodge-to-DeRham spectral sequence [**Pet25**]. The second result is proved in [**Yob20**, Theorem 4.1]. □

We will investigate various generalizations of these theorems for log pairs  $(X, \Delta)$  [**NY21**, **KTT<sup>+</sup>25**] and apply these vanishing theorems to birational geometry after discussing quasi- $F$ -splittings for pairs in chapter 3 section 4.

REMARK 2.10. *Just as log canonicity is a natural analog of (local)  $F$ -splittings in characteristic 0, one can ask whether there is a characteristic 0 analog of (local) quasi- $F$ -splittings. This also turns out to also be log-canonicity; the distinction between quasi- $F$ -splittings and  $F$ -splittings is a purely characteristic  $p$  phenomenon. This is highlighted in [**Yob23**], where Yobuko shows that a variety of characteristic 0 is of dense  $F$ -pure type (i.e. it is locally  $F$ -split for infinitely many reductions mod  $p$ ) if and only if it is of dense quasi- $F$ -pure type. These notions both imply log canonicity and are conjecturally equivalent to log canonicity. For hypersurfaces, this fact can easily be seen as a corollary of [**GJ25**] and results of Mustata Takagi and Watanabe relating the  $F$ -pure threshold to the log canonical threshold [**MTW05**].*

**1.4. Ordinarity and Hodge-Witt Conditions.** Recall that an elliptic curve is ordinary if and only if Frobenius acts bijectively on  $H^1(X, \mathcal{O}_X)$ ; otherwise the curve is supersingular. This notion can be generalized to Abelian varieties of dimension  $d$  quite easily; they are ordinary if and only if  $F$  acts bijectively on  $H^d(X, \mathcal{O}_X)$ . It was deduced that any Abelian variety is ordinary if and only if it  $F$ -split via theorem 1.26. More generally, any  $F$ -split variety of dimension  $d$  admits a bijective action on  $H^d(X, \mathcal{O}_X)$ , meaning that  $F$ -split varieties are so-called **weakly ordinary**.

More generally however, ordinarity and  $F$ -splitting can fail to imply each other. In this subsection we will introduce a more general notion of ordinarity for any variety, not just Calabi-Yau, (in the sense of Bloch and Kato [BK86]), which implies  $F$ -splitting, and a higher analogue of this definition known as Hodge-Witt, which implies quasi- $F$ -splitting (and is equivalent for K3 surfaces and Abelian varieties). We motivate this with Crystalline cohomology; we will not review this cohomology theory detail for reasons of brevity, but we encourage an interested reader to read notes of Ogus and Berthelot [BO78] for an introduction to and motivation for this cohomology theory.

Any smooth proper variety  $X$  over a perfect field of positive characteristic  $k$  one can study the (algebraic) deRham Cohomology groups of  $X$   $H_{\text{dR}}^i(X) = \mathbb{H}^i(X, \Omega_X^\bullet)$ . This cohomology theory can be inadequate in this setting, in part due to its cohomology groups being vector spaces over  $k$ . This implies that  $p \cdot H_{\text{dR}}^i(X) = 0$  for any  $i$ . One solution initially defined by Grothendieck [Gro68] and developed by Berthelot [Ber74], is to apply infinitesimal thickenings to  $X$  to define a  $p$ -adic cohomology theory on  $X$ . Crystalline Cohomology, as it has been defined, can be viewed as taking the de Rham Cohomology of a characteristic 0 lift of  $X$  that reduces to deRham cohomology when going mod  $p$ .

We can associate crystalline cohomology groups  $H_{\text{crys}}^*(X/W(k))$  to a smooth proper variety  $X$  over a perfect field  $k$  of characteristic  $p$  in multiple different equivalent ways. The first

way, developed by Berthelot, was site theoretic. One can construct the Crystalline site of  $X$  over  $W_n(k)$  via taking so called “divided power thickenings” as covers, and open immersions that respect this divided power structure as morphisms. Defining this site as  $\text{Crys}(X/W_n)$ , one can define  $H_{\text{crys}}^i(X/W(k)) = \varprojlim H^i R\Gamma(X, \mathcal{O}_{X, \text{Crys}(X/W_n)})$  Where  $\mathcal{O}_{X, \text{Crys}(X/W_n)}$  is the structure sheaf of  $X$  valued in the crystalline site. Through work of Deligne and Illusie [Ill79] this was equivalently formalized as the hypercohomology of a certain complex called the **DeRham-Witt Complex**:

$$W\Omega_X^0 \xrightarrow{d} W\Omega_X^1 \xrightarrow{d} \dots$$

Constructed by taking  $\Omega_X^\bullet$  (and associated differentials) and considering their Witt vector thickenings. Analogous to the definition of DeRham cohomology, they show that

$$H_{\text{crys}}^i(X/W(k)) = \mathbb{H}^i(X, W\Omega_X^\bullet)$$

And further, they show there exists a canonical map of complexes  $W\Omega_X^\bullet \rightarrow \Omega_X^\bullet$  such that  $W\Omega_X^\bullet/pW\Omega_X^\bullet$  is quasi-isomorphic to  $\Omega_X^\bullet$ . This establishes the Crystalline-DeRham comparison. Each cohomology group  $H_{\text{crys}}^i(X/W(k))$  is a module over  $W(k)$ , which by construction is an unramified DVR with uniformizer  $p$ . By studying the DeRham Witt complex, one can better understand the associated crystalline cohomology groups and their related invariants, such as the Newton polygon which encodes supersingularity data.

**DEFINITION 2.11.** Let  $X$  be a smooth proper scheme over a field  $k$  of characteristic  $p$ . Then  $X$  is **Ordinary** if  $F$  acts bijectively on  $H^i(X, W\Omega_X^j)$  for every  $i$  and  $j$  [BK86]. If  $H^i(X, W\Omega_X^j)$  is finitely generated for every  $i$  and  $j$ , then  $X$  is **Hodge-Witt** [IR83].

This above definition of ordinarity recovers the original definition given when  $X$  is an Abelian variety.

**THEOREM 2.12** ([Yob23]). *Let  $X$  be an Abelian variety or a K3 surface over a field  $k$  of characteristic  $p$ . Then,*

- (a)  $X$  is ordinary if and only if it is  $F$ -split.

(b)  $X$  is Hodge-Witt if and only if it is quasi- $F$ -split.

$F$ -splitting and quasi- $F$ -splitting can be defined for any Noetherian  $\mathbb{F}_p$  scheme whereas ordinary and Hodge-Witt require that  $X$  be smooth and proper. Nakkajima shows that (quasi-) $F$  split varieties satisfy a weakening of the ordinarity and Hodge-Witt conditions.

THEOREM 2.13 ([Nak22]). *Let  $X$  be a proper scheme a field  $k$  of characteristic  $p$ .*

(a) *If  $X$  is  $F$ -split, then Frobenius acts bijectively on  $H^i(X, W \mathcal{O}_X)$  for every  $i$ .*

(b) *If  $X$  is quasi- $F$ -split, then  $H^i(X, W \mathcal{O}_X)$  is finitely generated for every  $i$ .*

Thus, (quasi-) $F$ -splitting can be seen as applying control on one “face” of the DeRham-Witt version of the Hodge-diamond  $\{\dim H^i(X, W\Omega_X^j)\}_{i+j \leq d}$  (specifically where  $j = 0$ ) whereas ordinarity and Hodge-Wittness control the entire diamond. Equivalences of these conditions for K3 surfaces and Abelian varieties are a consequence of certain types of Hodge symmetries holding in these special cases. Outside of this case,  $F$ -split/quasi- $F$ -split is not necessarily implied by, nor implies, ordinary or Hodge-Witt in general. We highlight some examples listed in [Yob23]:

EXAMPLE 2.14. *Let  $X$  be  $\mathbb{P}^3$  blown up at every  $\mathbb{F}_p$ -rational point. This yields a rational 3-fold that by [AZ17] is ordinary but does not admit a  $W_2(k)$  lift. It follows that this variety is ordinary but not quasi- $F$ -split.*

EXAMPLE 2.15. *Let  $Y$  be a supersingular (e.g. non-quasi- $F$ -split) K3 surface contained in  $\mathbb{P}^3$  (for instance, the hypersurface cut out by the Fermat quartic  $x^4 + y^4 + z^4 + w^4$  when  $p \not\equiv 1 \pmod{4}$ ). Embedding this into  $\mathbb{P}^4$  via the assignment  $[x : y : z : w] \mapsto [x : y : z : w : 0]$ , let  $X$  be the blow-up of  $\mathbb{P}^4$  along  $Y$ . This is  $F$ -split but not Hodge-Witt by [JR03].*

For an Abelian variety  $X$  of dimension  $d$  over a field  $k$  of characteristic  $p$ , we define the **p-rank** of  $X$ ,  $f(X)$ , to be the rank of its  $p$ -torsion subgroup. If  $f(X) = d$ , i.e. the  $p$ -torsion subgroup of  $X$  has full rank, then  $X$  is  $F$ -split (or equivalently, ordinary). As a result of Yobuko, the  $p$ -rank of an Abelian variety completely determines its (quasi-) $F$ -splitting.

**THEOREM 2.16** ([Yob23, vdGK03]). *Let  $X$  be an Abelian variety of dimension  $d$  over a perfect field  $k$  of characteristic  $p$ . Then*

- (1) *If  $f(X) = d$ , then  $X$  is ordinary (or equivalently,  $F$ -split).*
- (2) *If  $f(X) = d - 1$ , then  $X$  is Hodge-Witt and quasi- $F$ -split of height 2.*
- (3) *If  $f(X) < d - 1$ , then  $X$  is not quasi- $F$ -split.*

For curves this shows that a supersingular elliptic curve  $E$ , or a curve of  $p$ -rank  $0 = 1 - 1$ , is quasi- $F$ -split of height 2. Moving up a dimension, Abelian surfaces  $S$  have  $p$ -rank of 0, 1, or 2, corresponding to being not quasi- $F$ -split, quasi- $F$ -split but not  $F$ -split, and  $F$ -split respectively. It is known that indecomposable Abelian surfaces  $S$  satisfy  $f(S) \geq 1$ , and hence they must be at the very least quasi- $F$ -split. If Abelian surfaces decompose as a product of elliptic curves  $S \cong E_1 \times E_2$ , additivity of the  $p$ -rank implies that  $f(S) = f(E_1) + f(E_2)$ . Thus, the only way to construct an Abelian surface that isn't quasi- $F$ -split (i.e. an Abelian surface for which  $f(S) = 0$ ) is to take the product of two supersingular elliptic curves.

**PROPOSITION 2.17.** *An Abelian surface  $S$  over a perfect field  $k$  of characteristic  $p$  is not quasi- $F$ -split if and only if  $S \cong E_1 \times E_2$ , where each  $E_i$  is a supersingular elliptic curve.*

More generally, additivity of  $p$ -rank forces that for any decomposable Abelian variety  $X = X_1 \times X_2$ , then  $X$  can only be  $F$ -split if  $X_1$  and  $X_2$  are both  $F$ -split (equivalently, both are of full  $p$ -rank), and  $X$  can only be quasi- $F$ -split if 1 of  $X_1$  and  $X_2$  is of full  $p$ -rank and the other has  $p$ -rank at most 1 less than maximal (i.e. one of  $X_1, X_2$  is  $F$ -split and the other is quasi- $F$ -split). These ideas can be generalized by work of Ekedahl:

**THEOREM 2.18** ([Eke85]). *Let  $X$  and  $Y$  be smooth proper schemes over a perfect field  $k$  of characteristic  $p > 0$ .*

- (a) *If  $X$  is ordinary and  $Y$  is Hodge-Witt, then  $X \times Y$  is Hodge-Witt.*
- (b) *If  $X \times Y$  is Hodge-Witt, then one of  $X$  or  $Y$  is ordinary and the other is Hodge-Witt.*

Yobuko proved an analogous statement characterizing (quasi-)  $F$ -splitting of direct products.

THEOREM 2.19 ([Yob23]). *Let  $X$  and  $Y$  be schemes over a perfect field  $k$  of characteristic  $p > 0$ .*

- (a) *If  $X$  is  $F$ -split and  $Y$  is quasi- $F$ -split, then  $X \times Y$  is quasi- $F$ -split.*
- (b) *Suppose that  $X$  and  $Y$  are in addition either affine  $F$ -finite schemes or are geometrically connected and proper. Then if  $X \times Y$  is quasi- $F$ -split, one of  $X$  or  $Y$  is  $F$ -split and the other is quasi- $F$ -split.*

This characterization is equivalent to Ekedahl's for Abelian varieties, but as (ordinarity/Hodge-Witt) are different conditions than ( $F$ -split/quasi- $F$ -split) in general, this theorem conveys distinct information. Further, this theorem holds without requiring that  $X$  and  $Y$  are smooth and proper.

**1.5. Examples.** To conclude this section, we provide numerous examples of quasi- $F$ -split and non-quasi- $F$ -split singularities.

EXAMPLE 2.20 (Enriques Surfaces,  $p \neq 2$ ). *We recall that a surface  $X$  of characteristic  $p > 2$  is an **Enriques Surface** if it is of irregularity zero,  $K_X \neq 0$ , but  $K_X^2 = 0$ . If  $p > 2$ , Enriques Surfaces are always realizable as a quotient of a K3 surface  $S$  by a group of order 2 that acts without any fixed points. The quasi- $F$ -splitting of an Enriques surface  $X$  is entirely determined by  $S$ ;  $X$  is quasi- $F$ -split if and only if  $S$  is [Yob20]. Enriques Surfaces are ordinary for  $p > 2$ , and hence, quotients of a supersingular K3 surface in characteristic  $p > 2$  form a class of Enriques Surfaces that are ordinary but not quasi- $F$ -split.*

EXAMPLE 2.21 (Enriques Surfaces,  $p = 2$ ). *Enriques Surfaces of characteristic 2 can no longer be defined as above; we instead define them to be surfaces  $X$  for which  $K_X \sim 0$  and the 2nd Betti Number  $b_2 = 10$ . They were classified in [BM77] and form 3 distinct classes:*

- **Classical** Enriques Surfaces satisfy  $h^1(X, \mathcal{O}_X) = 0$ .
- **Singular** Enriques Surfaces satisfy  $h^1(X, \mathcal{O}_X) \neq 0$  but have a non-trivial Frobenius action.
- **Supersingular** Enriques Surfaces satisfy  $h^1(X, \mathcal{O}_X) \neq 0$  but have a trivial Frobenius action.

Classical and singular Enriques surfaces are ordinary, but supersingular Enriques surfaces are not ordinary but are Hodge-Witt [III79]. Singular Enriques surfaces are  $F$ -split, but classical and supersingular Enriques surfaces are not quasi- $F$ -split [Yob20].

EXAMPLE 2.22 (Maximal explicit finite quasi- $F$ -height, [KTY22]). The 3-dimensional Calabi-Yau hypersurface over  $\mathbb{F}_2$  cut out by the following equation has Artin-Mazur height 60:

$$Z(x^5 + y^5 + z^5 + w^5 + u^5 + xz^3w + yzw^3 + x^2zu^2 + y^2z^2w + xy^2wu + yzwu^2) \subseteq \mathbb{P}_{\mathbb{F}_2}^4$$

This is the maximal known explicit finite height, though quasi- $F$ -split heights as large as 102 are theoretically possible for Calabi-Yau threefolds.

EXAMPLE 2.23. We recall that a property  $\mathcal{P}$  satisfies **Inversion of Adjunction** if  $R/(f)$  satisfying  $\mathcal{P}$  implies that  $R$  satisfies  $\mathcal{P}$  for any general  $f$ . Inversion of adjunction holds for  $F$ -splittings but does not hold for quasi- $F$ -splittings: we provide a counter-example due to [KTY22]. Let  $S := \mathbb{F}_2[x, y, z, w, u, s]$  and  $g = xys^2 + zwu^2 + y^3w + x^3z$ . Then  $\text{ht}_{\text{QFS}}(S/(g)) = \infty$  but  $\text{ht}_{\text{QFS}}(S/(s, g)) = 2$ .

EXAMPLE 2.24. If  $Y \rightarrow X$  is a finite étale morphism for which  $K_Y \sim_{\mathbb{Q}} 0$ , then  $Y$  is  $F$ -split if and only if  $X$  is  $F$ -split (cf. theorem 1.108, remark 1.110). The same statement need not hold for quasi- $F$ -splittings, via a counter example found in [KTY22]. They specifically found a bielliptic surface  $X$  in characteristic 3, written as  $(E_1 \times E_2)/G$  for  $E_i$  elliptic curves and  $G \subset E_2$  a finite group subscheme acting on  $E_1$ . They show  $\text{ht}(X) = \infty$ ;  $Y = E_1 \times E_2 \rightarrow X = (E_1 \times E_2)/G$  is étale, and hence as long as  $E_1$  and  $E_2$  are not both supersingular (proposition 2.17)  $Y$  is quasi- $F$ -split but  $X$  is not.

REMARK 2.25. The (non) quasi- $F$ -splitting of examples arising from [KTY22] were checked via a quasi version of Fedder's Criterion developed in said paper. This criterion has since been implemented in Julia using GPU acceleration techniques [BGP25] and can compute the quasi- $F$ -split heights in low characteristic. We will discuss this criterion, and its applications to computing  $F$ -pure thresholds, in detail in section 4.

## 2. Non-Quasi- $F$ -Splitting of Tate Algebras

As an easy consequence of Kunz's theorem, all regular rings are  $F$ -pure. For  $F$ -finite or complete local rings,  $F$ -purity is equivalent to  $F$ -splitting, and hence,  $F$ -finite or complete regular local rings are necessarily  $F$ -split and therefore quasi- $F$ -split. Outside of the  $F$ -finite setting, however, it was conjectured and eventually proven by Datta and Smith [DS16] (see chapter 1 subsection 2.2), that regular rings can fail to be  $F$ -split. This result was improved upon by Datta and Murayama [DM23], who devised a class of local rings reminiscent of Tate algebras which are excellent Henselian DVRs, yet still fail to be  $F$ -split.

We will review this construction of Datta and Murayama in this section and show that, for such rings, being  $F$ -split and quasi- $F$ -split are equivalent. In particular, there exists excellent Henselian DVRs (necessarily defined over a non- $F$ -finite field) that fail to be quasi- $F$ -split.

Let  $(\mathfrak{K}, |\cdot|)$  be a non-Archimedean field and  $n \in \mathbb{N}$ . Let  $T_n(\mathfrak{K})$  be the  $\mathfrak{K}$ -subalgebra of  $\mathfrak{K}[[x_1, \dots, x_n]]$  defined as follows:

$$T_n(\mathfrak{K}) := \left\{ \sum_{i_1, \dots, i_n \in \mathbb{Z}_{\geq 0}} c_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n} \mid c_{i_1, \dots, i_n} \in \mathfrak{K} \text{ and } |c_{i_1, \dots, i_n}| \rightarrow 0 \text{ as } \sum_{j=1}^n i_j \rightarrow \infty \right\}$$

This is the **Tate Algebra** (in  $n$  variables) over  $\mathfrak{K}$ . These rings are regular, excellent, Banach algebras, and form the basic building blocks of rigid analytic geometry [Tat71, Ray74]. We will not review such uses or properties here; instead we turn our attention to the following question: For what  $\mathfrak{K}$  is  $T_n(\mathfrak{K})$   $F$ -split?

This question was answered completely in [DM23], where they show that  $T_n(\mathfrak{K})$  is  $F$ -split if and only if there exists a nonzero continuous (with respect to  $|\cdot|$ )  $\mathfrak{K}$ -linear map  $\mathfrak{K}^{1/p} \rightarrow \mathfrak{K}$ . Further, they show that admitting an  $F$ -splitting is equivalent to admitting any  $T_n(\mathfrak{K})$ -linear map  $F_* T_n(\mathfrak{K}) \rightarrow T_n(\mathfrak{K})$ . This implies that, for Tate algebras, being  $F$ -split is equivalent to

being  $F$ -solid in the sense of [DMS20]. The following lemma shows that, for any Noetherian ring, being  $F$ -solid is equivalent to being “quasi”- $F$ -solid, defined in an analogous sense.

LEMMA 2.26. *Let  $R$  be any Noetherian ring of characteristic  $p > 0$ . The following are equivalent:*

- (1)  $\mathrm{Hom}_R(F_*R, R) = 0$ .
- (2)  $\mathrm{Hom}_R(Q_{R,n}, R) = 0$  for all  $n \in \mathbb{N}$ .
- (3)  $\mathrm{Hom}_R(Q_{R,n}, R) = 0$  for some  $n \in \mathbb{N}$ .

PROOF. We will prove (1)  $\Rightarrow$  (2) by induction utilizing the following short exact sequence:

$$0 \rightarrow F_*^{n-1}(F_*R/R) \rightarrow Q_{R,n} \rightarrow Q_{R,n-1} \rightarrow 0$$

Applying  $\mathrm{Hom}_R(-, R)$  yields the following left exact sequence

$$0 \rightarrow \underbrace{\mathrm{Hom}_R(Q_{R,n-1}, R)}_{=0 \text{ by inductive hyp.}} \rightarrow \mathrm{Hom}_R(Q_{R,n}, R) \rightarrow \mathrm{Hom}_R(F_*^{n-1}(F_*R/R), R)$$

It is thus sufficient to show that  $\mathrm{Hom}_R(F_*^{n-1}(F_*R/R), R) = 0$ . Again we consider an exact sequence

$$0 \rightarrow F_*^{n-1}R \xrightarrow{F} F_*^n R \rightarrow F_*^{n-1}(F_*R/R) \rightarrow 0$$

and applying  $\mathrm{Hom}_R(-, R)$  again:

$$0 \rightarrow \mathrm{Hom}_R(F_*^{n-1}(F_*R/R), R) \rightarrow \mathrm{Hom}_R(F_*^n R, R) \rightarrow \mathrm{Hom}_R(F_*^{n-1}R, R)$$

As  $\mathrm{Hom}_R(F_*R, R) = 0$ ,  $\mathrm{Hom}_R(F_*^n R, R) = 0$ . Thus (1)  $\Rightarrow$  (2). (2)  $\Rightarrow$  (3) is immediate, so it is sufficient to prove (3)  $\Rightarrow$  (1). Taking the restriction map  $Q_{R,n} \rightarrow F_*R$  and applying  $\mathrm{Hom}_R(-, R)$  yields an injection  $\mathrm{Hom}_R(F_*R, R) \hookrightarrow \mathrm{Hom}_R(Q_{R,n}, R) = 0$ , completing the proof.  $\square$

Before moving on to the main result of this section, we first recall a local  $\mathfrak{K}$ -algebra defined in [DM23] that was inspired by the Tate algebra construction:

DEFINITION 2.27.

$$K_n(\mathfrak{K}) := \left\{ \sum_{i_1, \dots, i_n \in \mathbb{Z}_{\geq 0}} c_{i_1, \dots, i_n} x_1^{i_1} \dots x_n^{i_n} \mid \exists r_1, \dots, r_n, M \in \mathbb{R}_{>0} \text{ such that } |c_{i_1, \dots, i_n}| r_1^{i_1} \dots r_n^{i_n} \leq M \forall i_j \right\}$$

This is an excellent, Henselian, local, regular  $\mathfrak{K}$ -subalgebra of  $\mathfrak{K}[[x_1, \dots, x_n]]$  [DM23, Theorem 4.3]. Further,  $K_n(\mathfrak{K})$  is  $F$ -split if and only if it is  $F$ -solid, if and only if there exists a nonzero continuous  $\mathfrak{K}$ -linear map  $\mathfrak{K}^{1/p} \rightarrow \mathfrak{K}$  [DM23, Theorem 4.4]. Unifying this with lemma 2.26 yields the following:

THEOREM 2.28. *Let  $(\mathfrak{K}, |\cdot|)$  be a non-Archimedean field and  $n \in \mathbb{N}$ , and let  $R = T_n(\mathfrak{K})$  or  $R = K_n(\mathfrak{K})$  as defined above. Then the following are equivalent:*

- (1)  $R$  is  $F$ -split.
- (2)  $R$  is quasi- $F$ -split.
- (3)  $\text{Hom}_R(F_*R, R) \neq 0$ .
- (4)  $\text{Hom}_R(Q_{R,n}, R) \neq 0$  for all  $n \in \mathbb{N}$ .
- (5)  $\text{Hom}_R(Q_{R,n}, R) \neq 0$  for some  $n \in \mathbb{N}$ .
- (6) There exists a nonzero continuous  $\mathfrak{K}$ -linear map  $\mathfrak{K}^{1/p} \rightarrow \mathfrak{K}$ .

PROOF. (1)  $\Leftrightarrow$  (3)  $\Leftrightarrow$  (6) by [DM23, Theorem 3.1] for  $R = T_n(\mathfrak{K})$  and by [DM23, Theorem 4.4] for  $R = K_n(\mathfrak{K})$ . (3)  $\Leftrightarrow$  (4)  $\Leftrightarrow$  (5) for any Noetherian ring by lemma 2.26. (1)  $\Rightarrow$  (2)  $\Rightarrow$  (5) immediately from the definitions, and (5)  $\Rightarrow$  (1) by the first two remarks.  $\square$

As discussed in [DM23, Theorem 5.2], one can construct an explicit  $\mathfrak{K}$  that admits no nonzero continuous  $\mathfrak{K}$ -linear maps  $\mathfrak{K}^{1/p} \rightarrow \mathfrak{K}$  due to an argument of Ofer Gabber.

EXAMPLE 2.29. *Let  $k$  be any field of characteristic  $p$  and  $\Gamma \subset \mathbb{R}$  an additive subgroup such that  $\Gamma/p\Gamma$  is infinite. Define*

$$k[t^\Gamma] := \left\{ \sum_{\gamma \in \Gamma} a_\gamma t^\gamma \mid \text{the set } \{\gamma \in \Gamma \mid a_\gamma \neq 0\} \text{ is finite} \right\}$$

And let  $k(t^\Gamma)$  denote its fraction field, which lies in

$$k((t^\Gamma)) := \left\{ \sum_{\gamma \in \Gamma} a_\gamma t^\gamma \mid \text{the set } \{\gamma \in \Gamma \mid a_\gamma \neq 0\} \text{ is well-ordered} \right\}$$

Let  $L$  denote the compositum of fields  $k((t^{p^\Gamma})).k(t^\Gamma)$  over  $k((t^{p^\Gamma}))$  sitting inside  $k((t^\Gamma))$ . Assign it the induced norm from  $k((t^\Gamma))$ . Then  $L$  has no non-zero maps  $L^{1/p} \rightarrow L$  via [DM23, Theorem 5.2].

Taken together these statements imply the existence of an excellent Henselian DVR that, while clearly being quasi- $F$ -pure, is not quasi- $F$ -split.

**COROLLARY 2.29.1.** *Let  $(\mathfrak{K}, |\cdot|)$  be a non-Archimedean field that does not admit a nonzero continuous  $\mathfrak{K}$ -linear map  $\mathfrak{K}^{1/p} \rightarrow \mathfrak{K}$ . Then  $K_n(\mathfrak{K})$  is an excellent Henselian DVR that is not quasi- $F$ -split and  $T_n(\mathfrak{K})$  is a regular excellent ring that is not quasi- $F$ -split for any  $n \in \mathbb{N}$ .*

### 3. Quasi- $F$ -Purity

**REMARK 2.30.** *Much of this section is a reproduction of the author's work in [Jag26].*

Following the prior section, it would be prudent to define a “quasi” version of  $F$ -purity. Given that all regular rings are  $F$ -pure, any weakening of  $F$ -purity would also be implied by regularity. Before defining quasi- $F$ -purity, however, we first highlight a significant obstruction to developing the theory of quasi- $F$ -singularities outside of the  $F$ -finite setting.

**LEMMA 2.31** ([Jag26], lemma 2.1). *Let  $R$  be a Noetherian ring of characteristic  $p$ . For  $n > 1$ ,  $W_n(R)$  is Noetherian if and only if  $R$  is  $F$ -finite.*

**PROOF.** It is well known that when  $R$  is  $F$ -finite,  $W_n(R)$  is Noetherian [LZ04, Proposition A.4]. For the converse, consider the following short exact sequence:

$$0 \rightarrow \ker(\mathfrak{R}) \rightarrow W_n(R) \xrightarrow{\mathfrak{R}} W_{n-1}(R) \rightarrow 0$$

Where  $\mathfrak{R}$  is the restriction map on Witt Vectors.  $\ker(\mathfrak{R}) = \{(0, \dots, 0, r) \in W_n(R) \mid r \in R\}$  with a  $W_n(R)$ -module action  $\alpha \cdot V^{n-1}([r]) = V^{n-1} \left( \begin{bmatrix} \alpha_0^{p^{n-1}} & r \end{bmatrix} \right)$ . When  $R$  is not  $F$ -finite,  $R$

is not a finitely generated  $R^{p^{n-1}}$ -Module for  $n > 1$ . It follows then that the ideal  $\ker(\mathfrak{A}) \subset W_n(R)$  is not finitely generated and hence,  $W_n(R)$  is not Noetherian.  $\square$

We will overcome this hurdle using the functoriality of the Witt Vector construction, it's well behaved nature over local rings, and the preservation of étale morphisms after applying  $W_n(-)$ . Following the  $F$ -purity and  $F$ -splitting comparison, we define quasi- $F$ -purity using the  $R$  module  $Q_{R,n}$  introduced in section 1.

**DEFINITION 2.32.** Let  $R$  be a Noetherian ring of characteristic  $p$ .  $R$  is **n-Quasi-F-pure** if  $R \rightarrow Q_{R,n}$  is a pure  $W_n(R)$ -Module homomorphism (or equivalently, is a pure  $R$ -module homomorphism). If such an  $n$  exists,  $R$  is **Quasi-F-pure**. Let  $\text{ht}(R)$  denote the **Quasi-F-pure height**, or the minimal  $n$  such that  $R$  is  $n$ -Quasi- $F$ -pure, where  $\text{ht}(R) = \infty$  if it is not  $n$ -Quasi- $F$ -pure for any  $n$ .

It is easy to see that if  $R$  is  $n$ -Quasi- $F$ -pure, then it is also  $n+1$ -Quasi- $F$ -pure, implying our definition of height is well founded. As  $F_*\overline{W_1}(R) \cong F_*R$ ,  $R$  is  $F$ -pure if and only if it is 1-quasi- $F$ -pure.

**LEMMA 2.33** ([**Jag26**], lemma 3.2).  *$R$  is reduced if and only if  $\Phi_{R,n}$  is an injective  $R$ -Module homomorphism for some (equivalently, all)  $n \in \mathbb{N}$ .*

**PROOF.** Fix  $n \in \mathbb{N}$ . We will prove the equivalent statement that  $F : R \rightarrow F_*R$  is not injective if and only if  $\Phi_{R,n}$  is not injective. We are most of the way there fairly easily:

$$0 \neq t \in \ker(F) \iff t^p = 0 \iff F_*[t^p] = 0 \Rightarrow F_*\overline{[t^p]} = 0 \iff 0 \neq t \in \ker(\Phi_{R,n})$$

Recall that  $\forall t \in R$  nonzero,  $[t] = (t, 0, \dots, 0) \notin (p) = ((0, 1, 0, \dots, 0))$ . It follows that  $F_*[t^p] = 0 \iff F_*\overline{[t^p]} = 0$  as desired.  $\square$

Notably, as pure maps must be injective to begin with, lemma 2.33 above has the following obvious (and useful) corollary:

**COROLLARY 2.33.1.** *If  $R$  is Quasi- $F$ -Pure then  $R$  is reduced.*

The converse is not true; there are various examples in [KTY22] of integral  $F$ -finite schemes that are not Quasi- $F$ -split, and hence not Quasi- $F$ -pure. Similar to  $F$ -Splitting and  $F$ -Purity, quasi- $F$ -splitting and quasi- $F$ -purity are equivalent when  $R$  is  $F$ -finite or is a complete local ring.

**THEOREM 2.34.** *Suppose  $R$  is a local ring. If  $R$  is either complete OR  $F$ -finite, then being  $n$ -Quasi- $F$ -Split and  $n$ -Quasi- $F$ -Pure are equivalent notions. In particular, the Quasi- $F$ -pure height and the Quasi- $F$ -split height of  $R$  are the same.*

**PROOF.** This is a direct application of proposition 1.11, and the fact that for any local ring  $R$ ,  $Q_{R,n}^\wedge \cong Q_{\widehat{R},n}$  as  $\widehat{R}$ -modules. This fact follows from lemma 1.103.  $\square$

**LEMMA 2.35.** *Suppose  $R \rightarrow S$  is a pure extension of rings. Then  $\text{ht}(R) \leq \text{ht}(S)$ .*

**PROOF.** It is sufficient to show that, if  $S$  is  $n$ -quasi- $F$ -pure, then so is  $R$ . This follows immediately from an argument identical to that of lemma 1.18.  $\square$

**3.1. Stability under Completion.** In this section we will show that quasi- $F$ -purity, under mild assumptions, completes. From the following proposition, this will follow from a seemingly minor purity condition:

**PROPOSITION 2.36** ([Jag26], lemma 3.6). *Let  $(R, \mathfrak{m})$  be a local ring. If  $W_n(R) \rightarrow W_n(\widehat{R}) = W_n(R)^\wedge$  is a pure map for all  $n$ , then  $\text{ht}(R) = \text{ht}(\widehat{R})$ .*

**PROOF.** Given  $Q_{R,n} = F_* \overline{W}_n(R)$ , one easily sees that purity of  $W_n(R) \rightarrow W_n(\widehat{R})$  implies the purity of  $Q_{R,n} \rightarrow Q_{\widehat{R},n}$ . Recall that  $E = E_R(k) = E_{\widehat{R}}(k)$ .  $Q_{R,n} \rightarrow Q_{\widehat{R},n}$  is pure, so  $E \otimes_R Q_{R,n} \rightarrow E \otimes_R Q_{\widehat{R},n}$  is injective. From this criterion we have the following chain of

equivalences:

$$\begin{aligned}
R \text{ is } n\text{-Quasi-}F\text{-pure} &\iff \Phi_{R,n} \text{ is pure} \\
&\iff \ker(E \rightarrow E \otimes_R Q_{R,n}) = 0 \\
&\iff \ker(E \rightarrow E \otimes_R Q_{R,n} \hookrightarrow E \otimes_R Q_{\widehat{R},n}) = 0 \\
&\iff \ker(E = E \otimes_{\widehat{R}} \widehat{R} \rightarrow E \otimes_{\widehat{R}} Q_{\widehat{R},n}) = 0 \\
&\iff \Phi_{\widehat{R},n} \text{ is pure} \\
&\iff \widehat{R} \text{ is } n\text{-Quasi-}F\text{-pure}
\end{aligned}$$

□

This purity condition is known to hold for  $F$ -finite rings, as  $W_n(R)$  is Noetherian, and thus has faithfully flat (hence pure) completion. When  $R$  is not  $F$ -finite,  $W_n(R)$  is necessarily non-Noetherian, and hence, we will need to verify this purity condition directly.

**PROPOSITION 2.37.**  *$W_n(-)$  takes smooth extensions of Noetherian rings to pure extensions of rings.*

**PROOF.** Both smoothness and purity are local properties, so we can assume our extensions are local extensions. Let  $\varphi : R \rightarrow S$  be a smooth extension of Noetherian local rings. Via [Sta24, 054L]  $\varphi : R \rightarrow S$  factors through  $R[x_1, \dots, x_d]$ , for  $d$  the relative dimension of  $S$  over  $R$ :

$$R \rightarrow R[x_1, \dots, x_d] \xrightarrow{\pi} S$$

Where  $\pi$  is étale. Passing this diagram through  $W_n(-)$  yields

$$W_n(R) \rightarrow W_n(R[x_1, \dots, x_d]) \xrightarrow{W_n\pi} W_n(S)$$

which composes to  $W_n(\varphi) : W_n(R) \rightarrow W_n(S)$ .  $R \rightarrow R[x_1, \dots, x_d]$  splits in the category of rings, so via functionality of  $W_n(-)$  we can conclude that  $W_n(R) \rightarrow W_n(R[x_1, \dots, x_d])$  also splits as a ring homomorphism. Thus, it splits as a morphism of  $W_n(R)$ -Modules, and is

hence pure.  $W_n(\pi)$  is étale by theorem 1.111 and hence is flat. As  $W_n(\pi)$  is a local morphism of local rings, this is equivalent to faithful flatness, and all faithfully flat maps are pure. It follows that  $W_n(\varphi)$  is a composition of pure maps, and hence it is pure.  $\square$

The author stresses that purity is a much weaker condition than flatness. Indeed,  $W_n(\varphi)$  as above is not flat outside of the case where  $\varphi$  is étale.

REMARK 2.38. *When  $d > 0$  as above (i.e.  $\varphi$  is smooth but not étale) then  $W_n(\varphi)$  is pure but necessarily not flat (nor even faithful).*

PROOF.  $W_n(R) \rightarrow W_n(R[x_1, \dots, x_d])$  is not flat for any choice of  $n > 1, d > 0$ , and thus

$$\mathrm{Tor}_{W_n(R)}^1(W_n(R[x_1, \dots, x_d]), M) \neq 0$$

for some  $W_n(R)$ -module  $M$ . As  $W_n(\pi)$  is étale, it is pure, so by [HR76] we have an embedding of Tors

$$0 \neq \mathrm{Tor}_{W_n(R)}^1(W_n(R[x_1, \dots, x_d]), M) \hookrightarrow \mathrm{Tor}_{W_n(R)}^1(W_n(S), M)$$

implying that  $W_n(S)$  is not a flat  $W_n(R)$  module.  $\square$

As a consequence of Popescu's Theorem and lemma 1.101, we can extend proposition 2.37 to geometrically regular maps:

THEOREM 2.39 ([Jag26], theorem 2.6).  *$W_n(-)$  takes geometrically regular extensions of Noetherian rings to pure extensions of rings.*

PROOF. Suppose  $R \rightarrow S$  is a geometrically regular extension as above. Popescu's theorem tells us that  $S$  is a filtered colimit of smooth  $R$ -algebras  $S = \varinjlim S_i$ .  $W_n(R) \rightarrow W_n(S_i)$  for each  $i$  is hence a pure morphism by proposition 2.37, and as purity is preserved under colimits, it follows that  $W_n(R) \rightarrow \varinjlim W_n(S_i)$  is pure. via lemma 1.101 it follows that  $\varinjlim W_n(S_i) = W_n(\varinjlim S_i) = W_n(S)$ . Thus we conclude that  $W_n(R) \rightarrow W_n(S)$  is pure.  $\square$

COROLLARY 2.39.1. *If  $(R, \mathfrak{m})$  is a local  $G$ -ring, then it follows from the above theorem that  $W_n(R) \rightarrow W_n(\widehat{R}) = W_n(R)^\wedge$  is pure. Thus by proposition 2.36, for a  $G$ -ring  $R$ ,  $\text{ht}(R) = \text{ht}(\widehat{R})$ .*

The author is unaware whether or not  $W_n(R) \rightarrow W_n(\widehat{R}) = W_n(R)^\wedge$  is flat when  $R$  is not  $F$ -finite.

**3.2. Stability under Direct Limit.** It is well known that within the category of  $R$ -Modules, purity is preserved under direct limit. We prove a slightly more general claim that purity is preserved even when the base ring changes under the limit. This will be necessary for confirming that quasi- $F$ -purity is preserved under direct limit.

LEMMA 2.40 ([Jag26], lemma 3.7). *Let  $M = \varinjlim M_i$  and  $N = \varinjlim N_i$  be two directed systems of  $\varinjlim R_i = R$ -Modules where  $M_i \rightarrow N_i$  is pure as a morphism of  $R_i$ -Modules for each  $i$ . Then  $M \rightarrow N$  is a pure map of  $R$ -Modules, i.e. purity is preserved under direct limit.*

PROOF. Choose any  $R$ -Module  $L$ . Letting  $R_i \rightarrow R \rightarrow L$  define an  $R_i$ -Module structure on  $L$ , we immediately see that  $L \otimes_{R_i} M_i \rightarrow L \otimes_{R_i} N_i$  is injective for all  $i$  by purity. Further,  $\varinjlim (L \otimes_{R_i} M_i) \rightarrow \varinjlim (L \otimes_{R_i} N_i)$  is injective, as direct limits are exact in the category of abelian groups and thus preserve injectivity. Thus this map is an injective morphism of  $R$ -Modules. Via tensoring the universal property diagram of  $\varinjlim M_i$  with  $L$  (as an  $R_i$ -Module for each  $M_i$  and as an  $R$ -Module for  $M$ ), we see that  $L \otimes_R \varinjlim M_i$  satisfies the universal property of direct limits with respect to the directed system  $\{L \otimes_{R_i} M_i\}_{i \in \mathbb{N}}$ . It follows that

$$\varinjlim (L \otimes_{R_i} M_i) \cong L \otimes_R \varinjlim M_i$$

As  $R$ -Modules, and similarly for the  $N_i$ . Thus

$$L \otimes_R M \cong L \otimes_R \varinjlim M_i \rightarrow L \otimes_R \varinjlim N_i \cong L \otimes_R N$$

is an injective map of  $R$ -Modules, and hence,  $M \rightarrow N$  is pure.  $\square$

THEOREM 2.41 ([Jag26], Theorem 3.8). *Let  $\{S_i\}$  be a directed system of rings where  $\varinjlim S_i = S$ . If  $\exists M \in \mathbb{N}$  such that  $\forall m > M$   $\text{ht}(S_m) \leq n$ , then  $\text{ht}(S) \leq n$ .*

PROOF. Fix  $M$  as above. If  $\text{ht}(S_m) \leq n$ , then  $S_m \rightarrow Q_{S_m, n}$  is a pure map of  $S_m$ -Modules for all  $m > M$ . Via the prior lemma, it follows that  $S = \varinjlim S_i \rightarrow \varinjlim Q_{S_i, n}$  is a pure map of  $S$ -Modules.

Further, recall that direct limits commute with the Witt Vector construction (lemma 1.101). This, along with the fact that pushouts are themselves colimits and thus commute with direct limits, implies that  $\varinjlim Q_{S_i, n} = Q_{S, n}$ . It follows that  $S \rightarrow Q_{S, n}$  is pure. Thus,  $\text{ht}(S) \leq n$ .  $\square$

REMARK 2.42. *if  $\text{ht}(S_i) = n$  and this is a pure direct limit (i.e.  $S_i \rightarrow S$  is a pure  $S_i$ -module homomorphism for all  $i$  sufficiently large), then via lemma 2.35 we get equality, i.e.  $\text{ht}(S) = n$  for  $n$  as above.*

**3.3. Stability under Étale Extension.** Utilizing the fact that the relative Witt Frobenius is an isomorphism for étale extensions (lemma 1.112), we show that quasi- $F$ -purity is preserved under étale extension.

THEOREM 2.43 ([Jag26], Theorem 3.11). *Let  $(R, \mathfrak{m})$  be a  $G$ -ring and  $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$  be a local étale extension of local rings. Then  $S$  is a  $G$ -ring and  $\text{ht}(R) = \text{ht}(S)$ .*

PROOF. First note that  $S$  is essentially of finite type over  $R$ ; it follows from [Sta24, 07PV] that  $S$  is also a  $G$ -ring. In this setting via corollary 2.39.1, we know that  $\text{ht}(R) = \text{ht}(\widehat{R})$ ,  $\text{ht}(S) = \text{ht}(\widehat{S})$ , and via [Sta24, 039M] that  $\widehat{R}_{\mathfrak{m}} \rightarrow \widehat{S}_{\mathfrak{n}}$  remains étale. Thus, we may reduce to the case where  $R$  and  $S$  are both complete local, where quasi- $F$ -split and quasi- $F$ -pure height coincide (Theorem 2.34).

We first note that étale maps are (faithfully) flat, hence pure. Thus  $\text{ht}(R) \leq \text{ht}(S)$ . As quasi- $F$ -purity and quasi- $F$ -splitting are equivalent notions over complete local rings, it is

sufficient to check that if  $R$  is  $n$ -quasi- $F$ -split, then so is  $S$ . If  $R$  is  $n$ -quasi- $F$ -split there exists the following dashed morphism of  $W_n(R)$ -modules:

$$\begin{array}{ccc} W_n(R) & \longrightarrow & F_*W_n(R) \\ \downarrow & \swarrow \text{dashed} & \\ R & & \end{array}$$

Applying the functor  $- \otimes_{W_n(R)} W_n(S)$  yields the diagram of  $W_n(S)$ -Modules

$$\begin{array}{ccc} W_n(S) & \longrightarrow & F_*W_n(R) \otimes_{W_n(R)} W_n(S) \\ \downarrow & \swarrow \text{dashed} & \\ R \otimes_{W_n(R)} W_n(S) & & \end{array}$$

Via lemma 1.112, we can see that  $F_*W_n(R) \otimes_{W_n(R)} W_n(S) \cong F_*W_n(S)$ . Further, we have a restriction morphism

$$\text{Id} \otimes_{W_n(R)} \mathfrak{A}^{n-1} : R \otimes_{W_n(R)} W_n(S) \rightarrow R \otimes_{W_n(R)} S$$

As  $R \rightarrow S$  is étale, this map is an isomorphism [LZ04, Proposition A.8]. Utilizing this alongside the  $W_n(R)$ -module structure of  $R$  and  $S$  yields the following isomorphism of  $W_n(S)$ -modules.

$$R \otimes_{W_n(R)} W_n(S) \cong R \otimes_{W_n(R)} S \cong R \otimes_R S \cong S$$

Plugging both of these isomorphisms into the above diagram yields

$$\begin{array}{ccc} W_n(S) & \longrightarrow & F_*W_n(S) \\ \downarrow & \swarrow \text{dashed} & \\ S & & \end{array}$$

Where the dashed morphism denotes the desired  $n$ -Quasi- $F$ -splitting. □

**COROLLARY 2.43.1.** *Suppose  $(R, \mathfrak{m})$  is a local  $G$ -ring and  $(S, \mathfrak{n})$  is a direct limit of local étale  $R$ -algebras. Then  $\text{ht}(R) = \text{ht}(S)$ .*

**PROOF.**  $R \rightarrow S$  is a direct limit of pure extensions, and hence pure. Thus by lemma 2.35  $\text{ht}(R) \leq \text{ht}(S)$ . Checking  $\text{ht}(S) \leq \text{ht}(R)$  is a combination of theorems 2.41 and 2.43. □

**COROLLARY 2.43.2.** *Let  $(R, \mathfrak{m})$  be a local  $G$ -ring and  $k$ -algebra, for  $k$  any field of characteristic  $p > 0$ . If  $L/k$  is an algebraic separable extension, then  $\text{ht}(R) = \text{ht}(L \otimes_k R)$ .*

PROOF.  $L = \varinjlim L_i$  where each  $L_i$  is a finite separable, hence étale, extension of  $k$ . It follows then  $R \rightarrow L_i \otimes_k R$  is a local étale extension, as étale extensions are preserved under base change. Thus,

$$L \otimes_k R \cong (\varinjlim L_i) \otimes_k R \cong \varinjlim (L_i \otimes_k R)$$

is a direct limit of local étale  $R$ -algebras, and we cite the preceding corollary.  $\square$

As  $\text{ht}(R) = \text{ht}(\widehat{R})$  and  $\widehat{R}$  is naturally an algebra over a coefficient field  $\mathfrak{K}$ , corollary 2.43.2 shows that  $\text{ht}(R) = \text{ht}(L \otimes_{\mathfrak{K}} \widehat{R})$  for  $L$  any algebraic separable extension of  $\mathfrak{K}$ . We caution that the separability condition is necessary; see chapter 3 section 5 for further discussion.

We can generalize theorem 2.43 outside of the local case, but will require that  $R \rightarrow S$  be finite.

**THEOREM 2.44** ([Jag26], Theorem 3.12). *Let  $(R, \mathfrak{m})$  be a local  $G$ -ring and  $R \rightarrow S$  be a finite étale extension. Then  $S$  is a  $G$ -ring and  $\text{ht}(R) = \text{ht}(S)$ .*

PROOF. First we base change to the completion  $\widehat{R}$ , where the corresponding map  $\widehat{R} \rightarrow \widehat{R} \otimes_R S$  remains étale. Via [Sta24, 04GH] we can decompose  $\widehat{R} \otimes_R S \cong S_1 \times \cdots \times S_\ell$  where each  $S_i$  is local (with maximal ideal denoted  $\mathfrak{n}_i$ ) and finite over  $\widehat{R}$ . From this description, it is easy to see that  $S_{\mathfrak{n}_i} \cong S_i$  for each  $i$ , and as  $\mathfrak{n}_i$  lies over  $\mathfrak{m}$ , the composition  $\widehat{R} \rightarrow S \rightarrow S_{\mathfrak{n}_i} \cong S_i$  is a local map. As localization is étale it follows that  $\widehat{R} \rightarrow S_i$  is étale local. Thus via corollary 2.39.1 and theorem 2.43,  $\text{ht}(R) = \text{ht}(\widehat{R}) = \text{ht}(S_i) \forall i \leq \ell$ . As purity is a local property that can be checked at the level of  $\max \text{Spec}$ , we see that

$$\text{ht}(S) = \sup_{i \leq \ell} \text{ht}(S_i) = \text{ht}(R)$$

$\square$

This result can also be globalized. For any scheme of positive characteristic  $X$ ,  $X$  is  $n$ -quasi- $F$ -pure if the morphism  $\mathcal{O}_X \rightarrow Q_{X,n}$  is a pure map of  $\mathcal{O}_X$ -modules. This can be verified on stalks, though first we provide an important characterization of stalks on  $Q_{X,n}$ :

LEMMA 2.45. *Let  $X$  be any scheme of positive characteristic. Then for all  $x \in X$ ,*

$$(Q_{X,n})_x = Q_{\mathcal{O}_{X,x},n}$$

PROOF. Recall that  $\mathcal{O}_{X,x} = \varinjlim_{U \ni x} \mathcal{O}_X(U)$ .  $W_n(-)$  and Frobenius pushforward commute with colimits (lemma 1.101), and hence we have that

$$(W_n \mathcal{O}_X)_x = W_n(\mathcal{O}_{X,x})$$

and similarly for  $F_* W_n \mathcal{O}_X$ .  $Q_{X,n}$  is itself a pushout, and hence colimit of, the diagram

$$\mathcal{O}_X \xleftarrow{\mathfrak{F}^{n-1}} W_n \mathcal{O}_X \xrightarrow{F} F_* W_n \mathcal{O}_X$$

So the lemma follows from the fact that colimits commute with each other.  $\square$

LEMMA 2.46. *Let  $X$  be any scheme of positive characteristic.  $X$  is  $n$ -quasi- $F$ -pure (resp. quasi- $F$ -pure) if  $\mathcal{O}_{X,x}$  is  $n$ -quasi- $F$ -pure (resp. quasi- $F$ -pure) for all  $x \in X$ .*

PROOF.  $\mathcal{O}_X \rightarrow Q_{X,n}$  is a pure map of  $\mathcal{O}_X$ -modules if and only if  $\mathcal{O}_{X,x} \rightarrow (Q_{X,n})_x = Q_{\mathcal{O}_{X,x},n}$  is pure for all  $x \in X$ .  $\square$

REMARK 2.47. *In chapter 3 we will define “iterate” modules  $Q_{X,n}^e, Q_{X,\Delta,n}^e$  and iterate modules for splitting along a perturbation by  $c$ :  $Q_{X,n}^{e,c}, Q_{X,\Delta,n}^{e,c}$ . The above arguments hold when replacing  $Q_{X,n}$  with any of these modules, as the only fact that we used about  $Q_{X,n}$  is that it is a pushout of modules compatible with colimits.*

This lemma yields the following global analogue of theorem 2.43:

THEOREM 2.48. *Let  $X$  be a scheme of characteristic  $p$  such that  $\mathcal{O}_{X,x}$  is a  $G$ -ring for all  $x \in X$  (e.g.  $X$  is an excellent scheme). Let  $Y \rightarrow X$  be a dominant étale map. Then  $\text{ht}(Y) = \text{ht}(X)$ .*

PROOF. As  $Y \rightarrow X$  is étale and dominant,  $\mathcal{O}_{Y,y}$  is a  $G$ -ring for all  $y \in Y$  and  $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,f(y)}$  is a local étale extension of local rings. The theorem follows from applying theorem 2.43 at each stalk and concluding by lemma 2.46.  $\square$

#### 4. Quasi- $F$ -Purity and $F$ -Pure Thresholds

REMARK 2.49. *Much of this section is a reproduction of the author and Jack J Garzella's work in [GJ25]. Garzella has graciously allowed for this material to be reproduced in the author's thesis.*

In [BS15], Bhatt and Singh calculate the  $F$ -pure threshold of a smooth Calabi-Yau hypersurface  $X = Z(f) \subset \mathbb{P}_k^n$ . If the characteristic of  $k$  is strictly greater than  $n^2 - n - 1$ , they show that

$$\text{fpt}(f) = 1 - \frac{a}{p}$$

where  $a = a(X)$  is the  $a$ -number of a variety in positive characteristic, a generalization of the  $a$ -number of an abelian variety, defined in full generality by van der Geer and Katsura [vdGK13]. Van der Geer and Katsura also show that the  $a$ -number has the following relationship with the Artin-Mazur height  $\text{AM}(X)$  of the hypersurface  $f$ : if  $1 < \text{AM}(X) < \infty$ , then  $a(X) = 1$ , and if  $1 = \text{AM}(X)$ , then  $a(X) = 0$  (see section 1.2 for a more detailed discussion of these invariants and their relationship). Finally, Yobuko in [Yob19] shows that the Artin-Mazur height is equal to the quasi- $F$ -split height in this case. Taken together, these statements in the literature prove that, if  $\text{char}(k) > n^2 - n - 1$  and  $X = Z(f) \subset \mathbb{P}_k^n$  is a smooth quasi- $F$ -split Calabi-Yau hypersurface, then

$$\text{fpt}(f) \geq 1 - \frac{1}{p}$$

Via theorem 1.32,  $1 - 1/p$  is the highest  $F$ -pure threshold that can occur that is not exactly 1 in this setting. Thus, this statement shows that a quasi- $F$ -pure Calabi-Yau hypersurface is as close to being  $F$ -pure as possible: either it is just  $F$ -pure and  $\text{fpt}(f) = 1$  or it is not and  $\text{fpt}(f) = 1 - 1/p$  is maximal.

The goal of this section is to extend this result outside of the Calabi-Yau case, and in the process, weaken the bound on  $p$ . The proof uses only the Fedder-type criterion for quasi- $F$ -purity described in [KTY22] (cf. [Yos25]), to be introduced in the following subsection, and

the aforementioned theorem on the discreteness of the  $F$ -pure threshold. The proof makes no mention of formal groups, deformation theory, or the Hodge and conjugate filtrations.

**REMARK 2.50.** *As we are in the affine and  $F$ -finite setting for this entire section, quasi- $F$ -purity and quasi- $F$ -splitting are equivalent (theorem 2.34). Thus we use these terms interchangeably for the remainder of this section.*

Given our focuses on hypersurfaces in this section, for a polynomial  $f \in k[x_1, \dots, x_n]$  we let  $\text{ht}(f)$  denote the quasi- $F$ -pure (or equivalently, by the remark, the quasi- $F$ -split height) of the ring  $\frac{k[x_1, \dots, x_n]}{f}$ .

**4.1. Quasi-Fedder's Criterion.** In this subsection we develop the machinery necessary to state the Fedder's style criterion for quasi- $F$ -purity in [KTY22], similarly to section 1.1 in their paper. We then use this criterion to compute the  $F$ -pure threshold of a quasi- $F$ -pure hypersurface singularity.

For simplicity, we take our regular ring  $A$  to be the localized polynomial ring  $k[x_1, \dots, x_n]_{\mathfrak{m}}$  over an  $F$ -finite field  $k$  of characteristic  $p$ . As quasi- $F$ -purity localizes and completes, this is not a significant simplification.  $F_*A$  is a free  $A$  module with a basis consisting of monomials  $F_*x_1^{i_1}x_2^{i_2}\dots x_n^{i_n}$ , where each  $i_j \leq p-1$ . Further, the dual space  $\text{Hom}_A(F_*A, A)$  is principally generated as a  $F_*A$ -module by the dual basis vector  $u$  of  $F_*(x_1x_2\dots x_n)^{p-1}$ . Explicitly,

$$\text{Hom}_A(F_*A, A) \cong F_*A \cdot u \quad u(F_*b) = \begin{cases} 1 & b = F_*(x_1x_2\dots x_n)^{p-1} \\ 0 & \text{else} \end{cases}$$

Where  $F_*b$  denotes an arbitrary  $A$ -basis element of  $F_*A$ , viewed as a free  $A$ -module. The  $u$  map will be used in the statement of the criterion, as will a certain map  $\Delta_1 : A \rightarrow A$  which we will now define. For any element  $a \in A$ , we can write  $a = \sum a_i M_i$  as a  $k$ -linear sum of monomials  $M_i = \prod_{j=1}^n x_j^{i_j}$ . In  $W_2(A)$ ,  $(a, 0)$  and  $\sum(a_i M_i, 0)$  do not differ in the 0th coordinate, but do differ in the 1st coordinate;  $\Delta_1(a)$  measures this difference. In particular, for a given  $a \in A$  we define  $\Delta_1(a) \in A$  to be the unique value that satisfies the equation

$(0, \Delta_1(a)) = (a, 0) - \sum (a_i M_i, 0)$ . Using the universal Witt polynomials, one obtains a more explicit description:

$$\Delta_1(a) := \frac{(\sum a_i M_i)^p - \sum (a_i M_i)^p}{p} = \sum_{0 \leq \alpha_1, \dots, \alpha_m \leq p-1, \sum \alpha_j = p} \frac{1}{p} \binom{p}{\alpha_1, \dots, \alpha_m} \prod_{i=1}^m (a_i M_i)^{\alpha_i}$$

As  $p$  divides  $\binom{p}{\alpha_1, \dots, \alpha_m}$  for any valid choice of  $(\alpha_j)$ , this division by  $p$  is purely formal. This is all we will need to state the theorem for hypersurfaces  $A/(f)$ . The authors of [KTY22] actually obtained a more general criterion for complete intersections, see [KTY22, Theorem 4.11]. We state the criterion for principal quotients of polynomial rings below.

**THEOREM 2.51** (Fedder Type Criterion for Quasi- $F$ -Purity [KTY22]). *Let  $A = k[x_1, \dots, x_n]_{\mathfrak{m}}$  and pick  $f \in A$ . Define  $\theta : F_* S \rightarrow S$  to be the map*

$$\theta(F_* a) := u(F_* \Delta_1(f^{p-1})a)$$

*Letting  $I_1 := (f^{p-1})$ , we define the increasing sequence of ideals  $\{I_m\}_m$  inductively as follows:*

$$I_{m+1} := \theta(I_m \cap \ker(u)) + (f^{p-1})$$

*Then*

$$\text{ht}(f) = \inf\{m \mid I_m \not\subseteq \mathfrak{m}^{[p]}\}$$

We utilize the convention that the infimum of the empty set is  $\infty$ . If  $f$  is homogeneous with respect to an  $\mathbb{N}$ -grading on  $A$ , we can replace  $A$  with  $k[x_1, \dots, x_n]$  (i.e. we don't need to assume that  $A$  is local, see [KTY22, corollary 5.4]). This theorem recovers the standard Fedder's criterion and can detect the exact quasi- $F$ -pure height of a given hypersurface. Though the statement looks cumbersome at first glance, it is suitable for implementation in a computer algebra system using GPU acceleration techniques (see [BGP25], [FGM<sup>+</sup>]) and has been used to compute many examples of quasi- $F$ -pure hypersurfaces.

**4.2.  $F$ -Pure-Threshold of a Quasi- $F$ -Pure Hypersurface.** Using their theorem, the authors obtained a useful criterion for determining when a hypersurface is not non-Quasi- $F$ -pure.

COROLLARY 2.51.1 ([KTY22], corollary 4.19). *Let  $A = k[x_1, \dots, x_n]_{\mathfrak{m}}$  and  $f \in A$ .*

(a) *If  $f^{p-2} \in \mathfrak{m}^{[p]}$ , then  $\text{ht}(f) = \infty$ .*

(b) *If  $f^{(p+1)(p-2)} \Delta_1(f) \in \mathfrak{m}^{[p^2]}$  for some representative  $\Delta_1(f) \in A$  and  $f^{p-1} \in \mathfrak{m}^{[p]}$ , then  $\text{ht}(f) = \infty$ .*

Corollary 2.51.1(a) immediately implies that, if  $A/f$  is quasi- $F$ -pure,  $f^{p-2} \notin \mathfrak{m}^{[p]}$ . This yields the inequality  $\text{fpt}(f) \geq \nu_f(p)/p \geq 1 - \frac{2}{p}$ . We will now use 2.51.1(b) to obtain a sharper bound.

THEOREM 2.52 ([GJ25], theorem 3.2). *Let  $A = k[x_1, \dots, x_n]$ , for  $k$  an  $F$ -finite field of characteristic  $p > 0$ , and  $f \in A$  define an isolated hypersurface singularity or be quasi-homogeneous. If  $A/(f)$  is quasi- $F$ -pure, then  $\text{fpt}(f) \geq 1 - \frac{1}{p} - \frac{2}{p^2}$ . If we assume  $f$  is an isolated hypersurface singularity AND quasi-homogeneous, and in addition that  $p > n - 2$  and  $p \neq 2$ , then  $\text{fpt}(f) \geq 1 - \frac{1}{p}$ .*

PROOF. If  $f$  has an isolated singularity, via a change of coordinates we can assume the isolated singularity is at the origin. We can localize at  $\mathfrak{m} = (x_1, \dots, x_n)$  and can thus assume that  $A = k[x_1, \dots, x_n]_{\mathfrak{m}}$ . If  $f$  is quasi-homogeneous, we can reduce to the local case via [KTY22, proposition 2.25]. If  $\text{ht}(f) < \infty$ , by the contrapositive of 2.51.1(b) we see that either  $f^{p-1} \notin \mathfrak{m}^{[p]}$  (in which case  $\text{fpt}(f) = 1$  as  $A/f$  is  $F$ -pure by Fedder's Criterion) or  $f^{(p+1)(p-2)} = f^{p^2-p-2} \notin \mathfrak{m}^{[p^2]}$ . The former case is immediate, so we reduce to the latter case. This case implies that

$$\text{fpt}(f) \geq \frac{\nu_f(p^2)}{p^2} \geq 1 - \frac{1}{p} - \frac{2}{p^2}$$

Assuming  $f$  is quasi-homogeneous, we fix an  $\mathbb{N}$ -grading on  $A$  such that  $f$  is homogeneous. In this setting, we note that if  $\sum \deg(x_i) < \deg(f)$  then  $A/(f)$  is of general type, and thus, never quasi- $F$ -pure. Thus, if  $\sum \deg(x_i) \geq \deg(f)$  by theorem 1.32  $\text{fpt}(f)$  takes the form

$1 - \frac{h}{p}$  for some  $h$ . The bound  $\text{fpt}(f) \geq 1 - \frac{1}{p} - \frac{2}{p^2}$  shows that  $h < 2$  provided that  $p \neq 2$ , so it follows that

$$\text{fpt}(f) \geq 1 - \frac{1}{p}$$

as desired.  $\square$

As  $\text{fpt}(f) = 1$  if and only if  $A/(f)$  is  $F$ -pure, we obtain the following equality.

**COROLLARY 2.52.1.** *Let  $A = k[x_1, \dots, x_n]$  for  $k$  a perfect field of odd characteristic  $p > n - 2$  and  $f \in A$  be a quasi-homogeneous isolated singularity. If  $\text{ht}(f) \in (1, \infty)$  (e.g.  $A/f$  is quasi- $F$ -pure at the isolated singularity but not  $F$ -pure), then  $\text{fpt}(f) = 1 - \frac{1}{p}$ .*

We note that  $1 - \frac{1}{p}$  is the highest attainable  $F$ -pure threshold for a quasi-homogeneous polynomial that is not  $F$ -pure by [**HNnBWZ16**, corollary 3.7]. Thus, in terms of the  $F$ -pure threshold and in view of theorem 2.52, quasi- $F$ -pure singularities are “as close to being  $F$ -pure” as possible. We caution that the converse of theorem 2.52 is false in many cases. The following examples were computed via sampling from a space of certain polynomials using the methods of [**BGP25**].

**EXAMPLE 2.53** ([**Jag26**], Example 3.3). *The following equations in  $\mathbb{F}_7[x_1, x_2, x_3, x_4]$  define supersingular K3 surfaces (i.e. K3 surfaces that are not quasi- $F$ -pure) which have  $F$ -pure threshold  $1 - 1/7$ .*

- $4x_1^4 + 5x_1^3x_3 + 3x_1^3x_4 + 6x_1^2x_2^2 + 2x_1^2x_2x_3 + 3x_1^2x_2x_4 + 3x_1^2x_3^2 + 3x_1^2x_3x_4 + 5x_1^2x_4^2 + 3x_1x_2^2x_4 + 6x_1x_2x_3^2 + x_1x_2x_4^2 + 2x_1x_3^2x_4 + 5x_1x_3x_4^2 + 4x_1x_4^3 + 5x_2^3x_3 + 5x_2^3x_4 + 5x_2^2x_3^2 + 2x_2^2x_3x_4 + 3x_2^2x_4^2 + 5x_2x_3^2x_4 + 4x_2x_3x_4^2 + 2x_2x_4^3 + 2x_3x_4^3$
- $5x_1^4 + 3x_1^3x_2 + x_1^3x_3 + 6x_1^3x_4 + 5x_1^2x_2^2 + 4x_1^2x_2x_4 + 2x_1^2x_3^2 + 5x_1^2x_3x_4 + 4x_1x_2^3 + 4x_1x_2^2x_3 + 4x_1x_2^2x_4 + 5x_1x_2x_3x_4 + 2x_1x_3^2x_4 + 5x_1x_3x_4^2 + x_1x_4^3 + 5x_2^4 + x_2^3x_3 + 3x_2^2x_3^2 + 4x_2^2x_3x_4 + 4x_2^2x_4^2 + 3x_2x_3^3 + 5x_2x_3^2x_4 + 5x_2x_4^3 + 4x_3^4 + 5x_3^3x_4 + x_3x_4^3 + 5x_4^4$

- $3x_1^4 + 3x_1^3x_2 + 3x_1^3x_3 + 6x_1^2x_2^2 + 3x_1^2x_2x_4 + 2x_1^2x_3^2 + 2x_1^2x_3x_4 + 3x_1^2x_4^2 + 6x_1x_2^3 + 5x_1x_2^2x_3 + x_1x_2x_3x_4 + 5x_1x_2x_4^2 + 5x_1x_3^3 + 4x_1x_3^2x_4 + 3x_1x_3x_4^2 + 6x_1x_4^3 + x_2^4 + 4x_2^3x_4 + 3x_2^2x_3^2 + 5x_2^2x_3x_4 + 5x_2x_3^3 + x_2x_3^2x_4 + 6x_2x_3x_4^2 + x_3^3x_4 + x_3^2x_4^2 + 3x_3x_4^3 + 4x_4^4$

As a consequence of [BS15], K3 surfaces (and abelian surfaces) must necessarily have  $F$ -pure threshold  $1$ ,  $1 - 1/p$ , or  $1 - 2/p$ . If  $\text{fpt} = 1$ , then the surface is  $F$ -split, and hence, ordinary. If  $\text{fpt} = 1 - 2/p$ , the contrapositive of theorem 2.52 implies that it cannot be quasi- $F$ -split. These examples imply that if  $\text{fpt} = 1 - 1/p$ , the final case, then there exists both quasi- $F$ -pure and non-quasi- $F$ -pure examples.

**4.3. Fermat Hypersurfaces.** The goal of this subsection is to investigate when a Fermat type hypersurface (e.g. a hypersurface defined by an equation of the form  $x_1^d + \dots + x_n^d$ ) is  $F$ -pure, quasi- $F$ -pure but not  $F$ -pure, or not quasi- $F$ -pure. This section is a reproduction of [GJ25, section 4]; we will not cite each result individually.

We first note that the characteristic 2 differs slightly from the general case: we outline this case below.

LEMMA 2.54. *For  $n \geq 2$ , set  $f = x_1^d + \dots + x_n^d \in k[x_1, \dots, x_n]$  over  $k$ , an  $F$ -finite field of characteristic 2. Then*

$$\text{ht}(f) = \begin{cases} 1 & d = 1 \\ 2 & d = 3 \text{ and } n \geq 3 \\ \infty & \text{otherwise} \end{cases}$$

PROOF. The case of  $d = 1$  is clear. If  $d = 2$  or  $d = 4$ ,  $R$  is non-reduced and hence not quasi- $F$ -pure. If  $d = 3$ , via [KTY22, Example 7.9] we see that the Fermat cubic has height 2 in characteristic 2 when  $n \geq 3$ ; if  $n < 3$  then  $R$  is of general type and hence not quasi- $F$ -pure. This just leaves the case where  $d \geq 5$ .

To prove that  $\text{ht}(f) = \infty$  when  $d \geq 5$ , we can apply corollary 2.51.1(b) when  $p = 2$ : It is thus sufficient to show that  $\Delta_1(f) \in \mathfrak{m}^{[4]}$ . Fortunately in characteristic 2,  $\Delta_1(f)$  is easily

computable.

$$\Delta_1(f) = \frac{(\sum_{i=1}^n x_i^d)^2 - \sum_{i=1}^n x_i^{2d}}{2} = \sum_{i < j} x_i^d \cdot x_j^d$$

When  $d \geq 5$  it immediately follows that  $\Delta_1(f) \in \mathfrak{m}^{[4]}$ .  $\square$

**COROLLARY 2.54.1.** *As this characterization holds for all  $n \geq 2$ , this lemma provides examples of smooth Fano varieties of any dimension  $\geq 5$  in characteristic 2 that are not quasi- $F$ -pure.*

In view of this we will primarily concern ourselves with the following setting in this subsection:

**SETTING 2.55.** *Let  $A = k[x_1, \dots, x_n]$  where  $k$  is a perfect field of characteristic  $p > 2$  and  $n \geq 3$ . Let  $f = x_1^d + \dots + x_n^d \in A$  for some  $d > 1$  and take  $R = A/(f)$ .*

In general,  $F$ -pure thresholds of any diagonal hypersurface have been computed in [Her15]; this can be used to classify whether a Fermat Hypersurface is  $F$ -pure or not. While these methods can partially recover the following results, it cannot detect for quasi- $F$ -purity. Further, one can classify Fermat hypersurfaces very explicitly using the Fedder's criterion for (quasi)- $F$ -purity [KTY22]: We spell these arguments out below.

**LEMMA 2.56.** *In setting 2.55, if  $d < \lceil \frac{p}{a} \rceil$  and  $n \geq \frac{p-1}{a}$  for some  $a \geq 1$ ,  $R$  is  $F$ -pure.*

**PROOF.** Via Fedder's criterion [Fed83], we need to show that  $f^{p-1} \notin \mathfrak{m}^{[p]}$ . It is sufficient to construct a monomial present in  $f^{p-1}$  which contains no  $p$ th powers. A general monomial of  $f^{p-1}$  is of the following form:

$$c \cdot x_1^{i_1 \cdot d} x_2^{i_2 \cdot d} \dots x_n^{i_n \cdot d}$$

For  $0 \neq c \in k$  and  $\sum i_j = p-1$ . Our goal is to pick  $(i_1, \dots, i_n)$  such that  $i_j \cdot d < p$  for all  $j$ . Well as  $d < \lceil \frac{p}{a} \rceil$ , this is equivalent to finding  $(i_1, \dots, i_n)$  such that  $i_j \leq a$ . As  $a \cdot n \geq p-1 = \sum i_j$ , it is necessarily possible to choose the  $i_j$  such that they are all at most  $a$ ; simply set them all to  $a$  then subtract ones from them arbitrarily (without making any  $i_j$  negative)  $a \cdot n - (p-1)$  times.  $\square$

LEMMA 2.57. *In setting 2.55, if  $d \geq \lceil \frac{p}{a+1} \rceil$  and  $n < \frac{p-1}{a}$ , then  $R$  is not  $F$ -pure. If we further assume  $n < \frac{p-2}{a}$ , then  $R$  is not quasi- $F$ -pure.*

PROOF. Assume that  $n < \frac{p-1}{a}$ . Via Fedder's criterion [Fed83], we need to show that  $f^{p-1} \in \mathfrak{m}^{[p]}$ . A general monomial of  $f^{p-1}$  is of the following form:

$$c \cdot x_1^{i_1 \cdot d} x_2^{i_2 \cdot d} \dots x_n^{i_n \cdot d}$$

For  $0 \neq c \in k$  and  $\sum i_j = p-1$ . As  $n < \frac{p-1}{a}$ , at least one of these  $i_1, \dots, i_n$  is at least  $a+1$ . Fix  $j$  such that  $i_j \geq a+1$ . As  $d \geq \lceil \frac{p}{a+1} \rceil$ , this implies that  $i_j \cdot d \geq p$ , and so  $c \cdot x_1^{i_1 \cdot d} x_2^{i_2 \cdot d} \dots x_n^{i_n \cdot d} \in \mathfrak{m}^{[p]}$ . As this is true for every monomial, it follows that  $f^{p-1} \in \mathfrak{m}^{[p]}$  and thus  $f$  is not  $F$ -pure.

For the quasi- $F$ -pure step, it is sufficient to show that  $f^{p-2} \in \mathfrak{m}^{[p]}$  via corollary 2.51.1(a). Given that  $n < \frac{p-2}{a}$ , the proof is identical.  $\square$

Combining these results yields the majority of our theorem:

THEOREM 2.58. *Let  $f = x_1^d + \dots + x_n^d \in A = k[x_1, \dots, x_n]$  where  $k$  is of characteristic  $p > 2$  and  $n \geq 3$ . Let  $R = A/(f)$ . Then for any  $a \geq 1$ :*

- (1) *If  $\lceil \frac{p}{a+1} \rceil \leq d < \lceil \frac{p}{a} \rceil$ , then we have separate cases depending on  $n$ :*
  - (a) *If  $n < \frac{p-2}{a}$ ,  $R$  is not quasi- $F$ -pure.*
  - (b) *If  $\frac{p-2}{a} \leq n < \frac{p-1}{a}$ ,  $R$  is not  $F$ -pure. Further, if  $R$  is quasi- $F$ -pure then  $d < n$  or  $d = n = 3$ .*
  - (c) *If  $n \geq \frac{p-1}{a}$ ,  $R$  is  $F$ -pure.*
- (2) *If  $d \geq p$ ,  $R$  is not quasi- $F$ -pure.*

If we chase through the cases we obtain an interesting criterion for which  $F$ -purity and quasi- $F$ -purity are equivalent notions for Fermat hypersurfaces.

COROLLARY 2.58.1. *If  $p \not\equiv 2 \pmod{n}$ , then any Fermat type hypersurface is quasi- $F$ -pure if and only if it is  $F$ -pure.*

PROOF. We note that if  $n \in \mathbb{N}$ ,  $\frac{p-2}{a} \leq n < \frac{p-1}{a}$  if and only if  $n = \frac{p-2}{a}$ . Thus if  $p > 2$  and  $p \not\equiv 2 \pmod n$ , case (1b) in the theorem above is vacuous, and in every other case,  $R$  is either  $F$ -pure or not quasi- $F$ -pure. If  $p = 2$  then it is always equivalent to  $2 \pmod n$ ; therefore the result follows.  $\square$

We now prove the theorem.

PROOF OF THEOREM 2.58. The following cases are immediate given the above lemmas:

- (1) (a) Lemma 2.57
- (c) Lemma 2.56
- (2) If  $d \geq p$  then  $f \in \mathfrak{m}^{[p]}$ . As  $p > 2$ , it follows that  $f^{p-2} \in \mathfrak{m}^{[p]}$ , so  $R$  is not quasi- $F$ -pure by corollary 2.51.1(a).

This leaves only case (1b). We note that this case is vacuous unless  $p \equiv 2 \pmod n$ ; thus we fix  $a \in \mathbb{N}$  such that  $n = \frac{p-2}{a}$  and  $\lceil \frac{p}{a+1} \rceil \leq d < \lceil \frac{p}{a} \rceil$ . We note that by lemma 2.57,  $R$  is not  $F$ -pure.

It is thus sufficient to check that  $R$  is quasi- $F$ -pure, for  $n, d, p$  as above, if  $d < n$  or  $d = n = 3$ . We prove this by contrapositive. If  $d > n$ ,  $R$  is of general type and is not quasi- $F$ -pure. If  $d = n > 3$ , as  $p \not\equiv 1 \pmod n$   $R$  is never quasi- $F$ -pure by [KTY22, Example 7.3].  $\square$

There is evidence to suggest that the converse of case (1b) is also true. If  $n = d = 3$ , then  $R$  is the affine cone of an elliptic curve. In this case  $p \equiv 2 \pmod 3 = n$ , so it is known that  $R$  is a supersingular elliptic curve and hence quasi- $F$ -pure of height 2. Thus, to complete the classification it is sufficient to affirmatively answer the following question:

QUESTION 2.59. Let  $f = x_1^d + \cdots + x_n^d \in A = k[x_1, \dots, x_n]$  where  $k$  is of characteristic  $p > 2$  and  $n \geq 3$ . Suppose that  $p \equiv 2 \pmod n$  and choose  $a \geq 1$  such that  $n = \frac{p-2}{a}$ . If  $\lceil \frac{p}{a+1} \rceil \leq d < n$ , is  $R = A/(f)$  quasi- $F$ -pure?

Via a standard computation, all Fermat type hypersurfaces as above have  $F$ -pure threshold  $1 - \frac{1}{p}$ . While the converse of Theorem 2.52 does not always hold, the following lemma suggests that the singularities in this case are well controlled.

LEMMA 2.60. Let  $f = x_1^d + \cdots + x_n^d \in A = k[x_1, \dots, x_n]$  where  $k$  is of characteristic  $p > 2$  and  $n \geq 3$ . Let  $R = A/(f)$ . Suppose that  $p \equiv 2 \pmod n$  and choose  $a \geq 1$  such that  $n = \frac{p-2}{a}$ . If  $d < n$ , then  $\text{fpt}(f) \geq 1 - \frac{1}{p}$ . If we further assume that  $d \geq \lceil \frac{p}{a+1} \rceil$ , then  $\text{fpt}(f) = 1 - \frac{1}{p}$ .

PROOF. We claim that for all  $e \gg 0$ ,

$$\nu_f(p^e) \geq p^e - 2p^{e-1} + 1$$

If this the claim is true, then  $\text{fpt}(f) \geq 1 - \frac{2}{p} + \frac{1}{p^e}$ . As  $p = an + 2 > n - 2$  for  $a \geq 1$ , via [HNnBWZ16], we can conclude that  $\text{fpt}(f) \geq 1 - \frac{h}{p}$  for some  $h \geq 0$ . From the above bound, it follows that  $\text{fpt}(f) \geq 1 - \frac{1}{p}$ . If  $d \geq \lceil \frac{p}{a+1} \rceil$  then  $R$  is not  $F$ -pure by Lemma 2.57; it follows that  $\text{fpt}(f) = 1 - \frac{1}{p}$  in this case as desired.

We now prove the claim. We've assumed that  $d < n$ , so it suffices to show this holds for the case  $d = n - 1 = \frac{p-2}{a} - 1$ , as for any lower  $d$  the  $F$ -pure threshold would only increase. It is thus sufficient to show that

$$\left( x_1^{\frac{p-2}{a}-1} + \cdots + x_{\frac{p-2}{a}}^{\frac{p-2}{a}-1} \right)^{p^e - 2p^{e-1} + 1} \notin \mathfrak{m}^{[p^e]}$$

Rewriting this polynomial as

$$\left( x_1^{\frac{p-2}{a}-1} + \cdots + x_{\frac{p-2}{a}}^{\frac{p-2}{a}-1} \right)^{p^{e-1}(p-2)} \left( x_1^{\frac{p-2}{a}-1} + \cdots + x_{\frac{p-2}{a}}^{\frac{p-2}{a}-1} \right)$$

We see this polynomial contains the monomials

$$M_j := c_j \cdot (x_1 \cdot x_2 \cdots x_n)^{\left(\frac{p-2}{a}-1\right)(p^{e-1})a} \cdot x_j^{\frac{p-2}{a}-1}$$

For  $0 \neq c_j \in k$  and each  $j = 1, \dots, n = \frac{p-2}{a}$ . For simplicity we look at  $M_1$ . Note that  $\deg_{x_1}(M_1) = \deg_{x_j}(M_1) + \frac{p-2}{a} - 1$  for each  $j > 1$ , so it is sufficient to show that  $\deg_{x_1}(M_1) < p^e$

to conclude that  $M_1 \notin \mathfrak{m}^{[p^e]}$ , which would prove the claim.

$$\begin{aligned} \deg_{x_1}(M_1) &= \left(\frac{p-2}{a} - 1\right)(p^{e-1})a + \frac{p-2}{a} - 1 \\ &= p^{e-1}(p-2-a) + \frac{p-2}{a} - 1 \\ &= p^e - (a+2)p^{e-1} + \frac{p-2}{a} - 1 \end{aligned}$$

For any  $e$  chosen sufficiently large (possibly dependent on  $a$ ) and fixed  $a \geq 1$ , this is less than  $p^e$ .  $\square$

The first example of such an  $f$  is of the form  $(n, d, p) = (5, 4, 7)$ . A Julia computation via the methods of [BGP25] shows that this is quasi- $F$ -pure:

EXAMPLE 2.61. *The fermat quartic threefold over  $\mathbb{F}_7$  has quasi- $F$ -pure height 2.*

The next most simple examples exist in characteristic 11, specifically

$$(n, d, p) = (9, 6, 11), (9, 7, 11), (9, 8, 11)$$

but we are unable to compute the quasi- $F$ -pure height of these examples due to limits on computing capacity. Example 2.61 is interesting in that it is an “unlikely intersection” of the equations defining the non- $F$ -split locus. We now explain this more precisely.

We conjecture the type of each of the unknown cases for  $n = 5$ ,  $d = 4$ . To do this, we recall a few facts about the moduli of hypersurfaces. One may compute a presentation for the moduli of projective hypersurfaces using a general recipe.

DEFINITION 2.62. A *weak integer composition* of length  $n$  and degree  $d$  is a  $n$ -tuple  $(a_1, \dots, a_n)$  of nonnegative integers such that  $a_1 + \dots + a_n = d$ . We denote the set of weak integer compositions of length  $n$  and degree  $d$  by  $\text{wics}(n, d)$

The set  $\text{wics}(n, d)$  is in bijection with the set of homogeneous monomials of degree  $d$  in  $n$  variables, by raising the  $i$ -th variable to the  $a_i$ -th power. To construct the moduli of

hypersurfaces in  $\mathbb{P}^{n-1}$  and degree  $d$ , we may consider the projective space parameterizing the coefficients of a generic homogeneous polynomial of degree  $d$  in  $n$  variables, i.e.  $\mathbb{P}_k^{\#\text{wics}(n,d)}$ . Then, we remove the locus cut out by the Jacobian ideal to get a space parameterizing smooth hypersurfaces. Finally, we quotient by the group  $\text{PGL}_n$  of changes of variables. As  $\text{PGL}_n$  has dimension  $n^2 - 1$  and  $\#\text{wics}(n,d) = \binom{d+n-1}{n-1}$ , we see that the dimension of this moduli space is  $\binom{d+n-1}{n-1} - n^2 + 1$ .

We can also determine the non- $F$ -split locus: if one takes general polynomial (in  $\binom{d+n-1}{n-1}$  variables), raises it to the  $p - 1$ -th power, then by Fedder's criterion the non- $F$ -split locus is given by the coefficients of the monomials which do not have a  $p$ -th power of any variable.

**DEFINITION 2.63.** Let  $\text{wics}(n,d)_{<k}$  denote the subset of  $\text{wics}(n,d)$  for which all of the parts  $a_1, \dots, a_n$  in the weak integer composition are strictly less than  $k$ .

The previous discussion gives

**PROPOSITION 2.64.** *The non- $F$ -split locus is cut out by  $\#\text{wics}(n,d(p-1))_{<p}$  equations.*

The author and Garzella do not know of a combinatorial expression for the quantity  $\#\text{wics}(n,d(p-1))_{<p}$ , but it is easily computable for small inputs. In the case of a quartic threefold ( $n = 5, d = 4$ ), we have that the dimension of the moduli space is 46. In this case, if  $11 \leq p$ , then by [HNnBWZ16, Theorem 3.12] all quartic threefolds are  $F$ -split. For  $p = 2, 3, 5$  and  $7$ , the number of equations that cut out the non- $F$ -split locus is 5, 15, 70, and 210 respectively.

Notice that for  $p = 5, 7$ , the number of equations cutting out the non- $F$ -split locus is larger than the dimension of the moduli space. Thus, one may expect that all such hypersurfaces are  $F$ -split. However, this turns out to be too much to hope for: Example 2.61 is an “unlikely intersection” of the equations which define the non- $F$ -split locus. This contrasts the  $p = 2$  or  $3$  case where one knows that non- $F$ -split surfaces exist for moduli reasons (i.e.

the non- $F$ -split locus is not cut out by enough equations to vanish completely).

Intuitively, it's possible for such a hypersurface in characteristic 7 to be non- $F$ -split because each monomial is very “tall” (i.e. each monomial has a single variable of a high power) rather than “wide” (i.e. a monomial having low powers of many variables) However, the Fedder-type criterion for quasi- $F$ -purity “mixes up” the variables. Motivated by this intuition we suggest the following question:

QUESTION 2.65. Let  $X$  be a projective hypersurface of degree  $d$  in  $\mathbb{P}^{n-1}$  over  $\mathbb{F}_p$ . Assume that

$$\binom{d+n-1}{n-1} - n^2 + 1 < \# \text{wics}(n, d(p-1))_{<p}$$

Then is  $X$  always quasi- $F$ -split?

If true, it would imply that any quartic threefold over  $\mathbb{F}_5$  or  $\mathbb{F}_7$  is quasi- $F$ -split. The author and Garzella do not know of a single example of a quartic threefold in characteristic 5 which is not  $F$ -split. In contrast, by corollary 2.51.1, the Fermat quartic threefold over  $\mathbb{F}_3$  is not quasi- $F$ -split.

## CHAPTER 3

**Quasi- $F$ -Regularity and Quasi- $F$ -Rationality**

In this chapter we will extend the theory of quasi- $F$ -splittings to the more refined invariant of quasi- $F^e$ -splitting and to log pairs similarly to the extension of  $F$ -singularities to log pairs discussed in chapter 1 section 1.8. Using these newer invariants (to be introduced in section 1) we will define quasi- $F$ -regularity and quasi- $F$ -rationality, weaker conditions than strong  $F$ -regularity and  $F$ -rationality. These newer invariants, much like quasi- $F$ -splittings were to  $F$ -splittings, recover key properties of their classical counterparts while being satisfied by a much broader class of varieties.

In these sections we will also introduce the non- $F$ -finite variants of these singularity types. In section 2 we will define quasi- $F^e$ -purity, a local analogue of quasi- $F^e$ -splittings that are extendable to the  $F$ -finite setting, and show that it is stable under completion and finite étale extension. In section 3 we will introduce quasi- $F$ -pure regularity, an analogous generalization of quasi- $F$ -regularity as  $F$ -pure regularity was to strong  $F$ -regularity (as in chapter 1 section 2). We will also define quasi- $F$ -rationality in the non- $F$ -finite setting and construct the quasi test ideal and the quasi-parameter test submodule, both used to test for quasi- $F$ -regularity and quasi- $F$ -rationality respectively, and show they can still be defined and record the same data outside of the  $F$ -finite setting.

In section 4 we will review applications of quasi- $F$ -singularities to birational geometry as in [KTT<sup>+</sup>25, KTT<sup>+</sup>24a, KTT<sup>+</sup>24b, TWY24]. Finally in section 5 we will make progress towards applying the  $\Gamma$ -construction (first discussed in chapter 1 section 2.3) to quasi- $F$ -singularities and highlight possible approaches to resolving the conjecture on the  $\Gamma$ -construction posed by the author in [Jag26, section 4].

## 1. Iterated Quasi- $F$ -Splittings and Witt Divisorial Sheaves

Before defining quasi- $F$ -splitting (and quasi- $F^e$ -splitting) for pairs, we first review the Witt divisorial sheaf construction of [Tan22].

**1.1. Introduction to Witt Divisorial Sheaves and Modules.** For a given  $\mathbb{R}$ -divisor  $D$  on  $X$ , we would like to construct an analogous object to  $\mathcal{O}_X(D)$  on the Witt sheaf  $W_n \mathcal{O}_X$ . This has been done in [Tan22]; we will review this construction below, both in the general and the  $X = \text{Spec}(R)$  affine setting. Let  $X$  be an integral normal noetherian  $\mathbb{F}_p$  scheme and  $D$  a  $\mathbb{Q}$ -divisor on  $X$ . We define the  **$n$ th-Witt-Divisorial Sheaf associated to  $D$**  to be the following subsheaf of  $W_n(K(X))$ , defined on any open set  $U \subset X$ :

$$W_n \mathcal{O}_X(D)(U) := \left\{ (s_0, \dots, s_{n-1}) \in W_n(K(X)) \mid (\text{div}(s_i) + p^i D)|_U \geq 0 \ \forall i \leq n-1 \right\}$$

For  $n = 1$ ,  $W_n \mathcal{O}_X(D)(U) = \mathcal{O}_X(D)(U)$ , i.e. this recovers the standard divisorial sheaf construction.

LEMMA 3.1. [Tan22] *For  $X$  and  $D$  as above,  $W_n \mathcal{O}_X(D)$  is a  $W_n \mathcal{O}_X$ -submodule of  $W_n K(X)$ , and in fact, is a quasi-coherent  $W_n \mathcal{O}_X$ -Module. If  $X$  is in addition  $F$ -finite, then  $W_n \mathcal{O}_X(D)$  is a coherent  $W_n \mathcal{O}_X$ -module.*

Similar to  $\mathcal{O}_X(D)$ ,  $W_n \mathcal{O}_X(D)$  is well defined up to  $\mathbb{Z}$ -linear equivalence.

LEMMA 3.2. [Tan22] *For  $X$  and  $D_1, D_2$  as above, if  $D_1 = D_2 + \text{div}(\varphi)$  for  $\varphi \in K(X)^\times$  (i.e.  $D_1 \sim_{\mathbb{Z}} D_2$ ), then  $W_n \mathcal{O}_X(D_1) \cong W_n \mathcal{O}_X(D_2)$  as  $W_n \mathcal{O}_X$ -Modules.*

To paint the local picture, we briefly introduce the notion of generalized divisors over a normal noetherian domain  $R$ . For a ring  $R$  We say  $I$  is a **fractional ideal** if it is a finitely generated  $R$ -submodule of  $K$ , the total ring of fractions of  $R$ . If  $I \subset R$ , we say it is **effective**. When  $X = \text{Spec}(R)$ , divisors on  $X$ , or codimension 1 subschemes of  $\text{Spec}(R)$  without embedded components, correspond precisely to effective fractional ideals  $I \subset R$  of pure height 1. To any fractional ideal  $I$  of pure height 1, we associate a **generalized divisor**

$D$ . When  $I$  is effective, we say that  $D \geq 0$ .

For an element  $f \in R$ , let  $\text{div}(f)$  denote the divisor associated to the principal ideal  $(f)$ . Let  $R(D) := \text{Hom}_R(I, R)$  be the divisorial module of  $R$  associated to  $D$ ; it can easily be checked that  $R(\text{div}(f)) = R[1/f]$ , for instance. Let  $\text{Div}(R)$  denote the free abelian group generated by the generalized divisors of  $R^1$ . We define the groups  $\text{Div}(R) \otimes_{\mathbb{Z}} \mathbb{Q}$  and  $\text{Div}(R) \otimes_{\mathbb{Z}} \mathbb{R}$  to be the groups of  $\mathbb{Q}$ -divisors and  $\mathbb{R}$ -divisors, respectively. We say that  $D_1 \sim D_2$  if  $D_1 = D_2 + \text{div}(f)$  for some  $f \in R$ ; using this divisorial equivalence we define  $\text{Cl}(R)$  the **class group** of  $R$  to be the group  $\text{Div}(R)/\sim$ . Further, this law of addition motivates an equivalent definition for divisorial modules:

$$R(D) := \{a \in K \mid \text{div}(a) + D \geq 0\}$$

Agreeing with the local description of  $\mathcal{O}_X(D)$ . Indeed, we can define the  $W_n(R)$ -module  $W_n(R)(D)$  identically to its global counterpart. Given this description, and the fact the the arguments were all made locally, analogous results to lemmas 3.1 and 3.2 apply to  $W_n(R)(D)$  as well. Via this description it is easy to see that the (Witt) divisorial module at the stalk is the same as the corresponding stalk of the (Witt) divisorial sheaf; i.e. for any  $x \in X, n \in \mathbb{N}$  and divisor  $D$ ,

$$W_n \mathcal{O}_X(D)_x = W_n(\mathcal{O}_{X,x})(D|_{\{x\}})$$

When  $R$  is a normal domain, any ideal of pure height 1 is a torsion free reflexive module of rank 1, and further, there is a bijection between elements of  $\text{Cl}(R)$  and isomorphism classes of reflexive rank 1 modules. When we further assume  $R$  admits a canonical module  $\omega_R$ , which is itself a reflexive rank 1 module, we can let  $K_R \in \text{Cl}(R)$  be the the divisor corresponding to  $\omega_R$  under the previously stated bijection; this would in particular imply that  $R(-K_R) = \omega_R$ . If  $nK_R = 0$  for some  $n$ ,  $R$  is  $\mathbb{Q}$ -Gorenstein, and if  $K_R = 0$   $R$  is quasi-Gorenstein (this is of

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<sup>1</sup>Despite this formal definition, we can explicitly write down  $D_1 + D_2$ . Indeed it can be constructed via taking the product of the representative ideals  $I_1 I_2$  and then taking the intersection over the primary components of the product that correspond to height 1 primes.

course equivalent to the condition that  $R \cong \omega_R$ ).

Finally, we remark that for a  $\mathbb{Q}$  or  $\mathbb{R}$ -divisor  $D$  on  $R$  (on  $X$ ), the floor  $\lfloor D \rfloor \in \text{Div}(R)$  ( $\in \text{Div}(X)$ ) is defined as one would expect. While it is true that  $R(D) \cong R(\lfloor D \rfloor)$  and  $\mathcal{O}_X(D) \cong \mathcal{O}_X(\lfloor D \rfloor)$ , we caution that  $W_n(R)(D) \not\cong W_n(R)(\lfloor D \rfloor)$  and  $W_n \mathcal{O}_X(D) \not\cong W_n \mathcal{O}_X(\lfloor D \rfloor)$  in general.

LEMMA 3.3. *Suppose  $R \rightarrow S$  is a flat extension of noetherian rings. Let  $\Delta$  be a  $\mathbb{Q}$ -divisor on  $R$  and  $\Delta_S$  a  $\mathbb{Q}$ -divisor on  $S$  such that  $\Delta_S \sim S \otimes_R \Delta$ . Then  $S(\Delta_S) \cong S \otimes_R R(\Delta)$ .*

PROOF.

$$S \otimes_R R(\Delta) \cong S \otimes_R \text{Hom}_R(\Delta, R) \cong \text{Hom}_S(\Delta \otimes_R S, S) = S(S \otimes_R \Delta) = S(\Delta_S)$$

This utilizes that Hom commutes with flat base change, given that  $S$  is a flat  $R$ -algebra and  $\Delta$  is finitely presented.  $\square$

If  $R$  admits a dualizing module  $\omega_R$ , given that

$$\widehat{R(K_R)} \cong \widehat{\omega_R} \cong \omega_{\widehat{R}} \cong \widehat{R}(K_{\widehat{R}})$$

The natural map  $R(K_R) \rightarrow \widehat{R}(K_{\widehat{R}})$  as in the lemma above is simply completing as an  $R$ -Module.

REMARK 3.4. *Let  $(R, \Delta)$  be a divisor pair.  $R(\Delta)$  has a natural  $W_n(R)$ -Module structure under the action of  $W_n(R) \xrightarrow{\mathcal{R}} R \rightarrow R(\Delta)$ .*

Witt divisorial sheaves over local rings behave well with respect to completion.

LEMMA 3.5. *Suppose  $(R, \mathfrak{m})$  is a local ring. Let  $(R, \Delta)$  be a  $\mathbb{Q}$ -divisor pair and  $\Delta^\wedge \in \text{Cl}(\widehat{R})$  the  $\mathbb{Q}$ -divisor which is the image of  $\Delta$  under the map  $\text{Cl}(R) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \text{Cl}(\widehat{R}) \otimes_{\mathbb{Z}} \mathbb{Q}$ . Then,*

$$W_n(\widehat{R})(\Delta^\wedge) \cong W_n(R)(\Delta)^\wedge$$

As  $W_n(\widehat{R})$ -Modules, where the right hand side denotes the completion of  $W_n(R)(\Delta)$  as a  $W_n(R)$ -Module.

PROOF. Recall that the unique maximal ideal of  $W_n(R)$  is of the form

$$W_n \mathfrak{m} = \{(a_1, \dots, a_n) \in W_n(R) \mid a_1 \in \mathfrak{m}\}$$

Now Consider the ideal

$$W_n(\mathfrak{m}^k) = \ker(W_n(R) \rightarrow W_n(R/\mathfrak{m}^k)) = \{(a_0, \dots, a_{n-1}) \in W_n(R) \mid a_i \in \mathfrak{m}^k \forall i\}$$

One easily sees that

$$W_n(\widehat{R})(\Delta^\wedge) \cong \varprojlim \frac{W_n(R)(\Delta)}{W_n(\mathfrak{m}^k)W_n(R)(\Delta)}$$

It is therefore sufficient to show that the following two directed systems of  $W_n(R)$ -modules are cofinal:

$$\{W_n(\mathfrak{m}^k)W_n(R)(\Delta)\}_k \quad , \quad \{(W_n \mathfrak{m})^k W_n(R)(\Delta)\}_k$$

To show this, we prove that  $\{W_n(\mathfrak{m}^k)\}_k$  and  $\{(W_n \mathfrak{m})^k\}_k$  are cofinal; this was shown in the proof given for lemma 1.103 showing that  $W_n(\widehat{R}) \cong \widehat{W_n(R)}$ ,  $\square$

**1.2. A First Attempt at Quasi- $F^e$ -Splittings for Pairs.** We recall that for any noetherian ring  $R$ ,  $F : R \rightarrow F_* R$  (assigning  $1 \mapsto F_* 1$ ) splits (resp. is pure) if and only if  $F^e : R \rightarrow F_*^e R$  splits (resp. is pure) for any  $e > 0$ , or equivalently, for all  $e > 0$ . For maps  $F : R \rightarrow F_* R$  (assigning  $1 \mapsto F_* c$ ), for  $c$  chosen outside of a minimal prime, it is *a priori* easier for an iterate  $F^e : R \rightarrow F_*^e R$  assigning  $1 \mapsto F_*^e c$  to split than  $F : R \rightarrow F_* R$  splitting.

For quasi- $F$ -splittings, one would initially hope for a similar trend to hold. Indeed, in the pushout defining  $Q_{X,n}$ , one replace the top copy of  $F : W_n \mathcal{O}_X \rightarrow F_* W_n \mathcal{O}_X$  with an iterate  $F^e : W_n \mathcal{O}_X \rightarrow F_*^e W_n \mathcal{O}_X$  to define the  $W_n \mathcal{O}_X$ -module  $Q_{X,n}^e$  below:

$$\begin{array}{ccc} W_n \mathcal{O}_X & \xrightarrow{F^e} & F_*^e W_n \mathcal{O}_X \\ \mathfrak{R}^{n-1} \downarrow & & \downarrow \\ \mathcal{O}_X & \dashrightarrow^{\Phi_{X,n}^e} & Q_{X,n}^e \end{array}$$

Using Witt divisorial sheaves defined in the previous subsection, one can also extend this diagram to one for pairs and define  $Q_{X,\Delta,n}^e$  as the pushout of the following diagram:

$$\begin{array}{ccc} W_n \mathcal{O}_X(\Delta) & \xrightarrow{F^e} & F_*^e W_n \mathcal{O}_X(p^e \Delta) \\ \mathfrak{A}^{n-1} \downarrow & & \downarrow \\ \mathcal{O}_X(\Delta) & \xrightarrow{\Phi_{X,\Delta,n}^e} & Q_{X,\Delta,n}^e \end{array}$$

For simplicity, when  $e = 1$  we omit the superscript and refer to these newly defined objects and morphisms as  $Q_{X,n}, Q_{X,\Delta,n}, \Phi_{X,n}, \Phi_{X,\Delta,n}$  in order to align with previous notation introduced in chapter 2.

**REMARK 3.6.** *While  $Q_{X,n}$  and  $Q_{X,\Delta,n}$  are  $\mathcal{O}_X$ -modules and  $\Phi_{X,n}$  and  $\Phi_{X,\Delta,n}$  are morphisms of  $\mathcal{O}_X$ -modules, for  $e > 1$  this is not the case. We can only view  $Q_{X,n}^e$  and  $Q_{X,\Delta,n}^e$  as  $W_n \mathcal{O}_X$ -modules and  $\Phi_{X,n}^e$  and  $\Phi_{X,\Delta,n}^e$  as morphisms of  $W_n \mathcal{O}_X$ -modules.  $Q_{X,n}^e$  and  $Q_{X,\Delta,n}^e$ , however, are still coherent  $W_n \mathcal{O}_X$ -modules when  $X$  is  $F$ -finite [TWY24, proposition 3.12].*

For any Cartier divisor  $D$ , one checks that  $Q_{X,\Delta,n}^e \otimes_{W_n \mathcal{O}_X} W_n \mathcal{O}_X(D)$  satisfies the universal property of pushouts where  $\Delta$  as above is replaced with  $\Delta + D$ ; this yields the isomorphism

$$Q_{X,\Delta,n}^e \otimes_{W_n \mathcal{O}_X} W_n \mathcal{O}_X(D) \cong Q_{X,\Delta+D,n}^e$$

It can be read from this diagram that if  $X$  is globally  $F$ -split (resp.  $(X, \Delta)$  is naively  $F$ -split) then  $\Phi_{X,n}^e$  (resp.  $\Phi_{X,\Delta,n}^e$ ) splits as a  $W_n \mathcal{O}_X$ -module homomorphism. This provides us with a candidate definition;  $X$  (resp.  $(X, \Delta)$ ) could be  $n$ -quasi- $F^e$ -split if  $\Phi_{X,n}^e$  (resp.  $\Phi_{X,\Delta,n}^e$ ) splits for some, or all,  $e$ . Unfortunately, this definition is not tractible.

**LEMMA 3.7** ([TWY24], Proposition 3.8). *Let  $X$  be an integral normal scheme over  $\mathbb{F}_p$ . For  $e \geq 2$  and  $n \geq 1$ , the following are equivalent:*

- (1)  $X$  is globally  $F$ -split.
- (2)  $\Phi_{X,n}^e$  splits as a morphism of  $W_n \mathcal{O}_X$ -modules.

PROOF. (1) implies (2), so it sufficient to show that (2) implies (1). Let  $\theta : Q_{X,n}^e \rightarrow \mathcal{O}_X$  be the  $W_n \mathcal{O}_X$ -module splitting of  $\Phi_{X,n}^e$ . As  $p \mathcal{O}_X = 0$ ,  $\theta$  factors through

$$\frac{Q_{X,n}^e}{pQ_{X,n}^e} = \frac{F_*^e W_n \mathcal{O}_X}{pF_*^e W_n \mathcal{O}_X} \cong F_*^e \frac{W_n \mathcal{O}_X}{pW_n \mathcal{O}_X}$$

yielding the sequence of maps  $\mathcal{O}_X \xrightarrow{\overline{\Phi_{X,n}^e}} F_*^e \frac{W_n \mathcal{O}_X}{pW_n \mathcal{O}_X} \xrightarrow{\bar{\theta}} \mathcal{O}_X$  composing to the identity. If we show this factors through  $\mathcal{O}_X \xrightarrow{F} F_* \mathcal{O}_X$ , restricting down will yield the desired Frobenius splitting. The existence of this can be seen from the following commutative diagram:

$$\begin{array}{ccccc} W_n \mathcal{O}_X & \xrightarrow{F} & F_* W_n \mathcal{O}_X & \xrightarrow{F^{e-1}} & F_*^e W_n \mathcal{O}_X \\ \mathfrak{R}^{n-1} \downarrow & & \downarrow F_* \mathfrak{R}^{n-1} & & \downarrow \\ \mathcal{O}_X & \longrightarrow & F_* \mathcal{O}_X & \longrightarrow & F_*^e \frac{W_n \mathcal{O}_X}{pW_n \mathcal{O}_X} \\ & \searrow & & \nearrow & \\ & & \overline{\Phi_{X,n}^e} & & \end{array}$$

The map  $F_* \mathcal{O}_X \rightarrow F_*^e \frac{W_n \mathcal{O}_X}{pW_n \mathcal{O}_X}$  is the only map here that need not necessarily exist; it follows from the fact that  $e \geq 2$  and that  $\ker(F^{e-1} F_* \mathfrak{R}^{n-1}) \subseteq \ker\left(F_*^e W_n \mathcal{O}_X \rightarrow F_*^e \frac{W_n \mathcal{O}_X}{pW_n \mathcal{O}_X}\right)$ .  $\square$

To develop a better definition, we will develop a Grothendieck-dual analogue definition in the style of theorem 1.26 and proposition 2.8. To do this, we must develop a ‘‘Witt’’ version of the dualizing complex.

**1.3. Witt Dualizing Sheaves.** Let  $X$  be an  $F$ -finite<sup>2</sup> Noetherian reduced scheme of characteristic  $p > 0$ . We also assume that the dualizing module  $\omega_X$ , which exists by the  $F$ -finite hypothesis, is supported everywhere (i.e.  $\text{Supp}(\omega_X) = X$ ). This is not a strong condition; it is guaranteed by equidimensionality for varieties or local rings, for instance [KTT<sup>+</sup>24b, remark 2.21]. In this setting, we define the  $W_n \mathcal{O}_X$ -complex  $W_n \omega_X^\bullet$ , the dualizing complex associated to the scheme  $W_n X := (X, W_n \mathcal{O}_X)$ , to be the **nth Witt dualizing complex** on  $X$ .

<sup>2</sup>This is an essential hypothesis; otherwise the objects we are constructing definitively won’t exist via lemma 2.31. We will discuss how to overcome this hurdle in section 2.

If  $R$  is an  $F$ -finite Noetherian ring of characteristic  $p$  and  $X$  is a separated scheme of finite type over  $R$ , then  $W_n\omega_X^\bullet$  exists for each  $n$  alongside a class of finite morphisms

$$\left\{ i_{X,m,n}^e W_m X \xrightarrow{F^e} W_m X \hookrightarrow W_n X \mid n \geq m \geq 1, e \geq 0 \right\}$$

where the inclusion  $W_m X \hookrightarrow W_n X$  is determined by the restriction morphism on sheaves  $W_n \mathcal{O}_X \xrightarrow{\mathfrak{R}^{n-m}} W_m \mathcal{O}_X$ . These maps determine isomorphisms in  $D^b(W_n \mathcal{O}_X)$ :

$$\left\{ \rho_{X,m,n}^e : W_m \omega_X^\bullet \xrightarrow{\cong} (i_{X,m,n}^e)^! W_n \omega_X^\bullet \mid n \geq m \geq 1, e \geq 0 \right\}$$

With compatibilities  $(i_{X,\ell,n}^{e'})^! \rho_{X,m,n}^e \circ \rho_{X,\ell,m}^{e'} = \rho_{X,\ell,n}^{e+e'}$  for  $e, e' \geq 0$  and  $n \geq m \geq \ell \geq 1$ . Existence results can be found in [KTT<sup>+</sup>24b, section 9] alongside more details about their construction; the author also recommend [TWY24] for additional exposition. For  $X = \text{Spec}(R)$ , we denote the Witt dualizing complex by  $W_n \omega_R^\bullet$  for simplicity; it can be checked [KTT<sup>+</sup>24b, remark 2.23] that  $W_n \omega_R^\bullet$  is compatible with localization and completion in the usual ways, under the hypotheses above. We will also let  $W_n \omega_X := H^{-d}(W_n \omega_X^\bullet)$  and  $W_n \omega_R := H^{-d}(W_n \omega_R^\bullet)$  denote the dualizing modules in the global and affine setting, respectively.

Applying  $\mathcal{H}om_{W_n \mathcal{O}_X}(-, W_n \omega_X)$  to  $V : F_* W_{n-1} \mathcal{O}_X \rightarrow W_n \mathcal{O}_X$  and  $R : W_n \mathcal{O}_X \rightarrow W_{n-1} \mathcal{O}_X$  yield dual maps

$$V^* : W_n \omega_X \rightarrow F_* W_{n-1} \omega_X \quad \mathfrak{R}^* : W_{n-1} \omega_X \rightarrow W_n \omega_X$$

And applying  $\mathcal{H}om_{W_n \mathcal{O}_X}(-, W_n \omega_X)$  to the Frobenius  $F^e : W_n \mathcal{O}_X \rightarrow F_*^e W_n \mathcal{O}_X$  yields a Witt variant of the Frobenius Trace:

$$\text{Tr}_n^e : F_*^e W_n \omega_X \rightarrow W_n \omega_X$$

By applying  $\mathcal{H}om_{W_n \mathcal{O}_X}(-, W_n \omega_X)$  to the short exact sequence

$$0 \rightarrow F_* W_{n-1} \mathcal{O}_X \xrightarrow{V} W_n \mathcal{O}_X \xrightarrow{\mathfrak{R}^{n-1}} \mathcal{O}_X \rightarrow 0$$

one can check that  $V^*$  is surjective when  $X$  is Cohen-Macaulay and  $\mathrm{Tr}_n^e$  is surjective when  $X$  is regular, using Frobenius iterates in the above as necessary. We conclude this subsection by introducing divisor twists by a  $\mathbb{Q}$ -divisor  $D$  (on  $X$ ) of Witt dualizing sheaves:

$$W_n\omega_X(D) := \mathcal{H}om_{W_n\mathcal{O}_X}(W_n\mathcal{O}_X(-D), W_n\omega_X)$$

The "negative" arises from contravariance of  $\mathcal{H}om_{W_n\mathcal{O}_X}(-, W_n\omega_X)$ , and implies that for  $\mathbb{Q}$ -divisors  $D_1 \leq D_2$  one has a natural inclusion  $W_n\omega_X(D_1) \hookrightarrow W_n\omega_X(D_2)$ . One can define log variants of  $V^*$ ,  $\mathfrak{A}^*$  and the Frobenius trace, though we omit the descriptions of these.

REMARK 3.8. *When  $R$  is an  $F$ -finite regular integral domain and  $\ell = \log_p([\mathrm{Frac}(R) : \mathrm{Frac}(R)^p])$ ,  $W_n\omega_R$  can be identified with  $W_n\Omega_R^\ell$ , or the top non-zero term of the De-Rham Witt complex [KTT<sup>+</sup>24b, Proposition 9.10] (cf. chapter 2 section 1.4).*

We now define the following (contravariant) functor:

$$(-)^* := \mathcal{H}om_{W_n\mathcal{O}_X}(-, W_n\omega_X(-K_X))$$

Chasing through the definition, we see that

$$\begin{aligned} \mathcal{O}_X^* &= \mathcal{H}om_{W_n\mathcal{O}_X}(\mathcal{O}_X, W_n\omega_X(-K_X)) \\ &= \mathcal{H}om_{W_n\mathcal{O}_X}(\mathcal{O}_X, \mathcal{H}om_{W_n\mathcal{O}_X}(W_n\mathcal{O}_X(K_X), W_n\omega_X)) \\ &= \mathcal{H}om_{W_n\mathcal{O}_X}(\mathcal{O}_X \otimes_{W_n\mathcal{O}_X} W_n\mathcal{O}_X(K_X), W_n\omega_X) \\ &= \mathcal{H}om_{W_n\mathcal{O}_X}(\mathcal{O}_X(K_X), W_n\omega_X) \\ &= \mathcal{H}om_{W_n\mathcal{O}_X}(\omega_X, W_n\omega_X) \\ &= \mathcal{O}_X \end{aligned}$$

So  $\mathcal{O}_X$  is self dual. Via a similar argument and applying Grothendieck duality, one sees that  $\mathcal{O}_X(\Delta)^* = \mathcal{O}_X(-[\Delta])$  for any  $\mathbb{Q}$ -divisor  $\Delta$ . Further for any coherent sheaf  $\mathcal{F}$ :

$$\begin{aligned} \mathcal{F}^{**} &= \mathcal{H}om_{W_n \mathcal{O}_X}(\mathcal{F}^*, W_n \omega_X(-K_X)) \\ &= \mathcal{H}om_{W_n \mathcal{O}_X}(\mathcal{H}om_{W_n \mathcal{O}_X}(\mathcal{F}, W_n \omega_X(-K_X)), W_n \omega_X(-K_X)) \\ &= \mathcal{H}om_{W_n \mathcal{O}_X}(\mathcal{H}om_{W_n \mathcal{O}_X}(\mathcal{F}, W_n \omega_X), W_n \omega_X) \end{aligned}$$

We note that if  $X$  is Calabi-Yau then  $\omega_X = \mathcal{O}_X$ . Hence if  $\mathcal{F}$  is reflexive coherent sheaf on a Calabi-Yau variety,  $\mathcal{F}^{**} = \mathcal{F}$ , though this operator need not satisfy this in general. We do however have the following:

$$\mathcal{H}om_{W_n \mathcal{O}_X}(-, W_n \omega_X(-K_X)) = \mathcal{H}om_{W_n \mathcal{O}_X}(-, \mathcal{H}om_{W_n \mathcal{O}_X}(W_n \omega_X, W_n \omega_X))$$

Where  $\mathcal{H}om_{W_n \mathcal{O}_X}(W_n \omega_X, W_n \omega_X)$  can be identified with the (S2)-ification of  $W_n \mathcal{O}_X$ .

**1.4. Definition of Quasi- $F^e$ -Splitting.** Utilizing this dualizing operator and the map  $\Phi_{X,\Delta,n}^e : \mathcal{O}_X(\Delta) \rightarrow Q_{X,\Delta,n}^e$  on a  $\mathbb{Q}$ -divisor pair  $(X, \Delta)$  introduced in the prior section, one sees the equivalence

$$\Phi_{X,\Delta,n} : \mathcal{O}_X(\Delta) \rightarrow Q_{X,\Delta,n} \text{ splits} \iff H^0(X, (\Phi_{X,\Delta,n})^*) \text{ is surjective}$$

when  $[\Delta] = 0$ . This can be seen from the fact the fact that these are actually morphisms of  $\mathcal{O}_X$ -modules. A map of  $\mathcal{O}_X$ -modules  $\mathcal{O}_X \rightarrow \mathcal{F}$  splits if and only if the evaluation map on sections  $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \rightarrow H^0(X, \mathcal{O}_X)$  is surjective; the preimage of 1 will determine the section.  $H^0(X, (\Phi_{X,\Delta,n})^*)$  is simply the morphism

$$\text{Hom}_{W_n \mathcal{O}_X}(Q_{X,\Delta,n}, W_n \omega_X(-K_X)) \rightarrow \text{Hom}_{W_n \mathcal{O}_X}(\mathcal{O}_X(\Delta), W_n \omega_X(-K_X))$$

Which after utilizing the  $\mathcal{O}_X$ -module structure of the underlying morphism  $\Phi_{X,\Delta,n}$  and  $[\Delta] = 0$  yields the map

$$\text{Hom}_{\mathcal{O}_X}(Q_{X,\Delta,n}, \mathcal{O}_X) \rightarrow H^0(X, \mathcal{O}_X)$$

However, when  $e > 1$  we have that

$$\Phi_{X,\Delta,n}^e : \mathcal{O}_X(\Delta) \rightarrow Q_{X,\Delta,n}^e \text{ splits} \Rightarrow H^0(X, (\Phi_{X,\Delta,n}^e)^*) \text{ is surjective}$$

But the converse does not hold. In particular, surjectivity on the dual yields a weaker notion than splitting on the nose; this is a good thing in view of lemma 3.7. We will find that the converse essentially never holds; if it does then  $X$  was  $F$ -split to begin with. This gives us the “correct” definition of iterative quasi- $F$ -splitting.

**DEFINITION 3.9.** Let  $X$  be an integral normal  $F$ -finite  $\mathbb{F}$ -scheme such that  $W_n X$  is an excellent scheme admitting a dualizing complex. Let  $\Delta$  be a  $\mathbb{Q}$ -divisor on  $X$  and  $e > 0$ . Then  $(X, \Delta)$  is **n-Quasi- $F^e$ -Split** if

$$H^0(X, (\Phi_{X,\Delta,n}^e)^*) : H^0(X, (Q_{X,\Delta,n}^e)^*) \rightarrow H^0(X, (\mathcal{O}_X(\Delta))^*)$$

is a surjective morphism of  $W_n \mathcal{O}_X$ -modules.  $(X, \Delta)$  is **Quasi- $F^e$ -split** if it is  $n$ -quasi- $F^e$ -split for some  $n > 0$ .

**DEFINITION 3.10.**  $(X, \Delta)$  as above is **locally n-Quasi- $F^e$ -Split** (resp. **locally Quasi- $F^e$ -Split**) if there exists an open cover  $X = \bigcup U_i$  for which  $(U_i, \Delta|_{U_i})$  is  $n$ -quasi- $F^e$ -split (resp. quasi- $F^e$ -split). In literature this is often referred to as being Quasi- $F^e$ -pure, which is equivalent in the  $F$ -finite setting, but given our focus on the non- $F$ -finite setting we refer to this condition precisely.

It can be readily seen that being  $n$ -quasi- $F^e$ -split implies that you are  $m$ -quasi- $F^e$ -split for any  $m \geq n$ , similarly to the  $e = 1$  case. This motivates the notion of the quasi- $F^e$ -split height as an invariant of  $X$ .

**DEFINITION 3.11.** Let  $\text{ht}_{\text{QFS}}^e(X, \Delta)$  denote the **quasi- $F^e$ -split height** of  $X$ , or the minimal  $n$  for which  $(X, \Delta)$  is  $n$ -quasi- $F^e$ -split. If  $X$  is not  $n$ -quasi- $F^e$ -split for any  $n$  (i.e. it is not quasi- $F^e$ -split) then  $\text{ht}_{\text{QFS}}^e(X, \Delta) = \infty$ .

As a consequence of the universal property of pushouts,  $\Phi_{X,\Delta,n}^{e+1}$  factors as  $\Phi_{X,\Delta,n}^{e+1} : \mathcal{O}_X \xrightarrow{\Phi_{X,\Delta,n}^e} Q_{X,\Delta,n}^e \rightarrow Q_{X,\Delta,n}^{e+1}$ . After applying  $(-)^*$ , it follows that  $\text{ht}_{\text{QFS}}^e(X, \Delta) \leq \text{ht}_{\text{QFS}}^{e+1}(X, \Delta)$ . Thus, being quasi- $F^e$ -split becomes a *stronger* condition as  $e$  grows; This contrasts the theory of  $F$ -splittings where splitting  $1 \mapsto F_*1$  is equivalent to the iterate  $1 \mapsto F_*^e 1$  splitting, and splitting  $1 \mapsto F_*^e c$  for an arbitrary  $c \in R^\circ$  becomes easier as  $e$  increases.

REMARK 3.12. *There exist many equivalent definitions of quasi- $F^e$ -splitting due to [TWY24]. Quasi- $F^e$ -splitting can be defined via a splitting criterion similar to Yobuko's original definition of quasi- $F^e$ -splitting [TWY24, Proposition 3.19] and can be defined via the theory of Cartier operators [TWY24, Section 3.4]. We won't discuss these alternatives in detail since they do not adequately generalize to the non- $F$ -finite setting; we will however use yet another alternate definition of quasi- $F^e$ -splitting [TWY24, Lemma 3.10] in section 2.*

**1.5. Quasi- $F^\infty$ -Splitting.** if  $(X, \Delta)$  is  $n$ -quasi- $F^e$ -split for all  $e$ , then we say that it is  $n$ -quasi- $F^\infty$ -split; however, different conditions arise for quasi- $F^\infty$ -splitting depending on if  $e$  can be chosen before  $n$ :

DEFINITION 3.13. Let  $X$  be an integral normal  $F$ -finite  $\mathbb{F}_p$ -scheme such that  $W_n X$  is an excellent scheme admitting a dualizing complex. Let  $\Delta$  be a  $\mathbb{Q}$ -divisor on  $X$ . Then,

- $(X, \Delta)$  is **n-Quasi- $F^\infty$ -Split** if for all  $e > 0$ ,  $(X, \Delta)$  is  $n$ -quasi- $F^e$ -split.
- $(X, \Delta)$  is **Quasi- $F^\infty$ -Split** if for all  $e > 0$ , there exists an  $n > 0$  for which  $(X, \Delta)$  is  $n$ -quasi- $F^e$ -split.
- $(X, \Delta)$  is **Uniformly Quasi- $F^\infty$ -Split** if there exists an  $n > 0$  for which  $(X, \Delta)$  is  $n$ -quasi- $F^e$ -split for all  $e > 0$ .
- $(X, \Delta)$  is **Locally n-Quasi- $F^\infty$ -Split** if there exists an open cover  $X = \bigcup U_i$  for which  $(U_i, \Delta|_{U_i})$  is  $n$ -quasi- $F^\infty$ -split.
- $(X, \Delta)$  is **Locally Quasi- $F^\infty$ -Split** if there exists an open cover  $X = \bigcup U_i$  for which  $(U_i, \Delta|_{U_i})$  is quasi- $F^\infty$ -split.
- $(X, \Delta)$  is **Locally Uniformly Quasi- $F^\infty$ -Split** if there exists an open cover  $X = \bigcup U_i$  for which  $(U_i, \Delta|_{U_i})$  is uniformly  $n$ -quasi- $F^\infty$ -split.

REMARK 3.14. *For  $F$ -finite normal Gorenstein domains, being quasi- $F$ -split, quasi- $F^e$ -split, and quasi- $F^\infty$ -split are equivalent, see [KTT<sup>+</sup>24b, corollary 8.6].*

We stress that uniform quasi- $F^\infty$ -splitting is a stronger condition than quasi- $F^\infty$ -splitting. An elliptic curve, for instance, is  $e + 1$ -quasi- $F^e$ -split for all  $e$ , meaning that it is quasi- $F^\infty$ -split but not uniform quasi- $F^\infty$ -split. This can be seen from a more general result of [TWY24]:

THEOREM 3.15 ([TWY24], Theorem 7.1). *Let  $k$  be an algebraically closed field of characteristic  $p > 0$  and  $X$  a smooth proper variety over  $k$  of dimension  $d$ . Assume that  $\omega_X \cong \mathcal{O}_X$  and that  $H^d(X, W \mathcal{O}_X)$  is a finitely generated free  $W(k)$ -module of rank  $h \geq 1$ . Then for any  $e > 0$ ,  $\text{ht}^e(X) = eh - e + 1$ .*

As  $\text{ht}(X) = \dim_{\text{Frac } W(k)} H^d(X, W \mathcal{O}_X) \otimes_{W(k)} \text{Frac } W(k)$  for any Calabi-Yau variety due to [Yob19], it follows that the quasi- $F^e$ -split height of any Calabi-Yau variety is  $eh(X) - e + 1$ , for  $h(X)$  the quasi- $F$ -split height. Thus any quasi- $F$ -split but not  $F$ -split abelian variety, for instance, has quasi- $F^e$ -split height  $e \cdot 2 - e + 1 = e + 1$  due to theorem 2.16.

## 2. Quasi- $F^e$ -Purity and (Uniform) Quasi- $F^\infty$ -Purity

Taking the Grothendieck dual to state the definition of quasi- $F^e$ -splitting requires the existence of a dualizing complex  $W_n \omega_X^\bullet$  of  $W_n X$ . This was included within the assumptions of [TWY24], however when  $X$  is not  $F$ -finite the stalks of  $W_n \mathcal{O}_X$  are never Noetherian (lemma 2.31). Thus,  $W_n \omega_X^\bullet$  cannot exist and the standard definition of Quasi- $F^e$ -splitting ill-formed in this new setting. One goal of this section is to reformulate the definition of quasi- $F^e$ -purity<sup>3</sup> to be compatible with schemes that may not be  $F$ -finite.

**2.1. Global Variant of Local Duality.** We have utilized Matlis duality for modules over a local ring  $(R, \mathfrak{m})$  throughout this thesis with exposition, as this formulation is standard commutative algebra knowledge. In what follows, we state a version of local duality for complexes of sheaves in  $D_{\text{Coh}}^b(X)$  for  $X$  an irreducible proper scheme over a local ring  $(R, \mathfrak{m})$ .

<sup>3</sup>For non- $F$ -finite schemes (even affine schemes), quasi- $F^e$ -splitting is not a good notion of singularity, see chapter 2 section 2. Thus, we focus on the local notion of quasi- $F^e$ -purity.

DEFINITION 3.16. Let  $X$  be a proper scheme over Noetherian local ring  $(R, \mathfrak{m})$  and  $\mathcal{F}$  a coherent sheaf on  $X$ . Then

$$H_{\mathfrak{m}}^i(\mathcal{F}) := H^i \mathrm{R}\Gamma_{\mathfrak{m}} \mathrm{R}\Gamma(X, \mathcal{F})$$

$\mathrm{R}\Gamma(X, \mathcal{F})$  is a complex of  $\mathcal{O}_X$ -modules and hence have a natural  $R$ -action;  $\mathrm{R}\Gamma_{\mathfrak{m}}$  is the local cohomology functor on  $R$ -modules, hence  $H_{\mathfrak{m}}^i(\mathcal{F})$  is an Artinian  $R$ -module. If  $X = \mathrm{Spec}(R)$  then this recovers standard local cohomology.

THEOREM 3.17 ([TWY24], Lemma 2.9). *Let  $(R, \mathfrak{m})$  be a noetherian local ring admitting a dualizing complex such that  $\mathrm{Spec}(R)$  is irreducible. Let  $X$  be an irreducible proper scheme of dimension  $d$  over  $\mathrm{Spec}(R)$  and  $\mathcal{F}$  a coherent sheaf on  $X$ . Then*

$$H_{\mathfrak{m}}^d(\mathcal{F})^{\vee} \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X)^{\wedge}$$

where  $\vee$  denotes Matlis duality over  $R$  and  $\wedge$  denotes  $\mathfrak{m}$ -adic completion. If  $X$  is Cohen Macaulay this correspondence extends to lower cohomologies:

$$H_{\mathfrak{m}}^i(\mathcal{F})^{\vee} \cong \mathrm{Ext}_{\mathcal{O}_X}^{d-i}(\mathcal{F}, \omega_X)^{\wedge}$$

PROOF. Applying Matlis Duality for complexes over  $R$  and Grothendieck duality, one gets the chain of isomorphisms

$$\begin{aligned} H_{\mathfrak{m}}^i(\mathcal{F})^{\vee} &= (H^i \mathrm{R}\Gamma_{\mathfrak{m}} \mathrm{R}\Gamma(X, \mathcal{F}))^{\vee} \\ &\cong (H^{-i} \mathrm{R}\mathrm{Hom}_R(\mathrm{R}\Gamma(X, \mathcal{F}), \omega_R^{\bullet}))^{\vee} \\ &\cong (H^{-i} \mathrm{R}\Gamma \circ \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X^{\bullet}))^{\vee} \\ &\cong (H^{-i} \mathrm{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X^{\bullet}))^{\wedge} \end{aligned}$$

If  $X$  is Cohen Macaulay then  $\omega_X^{\bullet}$  is concentrated in a single degree and the result follows. If  $X$  is not Cohen Macaulay but  $i = d$ , there are no higher cohomologies so we can conclude the basic statement.  $\square$

**2.2. Definition of Quasi- $F^e$ -Purity.** To define Quasi- $F^e$ -Purity, we provide a very helpful alternate criterion for quasi- $F^e$ -splitting introduced in [TWY24]:

PROPOSITION 3.18 ([TWY24], Lemma 3.10). *Let  $(R, \mathfrak{m})$  be a noetherian local  $F$ -finite ring admitting a dualizing complex such that  $\text{Spec}(R)$  is irreducible. Let  $X$  be an irreducible proper  $F$ -finite scheme of dimension  $d$  over  $\text{Spec}(R)$  such that  $W_n X$  is an excellent scheme admitting a dualizing complex  $W_n \omega_X^\bullet$ . If  $\Delta$  is a  $\mathbb{Q}$ -divisor on  $X$  for which  $[\Delta] = 0$  then  $(X, \Delta)$  is  $n$ -quasi- $F^e$ -split if and only if*

$$H_{\mathfrak{m}}^d(X, \Phi_{X, K_X + \Delta, n}^e) : H_{\mathfrak{m}}^d(X, \mathcal{O}_X(K_X + \Delta)) \rightarrow H_{\mathfrak{m}}^d(X, Q_{X, K_X + \Delta, n}^e)$$

is injective.

PROOF. Applying theorem 3.17 (over the category of  $W_n(R)$ -modules) to

$$0 \rightarrow \ker(H_{\mathfrak{m}}^d(X, \Phi_{X, K_X + \Delta, n}^e)) \rightarrow H_{\mathfrak{m}}^d(X, \mathcal{O}_X(K_X + \Delta)) \rightarrow H_{\mathfrak{m}}^d(X, Q_{X, K_X + \Delta, n}^e)$$

yields

$$H^0(X, (Q_{X, \Delta, n}^e)^*) \wedge \xrightarrow{H^0(X, (\Phi_{X, \Delta, n}^e)^*)^\wedge} H^0(X, (\mathcal{O}_X(\Delta)^*)^\wedge) \rightarrow \ker(H_{\mathfrak{m}}^d(X, \Phi_{X, K_X + \Delta, n}^e))^\vee \rightarrow 0$$

Utilizing the fact that

$$\mathcal{H}om(Q_{X, K_X + \Delta, n}^e, W_n \omega_X) \cong \mathcal{H}om(Q_{X, \Delta, n}^e, W_n \omega_X(-K_X)) = (Q_{X, \Delta, n}^e)^*$$

And taking cohomology and completing. As  $W_n(R)$  is Noetherian (as a consequence of  $R$  being  $F$ -finite) completing is a faithfully flat functor. Hence,  $H^0(X, (\Phi_{X, \Delta, n}^e)^*)$  is surjective if and only if  $H^0(X, (\Phi_{X, \Delta, n}^e)^*)^\wedge$  is.  $H^0(X, (\Phi_{X, \Delta, n}^e)^*)^\wedge$  is surjective if and only if  $\ker(H_{\mathfrak{m}}^d(X, \Phi_{X, K_X + \Delta, n}^e))^\vee = 0$ , if and only if  $\ker(H_{\mathfrak{m}}^d(X, \Phi_{X, K_X + \Delta, n}^e)) = 0$ , if and only if  $H_{\mathfrak{m}}^d(X, \Phi_{X, K_X + \Delta, n}^e)$  is injective. This completes the equivalence.  $\square$

While the original definition of quasi- $F^e$ -splitting requires this dualizing operation, which in turn requires the existence of a dualizing complex  $W_n \omega_X^\bullet$ . This equivalent characterization does not. Further, for local rings local cohomology is a very natural invariant to utilize, and

hence, would serve as a great tool with which to define quasi- $F^e$ -purity for local rings. Hence, we take this equivalent formulation to be our definition of quasi- $F^e$ -purity.

DEFINITION 3.19. Suppose  $(R, \mathfrak{m})$  is a normal local domain. We say that the  $\mathbb{Q}$ -divisor pair  $(R, \Delta)$  is **n-Quasi- $F^e$ -pure** if and only if  $[\Delta] = 0$  and the following map of abelian groups is injective:

$$H_{\mathfrak{m}}^d(R(K_R + \Delta)) \rightarrow H_{W_n \mathfrak{m}}^d(Q_{R, K_R + \Delta, n}^e)$$

We will provide alternative yet equivalent definitions later (remark 3.25). When  $R$  is  $F$ -finite, this is equivalent to the definition of (local) Quasi- $F^e$ -splitting via the preceding proposition. However, this definition does not rely on the existence of a Witt-dualizing complex  $W_n \omega_R^\bullet$ .

REMARK 3.20. *It can be easily seen from the definition and remark 1.104 that this is a local property, i.e. that  $R$  is  $n$ -quasi- $F^e$ -pure if and only if  $R_P$  is for all  $P \in \text{Spec}(R)$ .*

In the spirit of this remark, we say that an integral normal  $\mathbb{F}_p$ -scheme- $\mathbb{Q}$ -divisor pair  $(X, \Delta)$  is **n-quasi- $F^e$ -pure** (resp. **quasi- $F^e$ -pure**) if and only if  $(\mathcal{O}_{X,x}, \Delta|_{\mathcal{O}_{X,x}})$  is  $n$ -quasi- $F^e$ -pure (resp. quasi- $F^e$ -pure) for each  $x \in X$ . We also define (uniform)-quasi- $F^\infty$ -splittings analogously to section 1.5.

DEFINITION 3.21. For  $(X, \Delta)$  as above,

- $(X, \Delta)$  is **n-quasi- $F^\infty$ -pure** if  $\forall e > 0$ ,  $(X, \Delta)$  is  $n$ -quasi- $F^e$ -pure.
- $(X, \Delta)$  is **quasi- $F^\infty$ -pure** if  $\forall e > 0 \exists n > 0$  such that  $(X, \Delta)$  is  $n$ -quasi- $F^e$ -pure.
- $(X, \Delta)$  is **uniformly quasi- $F^\infty$ -pure** if  $\exists n > 0$  such that  $\forall e > 0$   $(X, \Delta)$  is  $n$ -quasi- $F^e$ -pure.

One can show that for any  $n$ ,  $n$ -quasi- $F^e$ -pure implies  $n + 1$ -quasi- $F^e$ -pure for  $e \in \mathbb{N} \cup \infty$ . Thus we define the Quasi- $F^e$ -pure heights:

DEFINITION 3.22. We define the **Quasi- $F^e$ -Pure Height** and **Quasi- $F^\infty$ -Pure Height** as follows:

$$\text{ht}^e(X, \Delta) = \inf\{n \mid X \text{ is } n\text{-quasi-}F^e\text{-pure}\}$$

$$\text{ht}^\infty(X, \Delta) = \inf\{n \mid X \text{ is } n\text{-quasi-}F^\infty\text{-pure}\}$$

We take the convention that the infimum of the empty set is  $\infty$ : hence  $\text{ht}^e(X, \Delta) = \infty$  if  $(X, \Delta)$  is not quasi- $F^e$ -pure, and similarly with  $\text{ht}^\infty(X, \Delta)$ .

Local cohomology over non-Noetherian rings is not well behaved, particularly when supported in an ideal  $W_n\mathfrak{m}$  that is not finitely generated. We will first overcome this hassle and provide alternate, yet equivalent, definitions of quasi- $F^e$ -purity.

LEMMA 3.23. *Let  $(R, \mathfrak{m})$  be a Noetherian local ring and set  $\mathfrak{m} = (x_1, \dots, x_\ell)$ . Let  $[\mathfrak{m}] := ([x_1], \dots, [x_\ell]) \subset W_n(R)$ . Then,*

- (a)  $W_n\mathfrak{m} = [\mathfrak{m}] + \text{im}(V)$
- (b)  $(W_n\mathfrak{m})^n \subset [\mathfrak{m}]$ .
- (c)  $W_n\mathfrak{m} = \sqrt{[\mathfrak{m}]}$
- (d)  $W_n\mathfrak{m}$  is finitely generated up to radical.

PROOF. Let us prove (a) first. Choose  $(a_0, \dots, a_{n-1}) \in W_n\mathfrak{m}$ , and decompose it as

$$\begin{aligned} (a_0, \dots, a_{n-1}) &= [a_0] + (0, a_1, \dots, a_{n-1}) \\ &= \left[ \sum_{i=1}^{\ell} b_{0,i} x_i \right] + (0, a_1, \dots, a_{n-1}) \\ &= \sum_{i=1}^{\ell} [b_{0,i}] [x_i] + (0, E_1^a, \dots, E_{n-1}^a) + (0, a_1, \dots, a_{n-1}) \end{aligned}$$

Where  $(0, E_1^a, \dots, E_{n-1}^a) := \left[ \sum_{i=1}^{\ell} b_{0,i} x_i \right] - \sum_{i=1}^{\ell} [b_{0,i}] [x_i]$  are the resulting error terms. It is clear that  $\sum_{i=1}^{\ell} [b_{0,i}] [x_i] \in [\mathfrak{m}]$  and that  $(0, E_1^a, \dots, E_{n-1}^a) + (0, a_1, \dots, a_{n-1}) \in \text{im}(V)$ , so  $W_n\mathfrak{m} \subset [\mathfrak{m}] + \text{im}(V)$ .  $[1] \notin [\mathfrak{m}] + \text{im}(V)$ , so the ideal is proper. Thus by maximality of  $W_n\mathfrak{m}$ , we conclude (a). Using this,

$$(W_n\mathfrak{m})^n = ([\mathfrak{m}] + \text{im}(V))^n \subset [\mathfrak{m}] + \text{im}(V)^n \subset [\mathfrak{m}] + \text{im}(V^n)$$

As  $W_n(R) := W(R)/V^n(W(R))$ , it is clear that  $\text{im}(V^n) = 0$  in  $W_n(R)$ . Thus (b) is proven. From (b) we see that  $W_n\mathfrak{m} \subset \sqrt{[\mathfrak{m}]}$ , and via maximality of  $W_n\mathfrak{m}$  we can conclude (c). (c)  $\Rightarrow$  (d) is clear.  $\square$

When computing local cohomology over a finitely generated ideal, expected properties are recovered even when the base ring  $R$  is not Noetherian. This will be covered rigorously in appendix A; the facts given there, coupled with the fact that  $W_n\mathfrak{m}$  and  $[\mathfrak{m}]$  are equal up to radical, allow us to change the base of local cohomology easily.

REMARK 3.24. *If  $M$  is both an  $R$ -Module and a  $W_n(R)$ -module, then*

$$H_{\mathfrak{m}}^i(M) \cong H_{[\mathfrak{m}]}^i(M) \cong H_{W_n\mathfrak{m}}^i(M)$$

*As  $W_n(R)$ -modules. If  $M$  is just a  $W_n(R)$ -module, the second isomorphism holds.*

Recall that  $R(K_R + rfg\Delta)$  is both a  $R$ -Module and a  $W_n(R)$  module, and  $Q_{R, K_R + \Delta, n}^e$  is a  $W_n(R)$ -Module. This yields various equivalent definitions of Quasi- $F^e$ -purity, via exchanging out the source and target in the definition for isomorphic objects.

REMARK 3.25. *Let  $R$  be a normal domain of dimension  $d$ , and  $\Delta$  a  $\mathbb{Q}$ -divisor on  $R$ . Then, the following are equivalent.*

- (a)  $(R, \Delta)$  is  $n$ -Quasi- $F^e$ -pure.
- (b)  $H_{W_n\mathfrak{m}}^d(R(K_R + \Delta)) \hookrightarrow H_{W_n\mathfrak{m}}^d(Q_{R, K_R + \Delta, n}^e)$  as  $W_n(R)$ -Modules.
- (c)  $H_{[\mathfrak{m}]}^d(R(K_R + \Delta)) \hookrightarrow H_{[\mathfrak{m}]}^d(Q_{R, K_R + \Delta, n}^e)$  as  $W_n(R)$ -Modules.
- (d)  $H_{\mathfrak{m}}^d(R(K_R + \Delta)) \hookrightarrow H_{[\mathfrak{m}]}^d(Q_{R, K_R + \Delta, n}^e)$  as  $W_n(R)$ -Modules.

**2.3. Stability under Completion.** A natural question would be whether quasi- $F^e$ -purity is invariant under completion. In the  $F$ -finite case this is immediate; Both  $R(K_R + \Delta)$  and  $Q_{R, K_R + \Delta, n}^e$  are finitely generated modules over a Noetherian ring, and thus we have the square

$$\begin{array}{ccc}
H_m^d(R(K_R + \Delta)) & \longrightarrow & H_{[m]}^d(Q_{R, K_R + \Delta, n}^e) \\
\downarrow \cong & & \downarrow \cong \\
H_m^d(R(K_R + \Delta)^\wedge) & \longrightarrow & H_{[m]}^d(Q_{R, K_R + \Delta, n}^e)^\wedge
\end{array}$$

Notice that the top horizontal arrow is injective if and only if the bottom arrow is. Therefore, it is sufficient to show that  $H_{[m]}^d(Q_{R, K_R + \Delta, n}^e)^\wedge \cong H_{[m]}^d(Q_{\widehat{R}, K_{\widehat{R}} + \Delta, n}^e)$ , though this is true even outside of the  $F$ -finite case.

LEMMA 3.26. *Suppose  $R$  is a Noetherian local ring. Then  $Q_{R, \Delta, n}^e \cong Q_{\widehat{R}, \Delta, n}^e$  as  $W_n(\widehat{R})$ -modules.*

PROOF. Via [TWY24, 3.1.1] we have the short exact sequence of  $W_n(R)$ -modules below:

$$0 \longrightarrow F_*W_{n-1}(R)(p\Delta) \xrightarrow{VF^e} F_*^eW_n(R)(p^e\Delta) \longrightarrow Q_{R, \Delta, n}^e \longrightarrow 0$$

We can get a right exact sequence by completing as  $W_n(R)$ -modules<sup>4</sup>. Further, one could construct this sequence with respect to  $\widehat{R}$ . This discussion yields the following diagram of  $W_n(\widehat{R})$  modules:

$$\begin{array}{ccccccc}
F_*W_{n-1}(R)(\Delta)^\wedge & \xrightarrow{VF^e} & F_*^eW_n(R)(p^e\Delta)^\wedge & \longrightarrow & Q_{R, \Delta, n}^e \wedge & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \\
0 & \longrightarrow & F_*W_{n-1}(\widehat{R})(p\Delta) & \xrightarrow{VF^e} & F_*^eW_n(\widehat{R})(p^e\Delta) & \longrightarrow & Q_{\widehat{R}, \Delta, n}^e \longrightarrow 0
\end{array}$$

As  $W_n(-)$  and  $F_*^e(-)$  are both compatible with completion, the first two vertical isomorphisms follow. Thus, the third vertical map must be an isomorphism.  $\square$

Outside of the  $F$ -finite case much of the argument still follows, but it is not immediately clear that

$$H_{[m]}^d(Q_{R, K_R + \Delta, n}^e) \cong H_{[m]}^d(Q_{R, K_R + \Delta, n}^e)^\wedge$$

as  $W_n(R)$  is necessarily non-Noetherian and  $Q_{R, K_R + \Delta, n}^e$  is not a finitely generated  $W_n(R)$ -Module. This isomorphism does hold, however, and it is sufficient to conclude  $\text{ht}^e(R, \Delta) = \text{ht}^e(\widehat{R}, \Delta)^\wedge$  outside of the  $F$ -finite setting.

<sup>4</sup>This is not because completion is a right exact functor; indeed in the non-Noetherian case, completion can fail to be exact, even right exact, even when completing a short exact sequence of finitely presented modules with respect to a finitely generated ideal [Sta24, 05JF]. In this case however,  $Q_{R, \Delta, n}^e$  is realized as a quotient of  $W_n(R)$ -modules, each of which is compatible with the completion operation.

THEOREM 3.27. *Suppose  $(R, \mathfrak{m})$  is a normal domain that completes at the maximal ideal to a normal domain and  $\Delta$  is a  $\mathbb{Q}$ -divisor on  $R$ . Then  $H_{[\mathfrak{m}]}^d(Q_{R,\Delta,n}^e) \cong H_{[\widehat{\mathfrak{m}}]}^d(Q_{R,\Delta,n}^e \wedge)$ .*

PROOF. Recall the short exact sequence of  $W_n(R)$ -modules below:

$$0 \longrightarrow F_*W_{n-1}(R)(p\Delta) \xrightarrow{VF^e} F_*^eW_n(R)(p^e\Delta) \longrightarrow Q_{R,\Delta,n}^e \longrightarrow 0$$

We will take long exact sequences in cohomology with respect to the ideal  $[\mathfrak{m}]$ , both of this short exact sequence and the corresponding sequence over  $\widehat{R}$ . This yields the following diagram:

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ H_{[\mathfrak{m}]}^d(F_*W_{n-1}(R)(p\Delta)) & \longrightarrow & H_{[\widehat{\mathfrak{m}}]}^d(F_*W_{n-1}(R)(p\Delta) \wedge) \\ \downarrow & & \downarrow \\ H_{[\mathfrak{m}]}^d(F_*^eW_n(R)(p^e\Delta)) & \xrightarrow{\cong} & H_{[\widehat{\mathfrak{m}}]}^d(F_*^eW_n(R)(p^e\Delta) \wedge) \\ \downarrow & & \downarrow \\ H_{[\mathfrak{m}]}^d(Q_{R,\Delta,n}^e) & \longrightarrow & H_{[\widehat{\mathfrak{m}}]}^d(Q_{R,\Delta,n}^e \wedge) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

Note we are implicitly citing lemma 3.5 for the right side of this diagram. As Frobenius pushforwards commute with local cohomology and completion, the middle horizontal map is an isomorphism. To conclude the bottom map is an isomorphism, it is sufficient to show that

$$H_{[\mathfrak{m}]}^d(F_*W_{n-1}(R)(p\Delta)) \cong H_{[\widehat{\mathfrak{m}}]}^d(F_*W_{n-1}(R)(p\Delta) \wedge)$$

Consider the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \longrightarrow & W_n(R) & \xrightarrow{\mathfrak{R}} & W_{n-1}(R) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \widehat{R} & \longrightarrow & W_n(\widehat{R}) & \xrightarrow{\mathfrak{R}} & W_{n-1}(\widehat{R}) \longrightarrow 0 \end{array}$$

Where  $\mathfrak{R}$  is the restriction map and all vertical maps are completion morphisms. We can twist by  $p\Delta$ , push forward via  $F$ , then take a long exact sequence in local cohomology with support in  $[\mathfrak{m}]$ . Utilizing lemma 3.5 again, we have the following diagram:

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
H_{[\mathfrak{m}]}^d(R(p\Delta)) & \xrightarrow{\cong} & H_{[\widehat{\mathfrak{m}}]}^d(\widehat{R}(p\Delta)) \\
\downarrow & & \downarrow \\
H_{[\mathfrak{m}]}^d(F_*W_n(R)(p\Delta)) & \xrightarrow{\cong} & H_{[\mathfrak{m}]}^d(F_*W_n(R)(p\Delta)^\wedge) \\
\downarrow & & \downarrow \\
H_{[\mathfrak{m}]}^d(F_*W_{n-1}(R)(p\Delta)) & \longrightarrow & H_{[\widehat{\mathfrak{m}}]}^d(F_*W_{n-1}(R)(p\Delta)^\wedge) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

The top map is an isomorphism because  $R(p\Delta)$  is a finitely generated  $R$ -Module, and thus

$$H_{\mathfrak{m}}^d(R(p\Delta)) \cong H_{\mathfrak{m}}^d(R(p\Delta)^\wedge) \cong H_{\widehat{\mathfrak{m}}}^d(\widehat{R}(p\Delta)^\wedge)$$

Via remark 3.24 we can change the base to  $[\mathfrak{m}]$  and the isomorphism is preserved. The middle map is an isomorphism following the preceding corollary. Thus, the bottom map is an isomorphism and we can conclude.  $\square$

**COROLLARY 3.27.1.** *Suppose  $(R, \mathfrak{m})$  is a normal domain that completes at the maximal ideal to a normal domain. Then for any  $\mathbb{Q}$ -divisor  $\Delta$ ,  $(R, \Delta)$  is  $n$ -Quasi- $F^e$ -pure  $\iff (\widehat{R}, \Delta^\wedge)$  is  $n$ -Quasi- $F^e$ -pure. In particular,  $\text{ht}^e(R, \Delta) = \text{ht}^e(\widehat{R}, \Delta^\wedge)$ .*

**PROOF.** See the argument in the  $F$ -finite case at the beginning of this section; the previous lemma implies that the diagram square 2.3 holds even outside of the  $F$ -finite setting.  $\square$

Tracing the definitions through then yields the analogous statements for other notions of Quasi- $F$ -purity.

COROLLARY 3.27.2. *Suppose  $(R, \mathfrak{m})$  is a local normal domain that completes at the maximal ideal to a normal domain. Then for any  $\mathbb{Q}$ -divisor  $\Delta$ ,*

- (a)  $(R, \Delta)$  is  $n$ -Quasi- $F^\infty$ -pure  $\iff (\widehat{R}, \Delta^\wedge)$  is  $n$ -Quasi- $F^\infty$ -pure, or in other words,  $h^\infty(R, \Delta) = h^\infty(\widehat{R}, \Delta^\wedge)$ .
- (b)  $(R, \Delta)$  is Quasi- $F^e$ -pure  $\iff (\widehat{R}, \Delta^\wedge)$  is Quasi- $F^e$ -pure.
- (c)  $(R, \Delta)$  is Quasi- $F^\infty$ -pure  $\iff (\widehat{R}, \Delta^\wedge)$  is Quasi- $F^\infty$ -pure.
- (d)  $(R, \Delta)$  is uniformly Quasi- $F^\infty$ -pure  $\iff (\widehat{R}, \Delta^\wedge)$  is uniformly Quasi- $F^\infty$ -pure.

REMARK 3.28. *This extends the results of chapter 2 section 3.1 to not-necessarily-excellent normal domains. The author suspects these arguments still hold for any Noetherian local ring  $R$  where divisors are well defined on it and its completion, i.e. reduced local rings that are G1 (e.g. Gorenstein in codimension 1) and S2 that remain reduced, G1 and S2 after completing.*

**2.4. Stability of Quasi- $F^e$ -Stable Sections.** Following the preceding section, one can reduce the study of quasi- $F^e$ -purity of normal domains to the case of complete local normal domains. In this section we will re-introduce the notion of Quasi- $F^e$ -stable sections, introduced in [TWY24, section 3.6], but reformulated to make sense outside of the  $F$ -finite setting. In section 5 we will use these modules of stable sections to reduce the application of the  $\Gamma$ -construction to quasi- $F^\infty$ -purity to applying the  $\Gamma$ -construction to quasi- $F^e$ -purity.

DEFINITION 3.29. For  $\mathbb{Q}$ -divisors  $\Delta$  and  $L$  of  $R$  a normal domain, we define the module of **Quasi- $F^e$ -Stable Sections** to be the following  $R$ -submodule  $q^e S_n^0(R, \Delta; L) \subset R([L - \Delta])$ :

$$q^e S_n^0(R, \Delta; L) := \text{im} \left( H_{\mathfrak{m}}^d(R([K_R + \Delta - L])) \xrightarrow{\Phi_{R, [K_R + \Delta - L], n}^e} H_{[\mathfrak{m}]}^d(Q_{R, K_R + \Delta - L, n}^e) \right)^\vee$$

Where  $(-)^{\vee}$  denotes the Matlis dual as an  $R$ -Module. When  $L = 0$  we will refer to this module as  $q^e S_n^0(R, \Delta)$  for simplicity.

LEMMA 3.30. *If  $R$  is complete,  $q^e S_n^0(R, \Delta; L)$  is a Noetherian  $R$ -Module.*

PROOF.  $\text{im}(\mathbf{H}_{\mathfrak{m}}^d(R(\lfloor K_R + \Delta - L \rfloor)) \rightarrow \mathbf{H}_{[\mathfrak{m}]}^d(Q_{R, K_R + \Delta - L, n}^e))$  is a submodule of  $\mathbf{H}_{\mathfrak{m}}^d(R(\lfloor K_R + \Delta - L \rfloor))$  which is Artinian, and hence, is Artinian itself. Via Matlis duality over complete local rings, we can conclude that its Matlis dual  $q^e S_n^0(R, \Delta; L)$  is Noetherian.  $\square$

When  $R$  is complete, it follows that  $q^e S_n^0(R, \Delta; L) \cong q^e S_n^0(R, \Delta; L)^\wedge$ .

LEMMA 3.31. *Let  $(R, \mathfrak{m})$  be a (complete) local normal domain and  $\Delta$  a  $\mathbb{Q}$ -divisor such that  $\lfloor \Delta \rfloor = 0$ . Then*

$$q^e S_n^0(R, \Delta) = R \iff (R, \Delta) \text{ is } n\text{-quasi-}F^e\text{-pure}$$

PROOF. As  $\lfloor \Delta \rfloor = 0$ , we can rewrite  $q^e S_n^0(R, \Delta)$  as

$$\text{im}(\mathbf{H}_{\mathfrak{m}}^d(R(K_R)) \rightarrow \mathbf{H}_{[\mathfrak{m}]}^d(Q_{R, K_R + \Delta, n}^e))^\vee$$

$(R, \Delta)$  is  $n$ -quasi- $F^e$ -pure if and only if this map is injective, or equivalently, the source injects onto its image. Thus,

$$q^e S_n^0(R, \Delta) = \mathbf{H}_{\mathfrak{m}}^d(R(K_R))^\vee \iff (R, \Delta) \text{ is } n\text{-quasi-}F^e\text{-pure}$$

As  $R(-K_R)$  uniquely represents the functor  $\mathbf{H}_{\mathfrak{m}}^d(-)^\vee$ , see [BH98, 3.5.10], one can then see that

$$\mathbf{H}_{\mathfrak{m}}^d(R(K_R))^\vee \cong \text{Hom}_R(R(K_R), R(-K_R)) \cong R$$

$\square$

In general we have a natural short exact sequence

$$0 \rightarrow \ker(\Phi_{R, \lfloor K_R + \Delta - L \rfloor, n}^e) \rightarrow \mathbf{H}_{\mathfrak{m}}^d(R(\lfloor K_R + \Delta - L \rfloor)) \rightarrow \text{im}(\Phi_{R, \lfloor K_R + \Delta - L \rfloor, n}^e) \rightarrow 0$$

Via the argument at the end of the previous proof, taking the Matlis dual yields the following short exact sequence

$$0 \rightarrow q^e S_n^0(R, \Delta; L) \rightarrow R(\lceil L - \Delta \rceil) \rightarrow \ker (\Phi_{R, \lceil K_R + \Delta - L \rceil, n}^e)^\vee \rightarrow 0$$

Which implies that  $q^e S_n^0(R, \Delta; L) \subset R(\lceil L - \Delta \rceil)$  in general. We now recall the below constructions from [TWY24]:

$$\begin{aligned} q^e S^0(R, \Delta; L) &:= \bigcup_{n=1}^{\infty} q^e S_n^0(R, \Delta; L) \\ q^\infty S_n^0(R, \Delta; L) &:= \bigcap_{e>0} q^e S_n^0(R, \Delta; L) \\ q^\infty S^0(R, \Delta; L) &:= \bigcap_{e>0} \bigcup_{n>0} q^e S_n^0(R, \Delta; L) \\ q_{\text{uni}}^\infty S^0(R, \Delta; L) &:= \bigcup_{n>0} \bigcap_{e>0} q^e S_n^0(R, \Delta; L) \end{aligned}$$

It's easily checked that for  $n \leq m$ ,

$$q^e S_n^0(R, \Delta; L) \subseteq q^e S_m^0(R, \Delta; L) \quad \text{and} \quad q^\infty S_n^0(R, \Delta; L) \subseteq q^\infty S_m^0(R, \Delta; L)$$

Implying via Noetherianity that for  $n \gg 0$ ,

$$q^e S_n^0(R, \Delta; L) = q^e S^0(R, \Delta; L) \quad \text{and} \quad q^\infty S_n^0(R, \Delta; L) = q^\infty S_{\text{uni}}^0(R, \Delta; L)$$

These Quasi- $F^e$ -stable sections characterize our formulations of Quasi- $F^e$ -purity and (Uniform) Quasi- $F^\infty$ -purity exactly as in [TWY24, Remarks 3.46]:

$$\begin{aligned} q^e S_n^0(R, \Delta) = R &\iff (R, \Delta) \text{ is } n\text{-quasi-}F^e\text{-pure} \\ q^e S^0(R, \Delta) = R &\iff (R, \Delta) \text{ is quasi-}F^e\text{-pure} \\ q^\infty S_n^0(R, \Delta) = R &\iff (R, \Delta) \text{ is } n\text{-quasi-}F^\infty\text{-pure} \\ q^\infty S^0(R, \Delta) = R &\iff (R, \Delta) \text{ is quasi-}F^\infty\text{-pure} \\ q_{\text{uni}}^\infty S^0(R, \Delta) = R &\iff (R, \Delta) \text{ is uniformly quasi-}F^\infty\text{-pure} \end{aligned}$$

We recall that a scheme  $X$  is **divisorially Cohen Macaulay** if  $\mathcal{O}_X(D)$  is Cohen Macaulay for every Weil divisor  $D$  on  $X$  (see [TWY24, Definition 2.1]).

LEMMA 3.32. *Let  $(R, \mathfrak{m})$  be a complete local divisorially Cohen Macaulay normal domain and  $A$  an ample  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor. Further let  $N$  be a nef  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor. Then,  $\exists e_0 > 0$  such that  $\forall r > 0, k \geq 0$  for  $r, k \in \mathbb{Z}$ , the kernel of*

$$F^e : H_{[\mathfrak{m}]}^d(W_n(R)(-(rA + kN))) \rightarrow H_{[\mathfrak{m}]}^d(F_*^e W_n(R)(-p^e(rA + kN)))$$

*Is independent of the choice of  $e \geq e_0$ .*

PROOF. See [TWY24, lemma 3.50]. The proof given does not assume that  $R$  is  $F$ -finite, and all other hypothesis are mentioned above. The change of base in cohomology is due to remark 3.24.  $\square$

LEMMA 3.33. *Let  $(R, \mathfrak{m})$  be a complete local divisorially Cohen Macaulay normal domain and  $A$  an ample  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor. Further let  $N$  be a nef  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor. Then,  $\exists e_0 > 0$  such that  $\forall r, k \geq 0, e \geq e_0$  for  $r, k, e \in \mathbb{Z}$ , and for every  $\mathbb{Q}$ -divisor  $\Delta$  such that  $K_R + \Delta$  is  $\mathbb{Q}$ -Cartier,*

$$q^e S_n^0(R, \Delta, L) = q^{e_0} S_n^0(R, \Delta, L)$$

*Where  $L := K_R + \Delta + rA + kN$ . In particular,  $q^\infty S_n^0(R, \Delta, L) = q^e S_n^0(R, \Delta, L) \forall e \geq e_0$ .*

PROOF. In the proof of [TWY24, proposition 3.51], the authors reduce to showing that there exists  $e_0$  such that the kernel of

$$H_{\mathfrak{m}}^d(R(-A)) \rightarrow H_{W_n, \mathfrak{m}}^d(Q_{R, -A, n}^e)$$

is independent of choice of  $e \geq e_0$ . via corollary 3.24 again, this is equivalent to the kernel of  $H_{\mathfrak{m}}^d(R(-A)) \rightarrow H_{[\mathfrak{m}]}^d(Q_{R, -A, n}^e)$  being independent of  $e$ , which is sufficient to conclude the lemma above utilizing our definitions of  $q^\infty S_n^0(R, \Delta, L)$  and  $q^e S_n^0(R, \Delta, L)$ . Again, the  $F$ -finite hypothesis was unnecessary in the proof given by the above authors.  $\square$

COROLLARY 3.33.1. *Let  $(R, \mathfrak{m})$  be a complete local divisorially Cohen Macaulay normal domain and  $\Delta, L$  be  $\mathbb{Q}$ -divisors such that  $L - (K_R + \Delta)$  is an ample  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -Divisor. Then:*

$$\begin{aligned} \forall_{n>0} \exists_{e_0>0} \forall_{e \geq e_0} q^\infty S_n^0(R, \Delta, L) &= q^e S_n^0(R, \Delta, L) \\ \exists_{n_0>0} \forall_{n \geq n_0} \exists_{e_0>0} \forall_{e \geq e_0} q_{uni}^\infty S^0(R, \Delta, L) &= q^e S_n^0(R, \Delta, L) \end{aligned}$$

PROOF. Proof is again identical to [TWY24, 3.52], we omit this argument (and prior arguments in this subsection) for brevity.  $\square$

**2.5. Stability Under Étale Extension.** In this subsection, we will let study the behavior of quasi- $F^e$ -purity under finite unrammified maps. We first recall key facts about such maps, then show they preserve quasi- $F^e$ -purity.

LEMMA 3.34. *Let  $\varphi : R \rightarrow S$  be a finite map of normal domains. Then  $\varphi$  is flat in codimension 1.*

PROOF. Viewing  $S_P$  as  $S$  localized at  $P \in \text{Spec}(R)$  as an  $R$ -Module, it is sufficient to check that  $R_P \rightarrow S_P$  is a flat map for all  $P \in \text{Spec}(R)$  of height 1. Fix such a  $P$ . As  $R$  is normal  $R_P$  is a DVR; we see that  $S_P$  is a torsion free, finitely generated module over a DVR  $R_P$ . Thus,  $S_P$  is a free  $R_P$ -module, and in particular, flat.  $\square$

LEMMA 3.35. *Let  $\varphi : R \rightarrow S$  be a finite map of normal domains unrammified in codimension 1, where  $R$  and  $S$  each admit dualizing complexes  $\omega_R^\bullet, \omega_S^\bullet$  and corresponding canonical divisors  $K_R, K_S$ . Then  $K_R \otimes_R S \cong K_S$ .*

PROOF. As  $\varphi$  is unrammified, and via lemma 3.34, flat, in codimension 1, it is étale in codimension 1. Thus,  $\omega_R^\bullet \otimes_R^{\mathbb{L}} S \cong \omega_S^\bullet$  in codimension 1, which implies our result about divisors.  $\square$

REMARK 3.36. *If  $\varphi$  is also flat (e.g. it is étale) then we can say something stronger.  $S \otimes_R R(K_R + \Delta) \cong S(K_S + \Delta_S)$  via lemma 3.3, so the map  $R(K_R + \Delta) \rightarrow S(K_S + \Delta_S)$  is*

simply this base change map. In particular, we have natural base change relations at the level of local cohomology:

$$S \otimes_R H_{\mathfrak{m}}^d(R(K_R + \Delta)) \cong H_{\mathfrak{m}}^d(S(K_S + \Delta_S))$$

$$W_n(S) \otimes_{W_n(R)} H_{[\mathfrak{m}]}^d(R(K_R + \Delta)) \cong H_{[\mathfrak{m}]}^d(S(K_S + \Delta_S))$$

For the remainder of this section we focus on the following setting:

**SETTING 3.37.**  $(R, \mathfrak{m})$  is an excellent normal local domain admitting a dualizing complex  $\omega_R^\bullet$  with associated canonical divisor  $K_R$ , and  $\Delta$  is a  $\mathbb{Q}$ -divisor on  $R$  such that  $[\Delta] = 0$ . Let  $\varphi : R \rightarrow S$  be a local étale map to a domain  $(S, \mathfrak{n})$ . Let  $\Delta_S = (K_R \otimes_R S) - K_S + \Delta \otimes_R S$ .

In the context of setting 3.37,  $S$  is automatically an excellent normal domain with dualizing complex  $\omega_S^\bullet = \omega_R^\bullet \otimes_R^{\mathbb{L}} S$  [Sta24, 033C,07PV], so  $K_S$  is well defined. Further, as  $\varphi$  is unramified  $(K_R \otimes_R S) - K_S = 0$  and hence one can just write  $\Delta_S = \Delta \otimes_R S$ ; it is clear from this description that  $[\Delta_S] = 0$ . In particular, the map  $R(K_R + \Delta) \rightarrow S(K_S + \Delta_S)$  is well defined and we can compare the quasi- $F^e$ -pure heights  $\text{ht}^e(R, \Delta)$  and  $\text{ht}^e(S, \Delta_S)$ .

In this setting, as well as the global analogue which will be discussed later, it is pertinent to ask when quasi- $F^e$ -pure height is preserved along  $\varphi$ . We will return to this setting in section 3.2 to show that quasi- $F$ -pure regularity is preserved under these conditions as well.

**LEMMA 3.38.** *In setting 3.37 if  $\varphi$  is just unramified, then  $\text{ht}^e(R, \Delta) \leq \text{ht}^e(S, \Delta_S)$ .*

**PROOF.** Via corollary 3.27.2, we can assume  $R$  and  $S$  are both complete local rings. As  $\varphi$  is an unramified map,  $\varphi(\mathfrak{m}) = \mathfrak{n}$ , so we have an extension of fields  $\mathfrak{K} \rightarrow \mathfrak{L}$ , and thus an injection of injective hulls  $E_R \hookrightarrow E_S$ . As  $R(K_R) \cong \omega_R$  and same for  $S$ , we have the natural isomorphisms

$$H_{\mathfrak{m}}^d(R(K_R + \Delta)) \cong H_{\mathfrak{m}}^d(\omega_R) \cong E_R$$

$$H_{\mathfrak{n}}^d(S(K_S + \Delta_S)) \cong H_{\mathfrak{n}}^d(\omega_S) \cong E_S$$

Where the first is an isomorphism of  $R$ -Modules, and the second of  $S$ -Modules. It follows that we have an injective map of Abelian Groups  $H_{\mathfrak{m}}^d(R(K_R + \Delta)) \hookrightarrow H_{\mathfrak{n}}^d(S(K_S + \Delta_S))$ .

On the other hand, as  $\varphi : R \rightarrow S$  is unramified it extends to a map  $R(K_R + \Delta) \rightarrow S(K_S + \Delta_S)$ , which extends further to a map of Abelian Groups  $Q_{R, K_R + \Delta, n}^e \rightarrow Q_{S, K_S + \Delta_S, n}^e$  for any  $e, n$ . These are both  $W_n(R)$ -Modules, and thus one can take the local cohomology to yield the map

$$H_{[\mathfrak{m}]}^d(Q_{R, K_R + \Delta, n}^e) \rightarrow H_{[\mathfrak{m}]}^d(Q_{S, K_S + \Delta_S, n}^e)$$

As  $\varphi(\mathfrak{m}) = \mathfrak{n}$ , it is clear that  $W_n(\varphi)([\mathfrak{m}]) = [\mathfrak{n}]$ , and in particular we can change the base of cohomology of the codomain from  $[\mathfrak{m}]$  to  $[\mathfrak{n}]$ . Fitting this together yields the commutative diagram

$$\begin{array}{ccc} H_{\mathfrak{m}}^d(R(K_R + \Delta)) & \xrightarrow{\Phi_{R, K_R + \Delta, n}^e} & H_{[\mathfrak{m}]}^d(Q_{R, K_R + \Delta, n}^e) \\ \downarrow & & \downarrow \\ H_{\mathfrak{n}}^d(S(K_S + \Delta_S)) & \xrightarrow{\Phi_{S, K_S + \Delta_S, n}^e} & H_{[\mathfrak{n}]}^d(Q_{S, K_S + \Delta_S, n}^e) \end{array}$$

As the first vertical map is injective, it follows that if  $\Phi_{S, K_S + \Delta_S, n}^e$  is injective, then so is  $\Phi_{R, K_R + \Delta, n}^e$ . In particular, if  $(S, \Delta_S)$  is  $n$ -Quasi- $F^e$ -pure, so is  $(R, \Delta)$ . The result follows.  $\square$

One can ask when this is an equality. If in addition to being unramified  $\varphi$  is flat (or equivalently,  $\varphi$  is étale), then via [Sta24, 08Z5]  $E_R \cong E_S$ , and in particular, the first vertical map in the diagram above is an isomorphism of abelian groups. If we further assume that  $\text{Frac}(S)/\text{Frac}(R)$  is Galois of degree not divided by  $p$ , via [TWY24, Proposition 3.21], the morphism  $Q_{R, K_R + \Delta, n}^e \rightarrow Q_{S, K_S + \Delta_S, n}^e$  splits<sup>5</sup>, and thus, the corresponding map on local cohomology is injective. In this setting the square diagram above can be rewritten as follows:

$$\begin{array}{ccc} H_{\mathfrak{m}}^d(R(K_R + \Delta)) & \xrightarrow{\Phi_{R, K_R + \Delta, n}^e} & H_{[\mathfrak{m}]}^d(Q_{R, K_R + \Delta, n}^e) \\ \downarrow \cong & & \downarrow \\ H_{\mathfrak{n}}^d(S(K_S + \Delta_S)) & \xrightarrow{\Phi_{S, K_S + \Delta_S, n}^e} & H_{[\mathfrak{n}]}^d(Q_{S, K_S + \Delta_S, n}^e) \end{array}$$

<sup>5</sup>We note that  $F$ -finiteness was not required for this proof, and thus, the lemma can be cited here.

It is easy to see here that  $\Phi_{S, K_S + \Delta_S, n}^e$  is injective if and only if  $\Phi_{R, K_R + \Delta, n}^e$  is. In particular,  $(R, \Delta)$  is  $n$ -quasi- $F^e$ -pure if and only if  $(S, \Delta_S)$  is, for any  $e, n$ , so  $\text{ht}^e(R, \Delta) = \text{ht}^e(S, \Delta_S)$ .

Our hope is to do away with this Galois assumption on the fraction fields. Indeed when  $e = 1$ , we know that quasi- $F$ -pure height is stable under local étale extension [Jag26]; we would hope that the same is true for Quasi- $F^e$ -pure height. To show this we first prove a powerful result about the  $W_n(R)$ -Module  $Q_{R, \Delta, n}^e$  base changing to the corresponding module over  $W_n(S)$  when  $R \rightarrow S$  is étale.

**PROPOSITION 3.39.** *In setting 3.37,  $Q_{R, \Delta, n}^e \otimes_{W_n(R)} W_n(S) \cong Q_{S, \Delta_S, n}^e$  for any  $e, n \in \mathbb{N}$ .*

**PROOF.** This proof is similar in style to that of Theorem 3.27. Via [Tan22, 4.6] we have the identity<sup>6</sup>

$$W_n(R)(\Delta) \otimes_{W_n(R)} W_n(S) \cong W_n(S)(\Delta_S)$$

via [Jag26, lemma 3.10] we further have that, for étale maps, the (iterated) relative Witt Frobenius is an isomorphism. Combining these results yields the isomorphism

$$F_*^e W_n(R)(\Delta) \otimes_{W_n(R)} W_n(S) \cong F_*^e W_n(S)(\Delta_S)$$

We now apply this isomorphism to the short exact sequence

$$0 \longrightarrow F_* W_{n-1}(R)(p\Delta) \xrightarrow{VF^e} F_*^e W_n(R)(p^e \Delta) \longrightarrow Q_{R, \Delta, n}^e \longrightarrow 0$$

with the exact functor  $-\otimes_{W_n(R)} W_n(S)$  applied, as well as its variant over  $S$ . This yields the morphism of short exact sequences

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<sup>6</sup>The statement of lemma 4.6 requires that the schemes involved be regular, though the author believes this hypothesis to be unnecessary.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
F_*W_{n-1}(R)(p\Delta) \otimes_{W_n(R)} W_n(S) & \longrightarrow & F_*W_{n-1}(S)(p\Delta) \\
\downarrow & & \downarrow \\
F_*^e W_n(R)(p^e \Delta) \otimes_{W_n(R)} W_n(S) & \xrightarrow{\cong} & F_*^e W_n(S)(p^e \Delta) \\
\downarrow & & \downarrow \\
Q_{R,\Delta,n}^e \otimes_{W_n(R)} W_n(S) & \longrightarrow & Q_{S,\Delta_S,n}^e \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

The first two horizontal morphisms are naturally defined, and thus descend to define the last horizontal map  $Q_{R,\Delta,n}^e \otimes_{W_n(R)} W_n(S) \rightarrow Q_{S,\Delta_S,n}^e$ . To conclude the lemma, it is thus sufficient to show the map

$$F_*W_{n-1}(R)(p\Delta) \otimes_{W_n(R)} W_n(S) \rightarrow F_*W_{n-1}(S)(p\Delta)$$

is an isomorphism. This similarly can be computed via the exact sequence

$$0 \rightarrow F_*R(\Delta) \rightarrow F_*W_n(R)(\Delta) \rightarrow F_*W_{n-1}(R)(\Delta) \rightarrow 0$$

via considering a similar diagram:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
F_*R(\Delta) \otimes_{W_n(R)} W_n(S) & \xrightarrow{\cong} & F_*S(\Delta_S) \\
\downarrow & & \downarrow \\
F_*W_n(R)(\Delta) \otimes_{W_n(R)} W_n(S) & \xrightarrow{\cong} & F_*W_n(S)(\Delta_S) \\
\downarrow & & \downarrow \\
F_*W_{n-1}(R)(\Delta) \otimes_{W_n(R)} W_n(S) & \longrightarrow & F_*W_{n-1}(S)(\Delta_S) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

The first isomorphism follows can easily be checked directly (given that as  $R \rightarrow S$  is étale  $F_*R \otimes_R S \cong F_*S$ ) and the second has already been discussed. The last isomorphism, and thus the lemma, follows.  $\square$

**THEOREM 3.40.** *Let  $(R, \mathfrak{m})$  be an excellent normal local domain admitting a dualizing complex  $\omega_R^\bullet$  with associated canonical divisor  $K_R$ , and  $\Delta$  is a  $\mathbb{Q}$ -divisor on  $R$  such that  $[\Delta] = 0$ . Let  $\varphi : R \rightarrow S$  be a finite étale map to a domain  $S$ . Let  $\Delta_S = (K_R \otimes_R S) - K_S + \Delta \otimes_R S$ .  $\text{ht}^e(R, \Delta) = \text{ht}^e(S, \Delta_S)$  for any  $e \in \mathbb{N}$ .*

**PROOF.** We first assume that  $\varphi$  is a local finite étale map  $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$  to a local ring  $S$ . Recall that, equivalently,  $(R, \Delta)$  is  $n$ -Quasi- $F^e$ -pure precisely when

$$H_{[\mathfrak{m}]}^d(R(K_R + \Delta)) \xrightarrow{\Phi_{R, K_R + \Delta, n}^e} H_{[\mathfrak{m}]}^d(Q_{R, K_R + \Delta, n}^e)$$

is an injective map of  $W_n(R)$ -Modules (see corollary 3.25). Combining lemmas A.4 and 3.39 yields the isomorphism

$$H_{[\mathfrak{m}]}^d(Q_{R, K_R + \Delta, n}^e) \otimes_{W_n(R)} W_n(S) \cong H_{[\mathfrak{n}]}^d(Q_{S, K_S + \Delta_S, n}^e)$$

As stated in remark 3.36,  $H_{[\mathfrak{m}]}^d(R(K_R + \Delta)) \otimes_{W_n(R)} W_n(S) \cong H_{[\mathfrak{n}]}^d(S(K_S + \Delta_S))$ .  $R \rightarrow S$  is a local étale map, and hence, so is  $W_n(R) \rightarrow W_n(S)$ . In particular, this makes  $W_n(S)$  a faithfully flat  $W_n(R)$ -Module and thus,  $\Phi_{R, K_R + \Delta, n}^e$  is injective if and only if the base change  $\Phi_{R, K_R + \Delta, n}^e \otimes_{W_n(R)} W_n(S)$  is. However, we've just shown that the latter morphism is simply  $\Phi_{S, K_S + \Delta_S, n}^e$ , whose injectivity defines the  $n$ -quasi- $F^e$ -purity of  $(S, \Delta_S)$ . Thus the result follows in the local case.

In the general case, as  $\varphi$  is finite there exists finitely many maximal ideals of  $S$  lying over  $R$ . Denote these  $\mathfrak{n}_1, \dots, \mathfrak{n}_\ell$ . As  $R \rightarrow S$  is étale the base change  $\widehat{R} \rightarrow \widehat{R} \otimes_R S$  is still étale, and by [Sta24, 04GH] one can decompose

$$\widehat{R} \otimes_R S \cong S_{\mathfrak{n}_1} \times \cdots \times S_{\mathfrak{n}_\ell}$$

Where each  $\widehat{R} \rightarrow \widehat{R} \otimes_R S \twoheadrightarrow S_{\mathfrak{n}_i}$  is a local étale map.  $\widehat{R} \otimes_R S$  has  $\ell$  maximal ideals, and localizing at each yields  $S_{\mathfrak{n}_i}$  for some  $i$ . As a consequence of corollary 3.27.1 and the fact

that quasi- $F^e$ -purity is a local property verifiable at the level of  $\max\text{Spec}$ , we have that

$$\text{ht}^e(R, \Delta) = \text{ht}^e(\widehat{R}, \Delta^\wedge) = \text{ht}^e(S_{n_i}, \Delta_S|_{S_{n_i}}) \ [\forall i \leq \ell] = \text{ht}^e(S, \Delta_S)$$

Where the second equality follows from the local case.  $\square$

Working through the definitions yields the following corollary:

**COROLLARY 3.40.1.** *For  $(R, \Delta)$  and  $(S, \Delta_S)$  as above,  $\text{ht}^\infty(R, \Delta) = \text{ht}^\infty(S, \Delta_S)$  for any  $e \in \mathbb{N}$  and  $(R, \Delta)$  is uniformly quasi- $F^\infty$  pure if and only if  $(S, \Delta_S)$  is.*

**REMARK 3.41.** *When  $e = 1$ , being  $n$ -Quasi- $F^e$ -split can be made explicit; the map  $R(\Delta) \rightarrow Q_{R, \Delta, n}^1 \cong F_* \overline{W}_n(R)(\Delta)$  assigning  $r \mapsto F_* \overline{[r^p]}$  admits a section. Given such a section  $\theta_{R, \Delta, n} : Q_{R, \Delta, n}^1 \rightarrow R(\Delta)$  exists on  $R$ , this lemma implies that such a section exists on  $S$ . Further, this section is explicitly the map  $W_n(S) \otimes_{W_n(R)} \theta_{R, \Delta, n}$ , i.e. the section can be constructed explicitly from base changing the section on  $R$ .*

Via parsing through definitions, similar results clearly hold for (uniform) Quasi- $F^\infty$ -purity. Quasi- $F^e$ -purity and (uniform) Quasi- $F^\infty$ -purity are local notions, so we can rephrase the above results globally as follows:

**THEOREM 3.42.** *Let  $f : Y \rightarrow X$  be a finite dominant morphism that is étale in codimension 1, where  $X$  and  $Y$  are excellent integral schemes which each admit dualizing sheaves  $\omega_Y, \omega_X$ . Choose  $\Delta \in \text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$  such that  $[\Delta] = 0$ . Choose  $\omega_Y, \omega_X$ , defining canonical divisors  $K_Y, K_X$ , and  $\Delta_Y := f^*(K_X + \Delta) - K_Y \in \text{Div}(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ .*

*Then,  $\text{ht}^e(X, \Delta) = \text{ht}^e(Y, \Delta_Y) \ \forall e \in \mathbb{N} \cup \{\infty\}$ . Further,  $(X, \Delta)$  is uniformly Quasi- $F^\infty$ -pure if and only if  $(Y, \Delta_Y)$  is.*

**PROOF.** We first remark that  $[\Delta_Y] = 0$  since  $f^*K_X \sim K_Y$ , as  $f$  is étale in codimension 1. Via [TWY24, remark 3.7], we can restrict  $X$  and  $Y$  to their étale loci and consider the corresponding restricted map  $f' : Y_{\text{étale}} \rightarrow X_{\text{étale}}$ ; thus without loss of generality, we can assume  $f$  is étale. Quasi- $F^e$ -purity is a local notion, so we can simply apply theorem 3.40 at

each point  $x \in X$ ,  $f(x) \in Y$  utilizing lemma 2.45. The statement for (uniform) Quasi- $F^\infty$  purity follows from their definitions and the fact that  $\text{ht}^e(X, \Delta) = \text{ht}^e(Y, \Delta_Y) \forall e \in \mathbb{N}$ .  $\square$

**COROLLARY 3.42.1.** *Let  $X$  be a  $\mathbb{Q}$ -Gorenstein integral normal excellent scheme where  $nK_X$  is Cartier for  $n$  relatively prime to  $p$ . Let  $\Delta_X$  be a  $\mathbb{Q}$ -divisor for which  $[\Delta_X] = 0$ . Taking the canonical cover  $X' \rightarrow X$ , with corresponding  $\mathbb{Q}$ -divisor  $\Delta_{X'}$  on  $X'$ , yields a Gorenstein normal excellent scheme for which  $\text{ht}^e(X', \Delta_{X'}) = \text{ht}^e(X, \Delta_X)$  for any  $e \in \mathbb{N} \cup \{\infty\}$ .*

**PROOF.** This follows from the fact that canonical covers are étale in codimension 1, see [Sin03].  $\square$

**REMARK 3.43.** *An analogous result does not hold in the global setting for Quasi- $F^e$ -Splittings (see example 2.24). Quasi- $F^e$ -Purity, unlike Quasi- $F^e$ -Splitting, is a local property that can be verified at the level of stalks. In our proof we crucially utilized the fact that étale morphisms are not only flat, but faithfully flat, a statement that holds only for local étale maps of local rings but not in general.*

### 3. Quasi- $F$ -Pure-Regularity and Quasi- $F$ -Rationality

Recall that an  $F$ -finite noetherian ring  $R$  is strongly  $F$ -regular if and only if, for any non-zerodivisor  $c$ , there exists an  $e > 0$  for which  $R \rightarrow F_*^e R$  assigning  $1 \mapsto F_*^e c$  splits. Requiring some large enough  $e > 0$  is a *weaker* condition here; in essentially every strongly  $F$ -regular ring  $1 \mapsto F_* c$  does not split for every  $c$ . When trying to define the “quasi” variant of strong  $F$ -regularity, however, splitting off of a higher iterate is a *stronger* condition. Thus, one must be careful when defining quasi- $F$ -regularity.

In this section we will define quasi- $F$ -regularity, show that 1-quasi- $F$ -regularity is equivalent to global  $F$ -regularity, and construct a quasi-test ideal in the affine setting. This is an analogue of the test ideal discussed in chapter 1 section 1.7 that will be used to test for quasi- $F$ -regularity. We will then define an analogue of quasi- $F$ -regularity for the non- $F$ -finite setting, analogous to  $F$ -pure regularity develop in chapter 1 section 2. Finally, we

will introduce quasi- $F$ -rationality and its associated quasi test submodule, showing that it is well defined in the non- $F$ -finite setting and compatible with localization, completion, and étale extension.

REMARK 3.44. *Many of the constructions in this section, alongside basic results about them, are similar to those in [KTT<sup>+</sup>24b]. The analogous results are cited in the lemma and theorem statements in this thesis for posterity, though many of the proofs differ in subtle ways to remove the  $F$ -finite requirements of their analogues.*

### 3.1. Quasi- $F$ -Regularity.

DEFINITION 3.45. Let  $X$  be an integral normal  $F$ -finite  $\mathbb{F}$ -scheme such that  $W_n X$  is an excellent scheme admitting a dualizing complex. Let  $\Delta$  be a  $\mathbb{Q}$ -divisor on  $X$  such that  $[\Delta] = 0$ . Then  $(X, \Delta)$  is **Globally  $n$ -Quasi- $F$ -Regular** if for any effective  $\mathbb{Q}$ -divisor  $E$  there exists  $\varepsilon \in \mathbb{Q}_{>0}$  such that  $(X, \Delta + \varepsilon E)$  is  $n$ -quasi- $F^e$ -split for every  $e > 0$ .  $(X, \Delta)$  is **Globally Quasi- $F$ -Regular** if it is  $n$ -quasi- $F$ -regular for some  $n$ .

DEFINITION 3.46.  $(X, \Delta)$  as above is **locally  $n$ -Quasi- $F$ -Regular** (resp. **locally Quasi- $F$ -Regular**) if there exists an open cover  $X = \bigcup U_i$  for which  $(U_i, \Delta|_{U_i})$  is  $n$ -Quasi- $F$ -Regular (resp. Quasi- $F$ -Regular).

The definition of global- $F$ -regularity, reminiscent of corollary 1.71.1, requires you to find an  $n$  such that there exists an  $E$  and  $\varepsilon$  satisfying certain conditions. An a priori weaker condition would be, for any effective  $\mathbb{Q}$ -divisor  $E$ , requiring that there exists an  $n$  and  $\varepsilon$  for which  $(X, \Delta + \varepsilon E)$  is  $n$ -quasi- $F^e$ -split for all  $e > 0$ . This is precisely the definition of feeble quasi- $F$ -regularity.

DEFINITION 3.47. Let  $X$  be an integral normal  $F$ -finite  $\mathbb{F}$ -scheme such that  $W_n X$  is an excellent scheme admitting a dualizing complex. Let  $\Delta$  be a  $\mathbb{Q}$ -divisor on  $X$  such that  $[\Delta] = 0$ . Then  $(X, \Delta)$  is **Feebly Quasi- $F$ -Regular** if for any effective  $\mathbb{Q}$ -divisor  $E$  there exists  $n \in \mathbb{N}, \varepsilon \in \mathbb{Q}_{>0}$  such that  $(X, \Delta + \varepsilon E)$  is  $n$ -quasi- $F^e$ -split for every  $e > 0$ .

Quasi- $F$ -regularity implies its feeble variant, but if  $X$  is affine and  $\mathbb{Q}$ -Gorenstein, then these notions are equivalent [KTT<sup>+</sup>24b, Theorem 4.34]; we will not focus too much on this feeble condition.

In [KTT<sup>+</sup>24b, section 4], the authors prove that quasi- $F$ -regularity of  $(X, \Delta)$  (when  $X = \text{Spec}(R)$  is affine) can be checked at a single effective  $\mathbb{Q}$ -divisor  $E$ , provided that  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier.

REMARK 3.48.  *$K_X + \Delta$  being  $\mathbb{Q}$ -Cartier is not a difficult condition to satisfy in practice. In [SS10], Schwede and Smith show that for any sharply globally  $F$ -split variety and  $\mathbb{Q}$ -divisor pair  $(X, \Delta)$ , there exists an effective  $\mathbb{Q}$ -divisor  $B$  for which  $(X, \Delta + B)$  is sharply globally  $F$ -split and  $K_X + \Delta + B$  is  $\mathbb{Q}$ -trivial. The author is not aware of an analogous result in the quasi setting.*

Quasi- $F$ -regularity can be computed in practice. Work of Yoshikawa [Yos25] has extended the quasi-Fedder's criterion (theorem 2.51) discussed in chapter 2 section 4.1 to a variant which detects iterative splittings and quasi- $F$ -regularity. This can be also be seen as a generalization of Glassbrenner's Criterion (theorem 1.42) to the quasi setting. Unlike the quasi version of Fedder's Criterion, however, the author is not aware of a public implementation of this criterion in a computer algebra system.

There are still some properties of strong- $F$ -regularity that are not known to hold for its weaker counterpart in quasi- $F$ -regularity. For instance, strongly  $F$ -regular rings are always Cohen-Macaulay (proposition 1.38), but it is not known that quasi- $F$ -regular rings are Cohen-Macaulay in dimensions larger than 3 [KTT<sup>+</sup>24b, Theorem 8.9].

Using the existence of these quasi versions of big test elements, the authors of [KTT<sup>+</sup>24b] construct a quasi-test ideal which tests for quasi- $F$ -regularity. We will define some intermediate objects (that they introduced in section 4.2 of their paper) in this subsection, careful not to use the  $F$ -finite hypothesis when not required. later, we will use these intermediate

objects to construct the test ideal for non- $F$ -finite rings, and show it detects the non- $F$ -finite analogue of quasi- $F$  regularity, dubbed quasi- $F$ -pure regularity.

Let  $R$  be an excellent normal ring and  $\Delta$  a  $\mathbb{Q}$ -divisor on  $R$ . Similarly to the definition of  $Q_{R,\Delta,n}^e$ , for a nonzero divisor  $c \in R^\circ$  we can define the  $W_n(R)$ -Module  $Q_{R,\Delta,n}^{e,c}$  as the pushout of the following diagram [KTT<sup>+</sup>24b, 4.13.1]:

$$\begin{array}{ccccc}
 W_n(R)(\Delta) & \xrightarrow{F^e} & F_*^e W_n(R)(p^e \Delta) & \xrightarrow{\cdot F_*^e [c]} & F_*^e W_n(R)(p^e \Delta) \\
 \mathfrak{A}^{n-1} \downarrow & & \downarrow & & \downarrow \\
 R(\Delta) & \xrightarrow{\Phi_{R,\Delta,n}^e} & Q_{R,\Delta,n}^e & \xrightarrow{\quad \quad \quad} & Q_{R,\Delta,n}^{e,c} \\
 & \searrow & & \nearrow & \\
 & & \Phi_{R,\Delta,n}^{e,c} & & 
 \end{array}$$

REMARK 3.49. Using the following short exact sequence associated to  $Q_{R,\Delta,n}^{e,c}$ ,

$$0 \longrightarrow F_* W_{n-1}(R)(p\Delta) \xrightarrow{(\cdot F_*^e [c]) \circ V F^e} F_*^e W_n(R)(p^e \Delta) \longrightarrow Q_{R,\Delta,n}^{e,c} \longrightarrow 0$$

One can mimic the proof of Theorem 3.27 to show that

$$H_{[m]}^d(Q_{R,\Delta,n}^{e,c}) \cong H_{[m]}^d(Q_{R,\Delta,n}^{e,c} \wedge)$$

As  $W_n(\widehat{R})$ -Modules, for  $R$  a normal domain admitting a dualizing complex that completes to a normal domain. Similarly, for an étale map  $R \rightarrow S$  as in setting 3.37 one can mimic the argument of proposition 3.39 to show that

$$Q_{R,\Delta,n}^{e,c} \otimes_{W_n(R)} W_n(S) \cong Q_{S,\Delta_S,n}^{e,c}$$

DEFINITION 3.50. Consider  $Q_{R,\Delta,n}^{e,c}$  for  $R$  an excellent normal ring,  $\Delta$  a  $\mathbb{Q}$ -divisor on  $R$ ,  $e, n \in \mathbb{N}$ , and  $c \in R^\circ$ . Using this module, we record the following definitions (see [KTT<sup>+</sup>24b, Definition 4.14]):

$$(1) \widetilde{K}_{\Delta,n}^{e,c} := \ker \left( \mathrm{H}_{[m]}^d(W_n(R)(\Delta)) \xrightarrow{F^e} \mathrm{H}_{[m]}^d(F_*^e W_n(R)(p^e \Delta)) \xrightarrow{\cdot F_*^e [c]} \mathrm{H}_{[m]}^d(F_*^e W_n(R)(p^e \Delta)) \right)$$

This is a  $W_n(R)$ -submodule of  $\mathrm{H}_{[m]}^d(W_n(R)(\Delta))$ . Using this, we define

$$\begin{aligned} \widetilde{0}_{R,\Delta,n}^* &:= \bigcup_{\substack{c \in R^\circ \\ e_0 \in \mathbb{Z}_{>0}}} \bigcap_{e \geq e_0} \widetilde{K}_{\Delta,n}^{e,c} \subseteq \mathrm{H}_{[m]}^d(W_n(R)(\Delta)) \\ &= \left\{ z \in \mathrm{H}_{[m]}^d(W_n(R)(\Delta)) \mid \exists c \in R^\circ, e_0 \in \mathbb{N} \text{ such that } z \in \widetilde{K}_{\Delta,n}^{e,c} \ \forall e \geq e_0 \right\} \end{aligned}$$

It can be checked that  $\widetilde{0}_{R,\Delta,n}^*$  is also a  $W_n(R)$ -submodule of  $\mathrm{H}_{[m]}^d(W_n(R)(\Delta))$  (cf.

[KTT<sup>+</sup>24b, Remark 3.17])

$$(2) K_{\Delta,n}^{e,c} := \ker \left( \mathrm{H}_m^d(R(\Delta)) \xrightarrow{\Phi_{R,\Delta,n}^{e,c}} \mathrm{H}_{[m]}^d(Q_{R,\Delta,n}^{e,c}) \right) \subseteq \mathrm{H}_m^d(R(\Delta))$$

This is an  $R$ -submodule of  $\mathrm{H}_m^d(R(\Delta))$ . Similarly to (1), we define the **n-quasi-**

**tight-closure**  $0_{R,\Delta,n}^*$  as follows:

$$\begin{aligned} 0_{R,\Delta,n}^* &:= \bigcup_{\substack{c \in R^\circ \\ e_0 \in \mathbb{Z}_{>0}}} \bigcap_{e \geq e_0} K_{\Delta,n}^{e,c} \subseteq \mathrm{H}_m^d(R(\Delta)) \\ &= \left\{ z \in \mathrm{H}_m^d(R(\Delta)) \mid \exists c \in R^\circ, e_0 \in \mathbb{N} \text{ such that } z \in K_{\Delta,n}^{e,c} \ \forall e \geq e_0 \right\} \end{aligned}$$

It can be checked that  $0_{R,\Delta,n}^*$  is also an  $R$ -submodule of  $\mathrm{H}_m^d(R(\Delta))$ .

When  $\Delta = 0$  we write  $\widetilde{K}_n^{e,c}$ ,  $K_n^{e,c}$  and  $\widetilde{0}_{R,n}^*$ ,  $0_{R,n}^*$  for simplicity.

The  $W_n(R)$ -module variants and  $R$ -module variants are related to each other via restriction.

PROPOSITION 3.51 ([KTT<sup>+</sup>24b], Remark 4.15, Proposition 4.16, 4.17). *Let  $(R, \mathfrak{m})$  be an excellent normal local ring,  $\Delta$  a  $\mathbb{Q}$ -divisor on  $R$ ,  $e, n \in \mathbb{N}$ , and  $c \in R^\circ$ .*

$$(a) \mathfrak{R}^{n-1} \left( \widetilde{K}_{\Delta,n}^{e,c} \right) = K_{\Delta,n}^{e,c}.$$

$$(b) \mathrm{H}_m^d(R(\Delta)) \supseteq 0_{R,\Delta,1}^* \supseteq 0_{R,\Delta,2}^* \supseteq \dots$$

$$(c) \widetilde{0}_{R,\Delta,1}^* = 0_{R,\Delta,1}^*, \text{ and this agrees with the tight closure module discussed in remark 1.58 for } \Delta = 0.$$

$$(d) \mathfrak{R}(\widetilde{0}_{R,\Delta,n+1}^*) \subseteq \widetilde{0}_{R,\Delta,n}^*.$$

$$(e) V^{-1}(\widetilde{0_{R,\Delta,n}^*}) \supseteq F_* \widetilde{0_{R,p\Delta,n-1}^*}.$$

If  $\Delta$  is in addition  $\mathbb{Q}$ -Cartier, (e) is an equality and for  $e, m \in \mathbb{N}$ , and  $t \in R^\circ$  chosen so that of  $[t] \cdot 0_{R,\Delta,1}^* = 0$  we have

$$(1) [t^2] \cdot \widetilde{0_{R,\Delta,n}^*} = 0.$$

$$(2) [t^2] \cdot \ker(F^m : H_m^d(W_n(R)(\Delta)) \rightarrow H_m^d(F_*^m W_n(R)(p^m \Delta))) = 0.$$

PROOF. To prove (a), the restriction morphism is always surjective. The exactness of the columns of the following diagram show injectivity.

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \downarrow & & \downarrow \\ & 0 & \longrightarrow & H_{[m]}^d(F_* W_{n-1}(R)) & \xrightarrow{\cong} & H_{[m]}^d(F_* W_{n-1}(R)) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \widetilde{K_{\Delta,n}^{e,c}} & \longrightarrow & H_{[m]}^d(W_n(R)(\Delta)) & \xrightarrow{(\cdot F_*^e c) \circ F^e} & H_{[m]}^d(F_*^e W_n(R)(p^e \Delta)) \\ & \downarrow \mathfrak{R}^{n-1} & & \downarrow \mathfrak{R}^{n-1} & & \downarrow \\ 0 & \longrightarrow & K_{\Delta,n}^{e,c} & \longrightarrow & H_m^d(R(\Delta)) & \longrightarrow & H_{[m]}^d(Q_{R,\Delta,n}^{e,c}) \end{array}$$

For (b), pick  $z \in 0_{R,\Delta,n+1}^*$ . choose  $c \in R^\circ, e_0 \in \mathbb{N}$  such that  $z \in K_{\Delta,n+1}^{e,c}$  for all  $e \geq e_0$ . This means that for all  $e \geq e_0$ ,  $\Phi_{R,\Delta,n+1}^{e,c}(z) = 0$ . Well,  $\Phi_{R,\Delta,n}^{e,c} = \mathfrak{R} \circ \Phi_{R,\Delta,n+1}^{e,c}$ , so  $\Phi_{R,\Delta,n}^{e,c}(z) = 0$  for all  $e \geq e_0$ . it follows that  $z \in 0_{R,\Delta,n}^*$  as desired. For (c), notice that  $0_{H_m^d(R)}^*$  consists of all elements  $[x] \in H_m^d(R)$  such that  $F^e[x]$  is annihilated by every  $c$  outside of a minimal prime and  $e$  large.  $\Phi_{R,1}^{e,c}$  is precisely the map assigning  $1 \mapsto F_*^e c$ , as we saw in the definition of strong- $F$ -regularity; these are exactly the same set after unpacking definitions.

We note that the analog of (c), [**KTT**<sup>+</sup>**24b**, Remark 4.15(3)], is stated without the  $\Delta = 0$  condition; we do not reproduce their variant for ease of exposition, but their proof does not require  $F$ -finiteness.

(d) follows immediately from the definitions, and (e) follows from the definitions as  $V(F_* \widetilde{0_{R,p\Delta,n-1}^*}) \subseteq \widetilde{0_{R,\Delta,n}^*}$ . Checking that (e) is an equality when  $\Delta$  is  $\mathbb{Q}$ -Cartier is a fairly complicated diagram chase which we omit, though we stress that the  $F$ -finite hypothesis was

not used.

$\ker(F^m : H_{\mathfrak{m}}^d(W_n(R)(\Delta)) \rightarrow H_{\mathfrak{m}}^d(F_*^m W_n(R)(p^m \Delta)))$  is contained in  $\widetilde{0_{R,\Delta,n}^*}$ , so (2) follows from (1). Thus, it is sufficient to show (1) to complete the proof of this proposition, which we will show via induction on  $n$ . The  $n = 1$  case follows from the hypothesis that  $[t] \cdot 0_{R,\Delta,1}^* = 0$  and the previous lemma, which implies that  $0_{R,\Delta,1}^* = \widetilde{0_{R,\Delta,1}^*}$ . Thus, we move forward with the inductive step. Fix  $z \in \widetilde{0_{R,\Delta,n}^*}$ ; our goal is to show it is annihilated by  $t^2$ .  $\mathfrak{R}^{n-1}(z) \in \widetilde{0_{R,\Delta,n}^*}$  via (d), and hence  $\mathfrak{R}^{n-1}([t] \cdot z) = t \cdot \mathfrak{R}^{n-1}(z) = 0$  by the inductive hypothesis.  $\ker(\mathfrak{R}^{n-1}) = \text{im}(V)$ , so there exists  $z'$  such that  $V(F_* z') = [t]z$ .  $[t]z \in \widetilde{0_{R,\Delta,n}^*}$  and hence  $F_* z' \in V^{-1}(\widetilde{0_{R,\Delta,n}^*}) = F_* \widetilde{0_{R,p\Delta,n-1}^*}$  by the more general form of (e), as  $\Delta$  is  $\mathbb{Q}$ -Cartier. Thus  $z' \in \widetilde{0_{R,p\Delta,n-1}^*}$ , so by the induction hypothesis  $[t^2] \cdot z' = 0$ . Therefore

$$[t^2]z = [t]V(F_* z') = V([t]F_* z') = V(F_* [t^p]z) = 0$$

And the claim follows.  $\square$

Claim (1) and (2) can be slightly augmented, as we will see in the following lemma. We omit the proof as it is quite similar to the proof of (1) and (2) above.

LEMMA 3.52 ([KTT<sup>+</sup>24b], Proposition 4.18). *Let  $(R, \mathfrak{m})$  be an excellent normal local ring,  $\Delta$  a  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor on  $R$ ,  $e, n \in \mathbb{N}$ , and  $t \in R^\circ \cap \tau(R)$ . Suppose there exists a  $\mathbb{Q}$ -Cartier Weil divisor  $D$  and  $g \in R^\circ$  such that  $\Delta \leq D \leq \Delta + \text{div}(g)$ . Then, the following hold:*

$$(1) [t^2 g] \cdot \widetilde{0_{R,\Delta,n}^*} = 0.$$

$$(2) [t^2 g] \cdot \ker(F^m : H_{\mathfrak{m}}^d(W_n(R)(\Delta)) \rightarrow H_{\mathfrak{m}}^d(F_*^m W_n(R)(p^m \Delta))) = 0.$$

REMARK 3.53. *In the original statement of lemma 3.52,  $t$  must be picked in  $\tau(R)$ , typically taken to be the (non-finitistic) test ideal of  $R$ . Due to the existence of strong test elements for  $F$ -pure regularity (theorem 1.89), we can take  $\tau(R)$  to be the ideal of such test elements in  $R$ , the  $F$ -pure regular test ideal of  $R$ . This agrees with the test ideal in the  $F$ -finite setting but lets us adequately generalize to the non- $F$ -finite setting. In an abuse*

of notation going forward, we will always let  $\tau(R)$  be this ideal of strong test elements for  $F$ -pure regularity.

PROPOSITION 3.54 ([KTT<sup>+</sup>24b], Proposition 4.19). *Let  $(R, \mathfrak{m})$  be an excellent normal local ring,  $\Delta = a \cdot \text{div}(g)$  a  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor on  $R$ ,  $t \in R^\circ \cap \tau(R)$ , and  $q$  chosen so that  $[q] \cdot 0_{R,\Delta,1}^* = 0$ . Fix  $c \in R^\circ \cap ((g^2t^4) \cup (q^4))$ . Then,  $H_{\mathfrak{m}}^d(W_n(R)(\Delta)) \supseteq \widetilde{K}_{\Delta,n}^{0,c} \supseteq \widetilde{K}_{\Delta,n}^{1,c} \supseteq \dots$  and  $\widetilde{0}_{R,\Delta,n}^* = \bigcap_{e \geq 0} \widetilde{K}_{\Delta,n}^{e,c}$ .*

PROOF. We first prove the case where  $c \in R^\circ \cap (g^2t^4)$ . Choose  $c' \in R^\circ$  such that  $c = c'g^2t^4$ . We first show that  $\widetilde{K}_{\Delta,n}^{e,c} \supseteq \widetilde{K}_{\Delta,n}^{e+1,c}$ . Set  $\Delta_e := [ap^e]\text{div}(g)$  and take  $z \in \widetilde{K}_{\Delta,n}^{e+1,c}$ . Setting  $c_1 := (c')^{p-1}g^{p-2}t^{2p-4}$ , we have that  $cc_1 = (c')^pg^pt^{2p}$ , yielding

$$0 = F_*^e[cc_1] \cdot F^{e+1}(z) = F_*^{e+1}[c'pg^pt^{2p}] \cdot F^{e+1}(z) = F(F_*^e[c'gt^2] \cdot F^e(z))$$

Where, by construction of  $z$ ,  $F_*^e[c'gt^2] \cdot F^e(z) \in H_{\mathfrak{m}}^d(F_*^e R(p^e\Delta))$ . As  $p^e\Delta \leq \Delta_e \leq p^e\Delta + \text{div}(g)$ , one can apply lemma 3.52 to get that  $[t^2g] \cdot \widetilde{0}_{R,p^e\Delta,n}^* = 0$ . Therefore,

$$0 = F_*^e[t^2g] \cdot F_*^e[c'gt^2] \cdot F_*^e(z) = F_*^e[c'g^2t^4] \cdot F_*^e(z) = F_*^e[c] \cdot F_*^e(z)$$

and thus  $z \in \widetilde{K}_{\Delta,n}^{e,c}$  as desired. We now prove the second claim: we now pick  $z \in \widetilde{0}_{R,\Delta,n}^*$  and  $e \geq 0$ . it follows that

$$\begin{aligned} F_*^e[c] \cdot F^e(z) &\in F_*^e[c] \cdot F^e(\widetilde{0}_{R,\Delta,n}^*) \\ &\subseteq_*^e [c] \cdot F_*^e \widetilde{0}_{R,p^e\Delta,n}^* \\ &= F_*^e \left( [c'g^2t^4] \cdot \widetilde{0}_{R,p^e\Delta,n}^* \right) \\ &= F_*^e \left( [c't^2] \cdot [gt^2] \cdot \widetilde{0}_{R,p^e\Delta,n}^* \right) \end{aligned}$$

Where  $F_*^e \left( [c't^2] \cdot [gt^2] \cdot \widetilde{0}_{R,p^e\Delta,n}^* \right) = 0$  as a consequence of lemma 3.52. Therefore  $z \in \bigcap_{e \geq 0} \widetilde{K}_{\Delta,n}^{e,c}$  as desired. The converse inclusion follows from the definition of  $\widetilde{0}_{R,\Delta,n}^*$ . The  $c \in R^\circ \cap (q^4)$  follows from applying proposition 3.51(1) rather than lemma 3.52.  $\square$

REMARK 3.55. In the original (*F*-finite) statement of [KTT<sup>+</sup>24b, Proposition 4.19], the authors prove that  $\widetilde{0_{R,\Delta,n}^*} = \widetilde{K_{\Delta,n}^{e,c}}$  for a single copy of  $\widetilde{K_{\Delta,n}^{e,c}}$  for  $e \gg 0$ , rather than taking the intersection. This crucially relied on  $H_{[\mathfrak{m}]}^d(W_n(R)(\Delta))$  being an Artinian  $W_n(R)$ -module, and hence  $\bigcap_{e \geq 0} \widetilde{K_{\Delta,n}^{e,c}} = \widetilde{K_{\Delta,n}^{e_0,c}}$  for  $e_0$  sufficiently large. This need not follow here: when  $R$  is not *F*-finite,  $W_n(R)(\Delta)$  need not be finitely generated as a module over  $W_n(R)$  (lemma 3.1), where  $W_n(R)$  itself is not Noetherian. Therefore,  $H_{[\mathfrak{m}]}^d(W_n(R)(\Delta))$  need not be Artinian. We will show that equality holds on the nose for a specific  $e$  when removing the “ $\widetilde{(-)}$ ” in proposition 3.57.

PROPOSITION 3.56 ([KTT<sup>+</sup>24b], Theorem 4.20). Let  $(R, \mathfrak{m})$  be an excellent normal local ring and  $\Delta$  a  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor on  $R$ . Then  $\mathfrak{R}^{n-1}(\widetilde{0_{R,\Delta,n}^*}) = 0_{R,\Delta,n}^*$ .

PROOF. We show the  $\subseteq$  direction first. Choose  $z \in \widetilde{0_{R,\Delta,n}^*}$ . As a result of this, we can choose  $e_0 \geq 0$  and  $c \in R^\circ$  such that  $z \in \bigcap_{e \geq e_0} \widetilde{K_{\Delta,n}^{e,c}}$ . It follows that

$$\mathfrak{R}^{n-1}(z) \in \mathfrak{R}^{n-1} \left( \bigcap_{e \geq e_0} \widetilde{K_{\Delta,n}^{e,c}} \right) \subseteq \bigcap_{e \geq e_0} \mathfrak{R}^{n-1} \left( \widetilde{K_{\Delta,n}^{e,c}} \right) = \bigcap_{e \geq e_0} K_{\Delta,n}^{e,c}$$

With the last equality an application of proposition 3.51(a). This implies that  $\mathfrak{R}^{n-1}(z) \in 0_{R,\Delta,n}^*$ . We now show the opposite direction. Take  $z \in 0_{R,\Delta,n}^*$ ,  $t \in \tau(R) \cap R^\circ$ , and  $\Delta = a \cdot \text{div}(g)$  by the fact that  $\Delta$  is  $\mathbb{Q}$ -Cartier. There exists a  $c \in R^\circ$  and  $e_1 > 0$  such that  $z \in \bigcap_{e \geq e_1} K_{\Delta,n}^{e,c}$ . Replacing  $c$  with  $cg^{2t^4}$  as necessary, we obtain that  $\widetilde{0_{R,\Delta,n}^*} = \bigcap_{e \geq 0} \widetilde{K_{\Delta,n}^{e,c}}$  by proposition 3.54.

$$z \in \bigcap_{e \geq e_1} K_{\Delta,n}^{e,c} = \bigcap_{e \geq e_1} \mathfrak{R}^{n-1} \left( \widetilde{K_{\Delta,n}^{e,c}} \right)$$

By applying proposition 3.51(a) again. We note that the following chains are descending chains of  $R$ -submodules of  $H_{\mathfrak{m}}^d(R(\Delta))$  :

$$\begin{aligned} \bigcap_{e_1+1 \geq e \geq e_1} \mathfrak{R}^{n-1} \left( \widetilde{K_{\Delta,n}^{e,c}} \right) &\supseteq \bigcap_{e_1+2 \geq e \geq e_1} \mathfrak{R}^{n-1} \left( \widetilde{K_{\Delta,n}^{e,c}} \right) \supseteq \dots \\ \mathfrak{R}^{n-1} \left( \bigcap_{e_1+1 \geq e \geq e_1} \widetilde{K_{\Delta,n}^{e,c}} \right) &\supseteq \mathfrak{R}^{n-1} \left( \bigcap_{e_1+2 \geq e \geq e_1} \widetilde{K_{\Delta,n}^{e,c}} \right) \supseteq \dots \end{aligned}$$

Because  $H_m^d(R(\Delta))$  is Artinian, both of these chains halt with an upper bound on the intersection index  $e_1 + n_1$  and  $e_1 + n_2$  for  $n_1, n_2 \geq 0$ . Let  $e_2 = e_1 + \max(n_1, n_2)$ . It follows that both of these chains stabilize at  $\mathfrak{R}^{n-1}(\widetilde{K_{\Delta,n}^{e_2,c}})$ , and hence:

$$z \in \bigcap_{e \geq e_1} \mathfrak{R}^{n-1}(\widetilde{K_{\Delta,n}^{e,c}}) = \mathfrak{R}^{n-1}(\widetilde{K_{\Delta,n}^{e_2,c}}) = \mathfrak{R}^{n-1}\left(\bigcap_{e \geq e_1} \widetilde{K_{\Delta,n}^{e,c}}\right) = \mathfrak{R}^{n-1}(\widetilde{0_{R,\Delta,n}^*})$$

Proving the result.  $\square$

We use this result to prove the following:

**PROPOSITION 3.57** ([**KTT<sup>+</sup>24b**], Proposition 4.21). *Let  $(R, \mathfrak{m})$  be an excellent normal local ring,  $\Delta = a \cdot \text{div}(g)$  a  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor on  $R$ ,  $t \in R^\circ \cap \tau(R)$ , and  $q$  chosen so that of  $[q] \cdot 0_{R,\Delta,1}^* = 0$ . Fix  $c \in R^\circ \cap ((g^2t^4) \cup (q^4))$ . Then, the following hold:*

- (1)  $H_m^d(R(D)) \supseteq K_{\Delta,n}^{0,c} \supseteq K_{\Delta,n}^{1,c} \supseteq \dots$
- (2) *There exists an integer  $e_0 \geq 0$  such that  $0_{R,\Delta,n}^* = K_{\Delta,n}^{e_0,c}$  for all  $e \geq e_0$ .*
- (3) *If there exists an  $e_2 \geq 0$  such that  $K_{\Delta,n}^{e_2,c} = 0$ , then  $0_{R,\Delta,n}^* = 0$ .*

**PROOF.** Part (1) follows from proposition 3.51(a) and proposition 3.54. To prove part (2), applying both the previous propositions yields

$$0_{R,\Delta,n}^* = \mathfrak{R}^{n-1}(\widetilde{0_{R,\Delta,n}^*}) = \bigcap_{e \geq 0} \mathfrak{R}^{n-1}(\widetilde{K_{\Delta,n}^{e,c}})$$

As  $\mathfrak{R}^{n-1}(\widetilde{K_{\Delta,n}^{e,c}})$  are submodule of an Artinian module for all  $e$ , it follows that for some  $e_0 \gg 0$ ,  $\bigcap_{e \geq 0} \mathfrak{R}^{n-1}(\widetilde{K_{\Delta,n}^{e,c}}) = \mathfrak{R}^{n-1}(\widetilde{K_{\Delta,n}^{e_0,c}})$ . Putting this together with proposition 3.51(a), we have that

$$0_{R,\Delta,n}^* = \mathfrak{R}^{n-1}(\widetilde{0_{R,\Delta,n}^*}) = \bigcap_{e \geq 0} \mathfrak{R}^{n-1}(\widetilde{K_{\Delta,n}^{e,c}}) = \mathfrak{R}^{n-1}(\widetilde{K_{\Delta,n}^{e_0,c}}) = K_{\Delta,n}^{e_0,c}$$

Thus (2) is proven. (1) and (2) immediately imply (3).  $\square$

Part (2) shows that  $c$  chosen as in proposition 3.54 is a “quasi”-strong test element in the same sense as in chapter 1 subsection 1.7.

**3.2. Definition of Quasi- $F$ -Pure Regularity and the Quasi Test Ideal.** As in the case with (Quasi)- $F$ -Purity, it would be pertinent in the non- $F$ -finite setting to define a notion of (Quasi)- $F$ -Regularity. For rings that are not necessarily  $F$ -finite or Noetherian, there exists various augmentations of the strong  $F$ -regularity definition in literature (see chapter 1 section 2.2).

This motivates a similar augmentation to our definition of Quasi- $F$ -Regularity.

DEFINITION 3.58. Let  $(R, \mathfrak{m})$  be an excellent normal local ring admitting a dualizing complex that need not be  $F$ -finite, and  $\Delta$  an effective  $\mathbb{Q}$ -divisor on  $R$ .

- $(R, \Delta)$  is  **$n$ -Quasi- $F$ -Pure Regular** if for every effective  $\mathbb{Q}$ -divisor  $E$  on  $R$ ,  $\exists \varepsilon > 0$  such that  $(R, \Delta + \varepsilon E)$  is  $n$ -Quasi- $F^e$ -Pure  $\forall \varepsilon > 0$ .
- $(R, \Delta)$  is **Quasi- $F$ -Pure Regular** if it is  $n$ -Quasi- $F$ -Pure Regular for some  $n$ .
- Let  $X$  be an excellent normal scheme admitting a dualizing complex  $\omega_X^\bullet$  and  $\Delta$  an effective  $\mathbb{Q}$ -divisor on  $X$ .  $(X, \Delta)$  is **Quasi- $F$ -Pure Regular** (is  **$n$ -Quasi- $F$ -Pure Regular**) if  $\mathcal{O}_{X,x}$  is Quasi- $F$ -Pure Regular (is  $n$ -Quasi- $F$ -Pure Regular)  $\forall x \in X$ .

REMARK 3.59. *We omit the “locally” adverb in the definition for quasi- $F$ -pure regularity for a scheme. We will show that quasi- $F$ -pure regularity localizes via a test ideals approach later in this subsection, so the property holding on stalks is equivalent to it holding on an affine open cover as a consequence of lemmas 2.45 and 2.46.*

When  $R$  is  $F$ -finite, equivalence between Quasi- $F^e$ -Purity and Quasi- $F^e$ -Splitting immediately implies that Quasi- $F$ -regularity and Quasi- $F$ -Pure Regularity coincide in the affine setting. In the global setting, local quasi- $F$ -pure regularity is analogous to local quasi- $F$ -regularity by similar reasoning and the previous remark.

DEFINITION 3.60. Using the modules from the preceding subsection we define the  **$n$ -quasi-test ideal**  $\tau_n(R, \Delta)$  and the **quasi-test ideal**  $\tau^q(R, \Delta)$  for a local excellent normal

ring  $R$  as follows:

$$\tau_n(R, \Delta) := \left( \frac{\mathrm{H}_m^d(R(K_R + \Delta))}{0_{R, K_R + \Delta, n}^*} \right)^\vee$$

$$\tau^q(R, \Delta) := \bigcup_{n > 0} \tau_n(R, \Delta) = \tau_N(R, \Delta) \quad (N \gg 0)$$

Where  $(-)^{\vee} = \mathrm{Hom}_R(-, E_R)$  is taking the Matlis dual.

Via [KTT<sup>+</sup>24b, Theorem 4.27] this is equivalent to the definition given in the  $F$ -finite setting, provided that  $K_R + \Delta$  is  $\mathbb{Q}$ -Cartier.  $0_{R, \Delta, 1}^* \supseteq 0_{R, \Delta, 2}^* \supseteq \dots$  by proposition 3.51, which determines a surjection  $\frac{\mathrm{H}_m^d(R(K_R + \Delta))}{0_{R, K_R + \Delta, n+1}^*} \twoheadrightarrow \frac{\mathrm{H}_m^d(R(K_R + \Delta))}{0_{R, K_R + \Delta, n}^*}$  for every  $n$ , and hence by Matlis duality, a series of inclusions

$$\tau_1(R, \Delta) \subseteq \tau_2(R, \Delta) \subseteq \dots \subseteq \tau_n(R, \Delta) \subseteq \tau_{n+1}(R, \Delta) \subseteq \dots \subseteq R(-\Delta)$$

Which by ACC implies there exists an  $n_0 \gg 0$  such that  $\tau^q(R, \Delta) = \tau_n(R, \Delta)$  for all  $n \geq n_0$ .

REMARK 3.61. *As a consequence of proposition 3.57, if we choose  $c$  well enough we can replace  $0_{R, K_R + \Delta, n}^*$  with a single copy of  $K_{K_R + \Delta, n}^{e, c}$ , provided that  $K_R + \Delta$  is  $\mathbb{Q}$ -Cartier. Explicitly, let  $K_R + \Delta = a \cdot \mathrm{div}(g)$  for  $a \in \mathbb{Q}$  and  $g \in R^\circ$ . For  $t \in \tau(R) \cap R^\circ$  (e.g. a test element that is a non-zero divisor) and  $q$  such that  $[q] \cdot 0_{R, K_R + \Delta, 1}^* = 0$ , if  $c \in R^\circ \cap ((g^2 t^4) \cup (q^4))$ , then  $\exists e_1 \geq 0$  such that  $0_{R, K_R + \Delta, n}^* = K_{K_R + \Delta, n}^{e, c}$  for every  $e \geq e_1$ . Thus, if  $K_R + \Delta$  is  $\mathbb{Q}$ -Cartier there exists  $e_0 \in \mathbb{N}$  such that*

$$\tau_n(R, \Delta) = \left( \frac{\mathrm{H}_m^d(R(K_R + \Delta))}{K_{K_R + \Delta, n}^{e, c}} \right)^\vee$$

for all  $e \geq e_0$  and  $c \in R^\circ$  chosen well.

LEMMA 3.62. *Let  $R$  be a local excellent normal domain admitting a dualizing complex, and  $\Delta$  a  $\mathbb{Q}$ -divisor on  $R$  such that  $\lfloor \Delta \rfloor = 0$ . Then*

$$K_{K_R + \Delta, n}^{e, c} \otimes_R R_P \cong K_{K_{R_P} + (R_P \otimes_R \Delta), n}^{e, c}$$

for any  $P \in \text{Spec}(R)$  as  $R_P$ -modules, and

$$K_{K_R + \Delta, n}^{e, c} \cong_{K_R + \Delta, n}^{e, c} \wedge \cong K_{\widehat{R} + \Delta, n}^{e, c}$$

as  $\widehat{R}$ -modules.

PROOF. These can be seen by similar exact sequence arguments. Let  $P \in \text{Spec}(R)$  and  $W_n P$  the image of  $P$  under the isomorphism  $\text{Spec}(R) \cong \text{Spec}(W_n(R))$ .

$$\begin{array}{ccccc} 0 & \longrightarrow & (K_{\Delta, n}^{e, c})_{W_n P} & \longrightarrow & H_{\mathfrak{m}}^d(R(\Delta))_{W_n P} & \longrightarrow & H_{[\mathfrak{m}]}^d(Q_{R, \Delta, n}^{e, c})_{W_n P} \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & (K_{\Delta, n}^{e, c})_P & \longrightarrow & H_{\mathfrak{m}}^d(R(\Delta))_P & \longrightarrow & H_{[\mathfrak{m}]}^d(Q_{R_P, \Delta \otimes R_P, n}^{e, c}) \\ & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & K_{\Delta \otimes R_P, n}^{e, c} & \longrightarrow & H_{\mathfrak{m}}^d(R_P(\Delta \otimes R_P)) & \longrightarrow & H_{[\mathfrak{m}]}^d(Q_{R_P, \Delta \otimes R_P, n}^{e, c}) \end{array}$$

Though  $W_n(R)$  is non-Noetherian, localization is still exact [Sta24, 00CS], yielding exactness the top row. Localization commutes with top local cohomology yielding the vertical isomorphisms, which in turn yield an isomorphism in the left column. For completion we proceed similarly:

$$\begin{array}{ccccc} 0 & \longrightarrow & K_{\Delta, n}^{e, c} & \longrightarrow & H_{\mathfrak{m}}^d(R(\Delta)) & \longrightarrow & H_{[\mathfrak{m}]}^d(Q_{R, \Delta, n}^{e, c}) \\ & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & K_{\Delta^\wedge, n}^{e, c} & \longrightarrow & H_{\mathfrak{m}}^d(\widehat{R}(\Delta^\wedge)) & \longrightarrow & H_{[\mathfrak{m}]}^d(Q_{\widehat{R}, \Delta^\wedge, n}^{e, c}) \end{array}$$

The middle isomorphism is a consequence of local cohomology being an  $\mathfrak{m}$ -torsion module and the right isomorphism is due to remark 3.49, inducing the left isomorphism. This left isomorphism implies that  $K_{\Delta, n}^{e, c}$  is a complete  $\widehat{R}$ -module, and hence  $K_{\Delta, n}^{e, c} \cong K_{\Delta, n}^{e, c} \wedge$ .  $\square$

LEMMA 3.63. *Let  $R$  be a local excellent normal domain admitting a dualizing complex, and  $\Delta$  a  $\mathbb{Q}$ -divisor on  $R$  such that  $[\Delta] = 0$  and  $K_R + \Delta$  is  $\mathbb{Q}$ -Cartier. Then for any  $P \in \text{Spec}(R)$ ,*

$$\tau_n(R, \Delta) \otimes_R R_P = \tau_n(R_P, \Delta_P)$$

$$\tau^q(R, \Delta) \otimes_R R_P = \tau^q(R_P, \Delta_P)$$

Where  $\Delta_P = \Delta \otimes_R R_P$  and

$$\tau_n(R, \Delta) \otimes_R \widehat{R} = \tau_n(\widehat{R}, \Delta^\wedge)$$

$$\tau^q(R, \Delta) \otimes_R \widehat{R} = \tau^q(\widehat{R}, \Delta^\wedge)$$

PROOF. We prove the above statements  $\tau_n(R, \Delta)$ , as the analogous results for  $\tau^q(R, \Delta)$  will follow for Choosing  $e$  and  $c$  as in the previous remark, set

$$\tau_n(R, \Delta) = \left( \frac{H_m^d(R(K_R + \Delta))}{K_{K_R + \Delta, n}^{e, c}} \right)^\vee$$

As  $\text{Hom}_R(-, E_R) \otimes_R R_P \simeq \text{Hom}_{R_P}(-, E_{R_P})$  and  $\text{Hom}_R(-, E_R) \simeq \text{Hom}_{\widehat{R}}(-, E_{\widehat{R}})$  as functors, it is sufficient to show that

$$\frac{H_m^d(R(K_R + \Delta))}{K_{K_R + \Delta, n}^{e, c}} \otimes_R R_P \cong \frac{H_m^d(R_P(K_{R_P} + \Delta_P))}{K_{K_{R_P} + \Delta_P, n}^{e, c}}$$

and show a similar result for completion. Local cohomology already satisfies such identities:

$$H_m^d(R(K_R + \Delta)) \otimes_R R_P \cong H_m^d(R_P(K_{R_P} + \Delta_P))$$

$$H_m^d(R(K_R + \Delta)) \otimes_R \widehat{R} \cong H_m^d(\widehat{R}(K_{\widehat{R}} + \widehat{\Delta}))$$

So we can conclude by this, the previous lemma, exactness of completion and localization, and the following short exact sequence:

$$0 \longrightarrow K_{K_R + \Delta, n}^{e, c} \longrightarrow H_m^d(R(K_R + \Delta)) \longrightarrow \frac{H_m^d(R(K_R + \Delta))}{K_{K_R + \Delta, n}^{e, c}} \longrightarrow 0$$

□

LEMMA 3.64. [KTT<sup>+</sup>24b, corollary 4.29] *Let  $R$  be a local excellent normal domain admitting a dualizing complex, and  $\Delta$  a  $\mathbb{Q}$ -divisor on  $R$  such that  $[\Delta] = 0$  and  $K_R + \Delta$  is  $\mathbb{Q}$ -Cartier. Then the following equalities hold:*

(1)

$$\tau_n(R, \Delta) = \bigcap_{c \in R^\circ} \bigcap_{e_0 > 0} \sum_{e \geq e_0} \left( \frac{H_m^d(R(K_R + \Delta))}{K_{K_R + \Delta, n}^{e, c}} \right)^\vee$$

(2)

$$\tau_n(R, \Delta) = \bigcap_{c \in R^\circ} \sum_{e > 0} \left( \frac{H_m^d(R(K_R + \Delta))}{K_{K_R + \Delta, n}^{e, c}} \right)^\vee$$

(3)

$$\tau_n(R, \Delta) = \bigcap_{c \in R^\circ} \sum_{e_0 > 0} \bigcap_{e \geq e_0} \left( \frac{H_m^d(R(K_R + \Delta))}{K_{K_R + \Delta, n}^{e, c^{p^{e-e_0}}}} \right)^\vee$$

PROOF. By proposition 3.57(1), there exists the following series of inclusions for  $c$  chosen as in remark 3.61:

$$\dots \supseteq K_{K_R + \Delta, n}^{e, c} \supseteq K_{K_R + \Delta, n}^{e+1, c} \supseteq \dots$$

Yielding a surjection  $\frac{H_m^d(R(K_R + \Delta))}{K_{K_R + \Delta, n}^{e+1, c}} \twoheadrightarrow \frac{H_m^d(R(K_R + \Delta))}{K_{K_R + \Delta, n}^{e, c}}$  for every  $e$ , and in turn, a series of inclusions of the Matlis duals:

$$\dots \subseteq \left( \frac{H_m^d(R(K_R + \Delta))}{K_{K_R + \Delta, n}^{e, c}} \right)^\vee \subseteq \left( \frac{H_m^d(R(K_R + \Delta))}{K_{K_R + \Delta, n}^{e+1, c}} \right)^\vee \subseteq \dots$$

As these are submodules of  $R(-\Delta) \cong R$ , this ascending chain stabilizes. It follows for any  $c$  chosen as in remark 3.61,

$$\bigcap_{e_0 > 0} \sum_{e \geq e_0} \left( \frac{H_m^d(R(K_R + \Delta))}{K_{K_R + \Delta, n}^{e, c}} \right)^\vee = \sum_{e > 0} \left( \frac{H_m^d(R(K_R + \Delta))}{K_{K_R + \Delta, n}^{e, c}} \right)^\vee$$

As the intersections taken in (1) and (2) can be taken over  $c \in (g^2 t^4)R^\circ$  instead of over all of  $c \in R^\circ$ , this equality implies that (1) and (2) are equivalent statements. Therefore, we just prove (2). The right hand side of the above equation can be rewritten as  $\left( \frac{H_m^d(R(K_R + \Delta))}{K_{K_R + \Delta, n}^{e, c}} \right)^\vee$  for an  $e$  chosen sufficiently large and  $c$  chosen as in remark 3.61, and thus:

$$\bigcap_{c \in R^\circ} \sum_{e > 0} \left( \frac{H_m^d(R(K_R + \Delta))}{K_{K_R + \Delta, n}^{e, c}} \right)^\vee = \bigcap_{c \in (g^2 t^4)R^\circ} \left( \frac{H_m^d(R(K_R + \Delta))}{K_{K_R + \Delta, n}^{e, c}} \right)^\vee = \tau_n(R, \Delta).$$

By remark 3.61. We now prove (3). By [KTT<sup>+</sup>24b, lemma 4.22] we have that  $K_{K_R + \Delta, n}^{e, c^{p^{e-e_0}}} \subseteq K_{K_R + \Delta, n}^{e_0, cq^2}$ , for  $e_0, q, c$  as in remark 3.61, yielding the inclusion

$$\left( \frac{H_m^d(R(K_R + \Delta))}{K_{K_R + \Delta, n}^{e_0, cq^2}} \right)^\vee \subseteq \left( \frac{H_m^d(R(K_R + \Delta))}{K_{K_R + \Delta, n}^{e, c^{p^{e-e_0}}}} \right)^\vee$$

And therefore,

$$\begin{aligned}\tau_n(R, \Delta) &= \bigcap_{c \in R^\circ} \sum_{e_0 > 0} \left( \frac{H_m^d(R(K_R + \Delta))}{K_{K_R + \Delta, n}^{e_0, cq^2}} \right)^\vee \\ &\subseteq \bigcap_{c \in R^\circ} \sum_{e_0 > 0} \bigcap_{e \geq e_0} \left( \frac{H_m^d(R(K_R + \Delta))}{K_{K_R + \Delta, n}^{e, c p^{e-e_0}}} \right)^\vee \\ &\subseteq \tau_n(R, \Delta)\end{aligned}$$

Where ( $=$ ,  $\subseteq$ ,  $\subseteq$ ) are due to (2), the above inclusion, and (2) again, respectively.  $\square$

**THEOREM 3.65.** [**KTT<sup>+</sup>24b**, Proposition 4.30] *Let  $R$  be a local excellent normal domain admitting a dualizing complex, and  $\Delta$  a  $\mathbb{Q}$ -divisor on  $R$  such that  $[\Delta] = 0$  and  $K_R + \Delta$  is  $\mathbb{Q}$ -Cartier.*

- (1)  $\tau_n(R, \Delta) = R$  if and only if  $R$  is  $n$ -quasi- $F$ -pure regular.
- (2)  $\tau^q(R, \Delta) = R$  if and only if  $R$  is quasi- $F$ -pure regular.

*In particular, quasi- $F$ -pure regularity commutes with localization and completion due to lemma 3.63.*

**PROOF.** From lemma 3.64(3), we see that  $\tau_n(R, \Delta) = R$  precisely when, for any  $c \in R^\circ$ , there exists an  $e_0 > 0$  such that  $K_{K_R + \Delta, n}^{e, c p^e} = 0$  for all  $e \geq e_0$ . This is equivalent to, for any  $c \in R^\circ$ ,  $(R, \Delta + \frac{1}{p^{e_0}} \text{div}(c))$  being  $n$ -quasi- $F^{e_0}$ -pure for every  $e \geq e_0$ , which is an equivalent formulation of the definition of  $n$ -quasi- $F$ -pure regularity. This proves (1), and (2) follows from (1).  $\square$

**THEOREM 3.66.** [**KTT<sup>+</sup>24b**, Theorem 4.31] *Let  $R$  be a local excellent normal domain admitting a dualizing complex, and  $\Delta$  a  $\mathbb{Q}$ -divisor on  $R$  such that  $[\Delta] = 0$  and  $K_R + \Delta$  is  $\mathbb{Q}$ -Cartier. Choose  $a \in \mathbb{Q}, g \in R^\circ$  such that  $K_R + \Delta = a \cdot \text{div}(g)$ . For  $t \in \tau(R) \cap R^\circ$  and  $q$  such that  $[q] \cdot 0_{R, K_R + \Delta, 1}^* = 0$ , pick  $c \in R^\circ \cap ((g^2 t^4) \cup (q^4))$ . Then for any  $n > 0$  the following are equivalent:*

- (1)  $(R, \Delta)$  is  $n$ -quasi- $F$ -pure regular.
- (2) There exists  $e \in \mathbb{N}$  such that  $(R, \Delta + (1/p^e) \text{div}(c))$  is  $n$ -quasi- $F^e$ -pure.

(3) There exists an  $e_0 > 0$  such that for all  $e \geq e_0$ ,

$$\Phi_{R, K_R + \Delta, n}^{e, c} : H_m^d(R(K_R + \Delta)) \rightarrow H_{[m]}^d(Q_{R, K_R + \Delta, n}^{e, c})$$

is injective.

(4)  $0_{R, K_R + \Delta, n}^* = 0$ .

PROOF. (3) and (4) are equivalent by proposition 3.57(3), (2) and (3) are equivalent by a careful application of lemma 3.64, and (1) and (3) is a direct consequence of  $c$  being a strong test element in the sense of proposition 3.57.  $\square$

**3.3. Stability of Quasi- $F$ -Pure Regularity Under Étale Extension.** In this section we show that, for  $R, S, \Delta$ , and  $\Delta_S$  in setting 3.37:

$$\tau_n(R, \Delta) \otimes_R S \subseteq \tau_n(S, \Delta_S)$$

$$\tau^q(R, \Delta) \otimes_R S \subseteq \tau^q(S, \Delta_S)$$

With equality if  $K_R + \Delta$  is  $\mathbb{Q}$ -Cartier. In particular, If  $(R, \Delta)$  is  $(n)$ -quasi- $F$ -pure regular and  $K_R + \Delta$  is  $\mathbb{Q}$ -Cartier, then so is  $(S, \Delta_S)$ . The converse of this implication holds by without the  $\mathbb{Q}$ -Cartier hypothesis by reducing to  $(n)$ -quasi- $F^e$ -purity: we sketch out the proof of this below.

LEMMA 3.67. *In setting 3.37, if  $(S, \Delta_S)$  is  $n$ -quasi- $F$ -pure regular, then so is  $(R, \Delta)$ .*

PROOF. Assume that  $S$  is  $n$ -quasi- $F$ -pure regular. Let  $E$  be an effective  $\mathbb{Q}$ -divisor on  $R$ , and let  $E_S := E \otimes_R S$  be the associated  $\mathbb{Q}$ -divisor on  $S$ . By hypothesis, there exists an  $\varepsilon > 0$  such that  $(S, \Delta_S + \varepsilon E_S)$  is  $n$ -quasi- $F^e$ -pure for all  $e > 0$ . By virtue of  $- \otimes_R S$  defining a map of abelian groups  $\text{Cl}(R) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \text{Cl}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$ , it is clear that

$$\Delta_S + \varepsilon E_S = \Delta \otimes_R S + \varepsilon(E \otimes_R S) \sim (\Delta + \varepsilon E) \otimes_R S$$

In fact, these are equal by flatness of  $S$  as an  $R$ -module. In particular,  $(S, (\Delta + \varepsilon E) \otimes_R S)$  is  $n$ -quasi- $F^e$ -pure for all  $e$ , and thus by theorem 3.40,  $(R, \Delta + \varepsilon E)$  is  $n$ -quasi- $F^e$ -pure for all  $e$ , completing the proof.  $\square$

The converse does not immediately follow from this logic since every  $\mathbb{Q}$ -divisor on  $S$  cannot necessarily be represented as the base change of a  $\mathbb{Q}$ -divisor on  $R$ .

LEMMA 3.68. *In setting 3.37, for any  $n, e \in \mathbb{N}$  and  $c \in R^\circ$ ,*

$$(1) K_{\Delta, n}^{e, c} \otimes_R S = K_{\Delta_S, n}^{e, c}$$

$$(2) 0_{R, \Delta, n}^* \otimes_R S \subseteq 0_{S, \Delta_S, n}^*, \text{ with equality if } \Delta \text{ is } \mathbb{Q}\text{-Cartier.}$$

PROOF. We prove (1) first. As  $R \rightarrow S$  is étale, so is  $W_n(R) \rightarrow W_n(S)$ . Notably,  $W_n(S)$  is a flat  $W_n(R)$ -Module, yielding the following map of left exact sequences:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ K_{\Delta, n}^{e, c} \otimes_{W_n(R)} W_n(S) & \longrightarrow & K_{\Delta_S, n}^{e, c} \\ \downarrow & & \downarrow \\ H_m^d(R(\Delta)) \otimes_{W_n(R)} W_n(S) & \longrightarrow & H_m^d(S(\Delta_S)) \\ \downarrow & & \downarrow \\ H_{[m]}^d(Q_{R, \Delta, n}^{e, c}) \otimes_{W_n(R)} W_n(S) & \longrightarrow & H_{[m]}^d(Q_{S, \Delta_S, n}^{e, c}) \end{array}$$

The bottom horizontal map is an isomorphism via remark 3.49 and lemma A.4. The middle map is an isomorphism as  $H_m^d(R(\Delta)) \otimes_{W_n(R)} W_n(S) \cong H_m^d(R(\Delta)) \otimes_R S \cong H_m^d(S(\Delta_S))$  again via lemma A.4. Thus, the top horizontal map is an isomorphism. As  $K_{\Delta, n}^{e, c}$  is a  $W_n(R)$  module via the restriction map  $W_n(R) \rightarrow R$ , it is clear that  $K_{\Delta, n}^{e, c} \otimes_{W_n(R)} W_n(S) \cong K_{\Delta, n}^{e, c} \otimes_R S$  as  $R$ -Modules, so the first claim is proved.

As for (2), note that (1) and  $H_m^d(R(\Delta)) \otimes_R S \cong H_m^d(S(\Delta_S))$  imply the following sequence of inclusions:

$$\begin{aligned} 0_{R,\Delta,n}^* \otimes_R S &= \left\{ z \in H_m^d(R(\Delta)) \mid \exists 0 \neq c \in R, e_0 \in \mathbb{N} \text{ such that } z \in K_{\Delta,n}^{e,c} \ \forall e \geq e_0 \right\} \otimes_R S \\ &= \left\{ z \in H_m^d(S(\Delta_S)) \mid \exists 0 \neq c \in R \subset S, e_0 \in \mathbb{N} \text{ such that } z \in K_{\Delta_S,n}^{e,c} \ \forall e \geq e_0 \right\} \\ &\subseteq 0_{S,\Delta_S,n}^* \end{aligned}$$

□

If  $\Delta$  is  $\mathbb{Q}$ -Cartier, we can write  $\Delta = a \cdot \text{div}(g)$  for  $a \in \mathbb{Q}$ ,  $g \in R^\circ$ . By remark 3.61, a test element  $t \in \tau(R) \cap R^\circ$ ,  $c \in (g^2 t^4)R^\circ$ , and  $e \gg 0$ ,  $0_{R,\Delta,n}^* = K_{\Delta,n}^{e,c}$ . Further,  $\Delta_S = \Delta \otimes_R S = a \cdot \text{div}(g)$ , for  $g$  viewed as an element in  $S$ ; it follows that  $c \in (g^2 t^4)S^\circ$  and that  $t$ , viewed as an element of  $S$ , is contained in  $\tau(S) \cap S^\circ$ . Therefore  $0_{S,\Delta_S,n}^* = K_{\Delta_S,n}^{e,c}$ , enlarging  $e$  if necessary. It follows from this argument and (1) that  $0_{R,\Delta,n}^* \otimes_R S = 0_{S,\Delta_S,n}^*$ .

**THEOREM 3.69.** *In setting 3.37,*

- (1)  $\tau_n(R, \Delta) \otimes_R S \subseteq \tau_n(S, \Delta_S)$
- (2)  $\tau^q(R, \Delta) \otimes_R S \subseteq \tau^q(S, \Delta_S)$

with equality when  $K_R + \Delta$  is  $\mathbb{Q}$ -Cartier.

**PROOF.** (2) follows from (1), so just we prove (1). Via lemma 3.68(2) and flatness of  $S$  over  $R$ , we get the following diagram of  $S$ -Modules:

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0_{R,K_R+\Delta,n}^* \otimes_R S & \longrightarrow & H_m^d(R(K_R + \Delta)) \otimes_R S & \longrightarrow & \frac{H_m^d(R(K_R+\Delta))}{0_{R,K_R+\Delta,n}^*} \otimes_R S \longrightarrow 0 \\ & & \downarrow & & \downarrow \cong & & \downarrow \\ 0 & \longrightarrow & 0_{S,K_S+\Delta_S,n}^* & \longrightarrow & H_m^d(S(K_S + \Delta_S)) & \longrightarrow & \frac{H_m^d(S(K_S+\Delta_S))}{0_{S,K_S+\Delta_S,n}^*} \longrightarrow 0 \end{array}$$

Implying that the rightmost vertical map is a surjection. Applying Matlis duality (as  $S$ -modules) yields an inclusion

$$\left( \frac{H_m^d(R(K_R + \Delta))}{0_{R,K_R+\Delta,n}^*} \otimes_R S \right)^\vee \hookrightarrow \left( \frac{H_m^d(S(K_S + \Delta_S))}{0_{S,K_S+\Delta_S,n}^*} \right)^\vee = \tau_n(S, \Delta_S)$$

So it is sufficient to show that the source is isomorphic to  $\tau_n(R, \Delta) \otimes_R S$ . This follows from the more general fact that finite étale base change commutes with Matlis duality for any  $R$ -Module  $M$ : For the benefit of the reader we sketch this proof below:

$$\begin{aligned}
M^\vee \otimes_R S &\cong \text{Hom}_R(M, E_R) \otimes_R S \\
(S \text{ is finite + projective over } R) &\cong \text{Hom}_S(M \otimes_R S, E_R \otimes_R S) \\
(R \rightarrow S \text{ is unrammified}) &\cong \text{Hom}_S(M \otimes_R S, E_S) \\
&\cong (M \otimes_R S)^\vee
\end{aligned}$$

When  $K_R + \Delta$  is  $\mathbb{Q}$ -Cartier, lemma 3.68(2) shows that  $0_{R, K_R + \Delta, n}^* \otimes_R S = 0_{S, K_S + \Delta_S, n}^*$ . In this case the leftmost vertical map in the diagram above is an isomorphism, and hence the rightmost vertical map is as well, yielding an isomorphism on Matlis duals. Equality then follows from a similar argument.  $\square$

Putting this together with lemma 3.67 yields the desired result:

**COROLLARY 3.69.1.** *In setting 3.37, if  $(S, \Delta_S)$  is  $(n)$ -quasi- $F$ -pure regular, then so is  $(R, \Delta)$ . If  $K_R + \Delta$  is  $\mathbb{Q}$ -Cartier, then the converse holds.*

**THEOREM 3.70.** *Let  $(R, \mathfrak{m})$  be an excellent normal local domain admitting a dualizing complex  $\omega_R^\bullet$  with associated canonical divisor  $K_R$ , and  $\Delta$  is a  $\mathbb{Q}$ -divisor on  $R$  such that  $[\Delta] = 0$  and  $K_R + \Delta$  is  $\mathbb{Q}$ -Cartier. Let  $\varphi : R \rightarrow S$  be a finite étale map to a domain  $S$ . Let  $\Delta_S = (K_R \otimes_R S) - K_S + \Delta \otimes_R S$ . Then  $(R, \Delta)$  is  $n$ -quasi- $F$ -pure regular if and only if  $(S, \Delta_S)$  is.*

**PROOF.** In this context, similarly to that of setting 3.37 but where  $\varphi$  is a finite map and  $S$  is not necessarily local,  $S$  is still an excellent normal domain admitting a dualizing complex  $\omega_S^\bullet = \omega_R^\bullet \otimes_R^\mathbb{L} S$ . As  $\varphi$  is finite there exists finitely many maximal ideals of  $S$  lying over  $R$ . Denote these  $\mathfrak{n}_1, \dots, \mathfrak{n}_\ell$ . As  $R \rightarrow S$  is étale the base change  $\widehat{R} \rightarrow \widehat{R} \otimes_R S$  is still

étale, and by [Sta24, 04GH] one can decompose

$$\widehat{R} \otimes_R S \cong S_{\mathfrak{n}_1} \times \cdots \times S_{\mathfrak{n}_\ell}$$

Where each  $\widehat{R} \rightarrow \widehat{R} \otimes_R S \twoheadrightarrow S_{\mathfrak{n}_i}$  is a local étale map.  $\widehat{R} \otimes_R S$  has  $\ell$  maximal ideals, and localizing at each yields  $S_{\mathfrak{n}_i}$  for some  $i$ . As a consequence of lemma 3.63 we have that  $(R, \Delta)$  is  $n$ -quasi- $F$ -pure regular if and only if  $(\widehat{R}, \Delta^\wedge)$  is. By the previous corollary,  $(\widehat{R}, \Delta^\wedge)$  is  $n$ -quasi- $F$ -pure regular if and only if  $(S_{\mathfrak{n}_i}, \Delta_S|_{S_{\mathfrak{n}_i}})$  is for each  $\mathfrak{n}_i$ , which is  $n$ -quasi- $F$ -pure regular if and only if  $(S, \Delta_S)$  is since quasi- $F$ -pure regularity is a local property verifiable at the level of  $\max\text{Spec}$ .  $\square$

REMARK 3.71. *The  $K_R + \Delta$  being  $\mathbb{Q}$ -Cartier assumption is only required to find a quasi-test element. If one could show that quasi-test elements exist without this assumption (as they do for  $F$ -pure regularity), then the assumption can be removed everywhere in this section.*

**3.4. Quasi- $F$ -Rationality and the Parameter Test Ideal.** Defining a weakening of quasi- $F$ -pure regularity in the same vein as the strong  $F$ -regularity to  $F$ -rationality comparison can be done from multiple equivalent approaches. In view of remark 1.58 and proposition 3.51(c), one can define quasi- $F$ -rationality as follows:

DEFINITION 3.72. Let  $R$  be an excellent normal local ring of dimension  $d$  and  $\Delta$  a  $\mathbb{Q}$ -divisor on  $R$ . Then  $(R, \Delta)$  is **n-Quasi-F-Rational** if  $R$  is Cohen-Macaulay and  $\mathfrak{R}^{n-1} \left( \widetilde{0_{R,\Delta,n}^*} \right) = 0$ .  $(R, \Delta)$  is **Quasi-F-Rational** if it is  $n$ -quasi- $F$ -rational for some  $n$ .

Quasi- $F$ -rationality was first defined in [KTT<sup>+</sup>24b], though we use the equivalent definition given in [Yos24]. Proposition 3.51(c) and remark 1.58 imply that 1-quasi- $F$ -rational is equivalent to being  $F$ -rational. If  $\Delta$  is  $\mathbb{Q}$ -Cartier then proposition 3.56 implies that  $(R, \Delta)$  is  $n$ -quasi- $F$ -rational (resp. quasi- $F$ -rational) if and only if  $0_{R,\Delta,n}^* = 0$  (resp.  $0_{R,\Delta,n}^* = 0$  for some  $n$ ). We can take this as the definition of quasi- $F$ -rationality in the no boundary case, relaxing our assumptions slightly.

DEFINITION 3.73. Let  $R$  be an excellent local ring of dimension  $d$ . Then  $R$  is **n-Quasi-F-Rational** if  $R$  is Cohen-Macaulay and  $0_{R,n}^* = 0$ .  $R$  is **Quasi-F-Rational** if it is  $n$ -quasi- $F$ -rational for some  $n$ .

If  $\Delta$  is  $\mathbb{Q}$ -Cartier then by proposition 3.57 there exists  $e \gg 0$  and  $c \in R^\circ$  chosen well for which  $0_{R,\Delta,n}^* = K_{\Delta,n}^{e,c}$ . This yields the following equivalent definitions for quasi- $F$ -rationality.

LEMMA 3.74. *Let  $R$  be an excellent local normal domain of dimension  $d$ . Choose  $q \in R^\circ$  such that  $[q] \cdot 0_{R,1}^* = 0$  and pick  $c \in R^\circ \cap (q^4)$  Then the following are equivalent:*

- (a)  $R$  is  $n$ -quasi- $F$ -rational.
- (b)  $0_{R,n}^* = 0$ .
- (c) There exists an  $e_0 \geq 0$  for which  $K_n^{e,c} = 0$  for all  $e \geq e_0$ .
- (d) There exists an  $e_0 \geq 0$  for which

$$\Phi_{R,n}^e : H_m^d(R) \rightarrow H_{[m]}^d(Q_{R,n}^{e,c})$$

*is injective as a map of  $W_n(R)$ -modules for all  $e \geq e_0$ .*

Let  $\Delta = a \cdot \text{div}(g)$  be a  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor on  $R$ . Pick  $\ell \in R^\circ$  such that  $[\ell] \cdot 0_{R,\Delta,1}^* = 0$  and  $t \in R^\circ \cap \tau(R)$ . For any  $c \in R^\circ \cap ((g^2 t^4) \cup (\ell^4))$  the following are equivalent:

- (a)  $(R, \Delta)$  is  $n$ -quasi- $F$ -rational.
- (b)  $0_{R,\Delta,n}^* = 0$ .
- (c) There exists an  $e_0 \geq 0$  for which  $K_{\Delta,n}^{e,c} = 0$  for all  $e \geq e_0$ .
- (d) There exists an  $e_0 \geq 0$  for which

$$\Phi_{R,\Delta,n}^e : H_m^d(R(\Delta)) \rightarrow H_{[m]}^d(Q_{R,\Delta,n}^{e,c})$$

*is injective  $W_n(R)$ -modules for all  $e \geq e_0$ .*

REMARK 3.75. *We recall that  $F$ -rationality is equivalent to parameter ideals being tightly closed (see remark 1.53). The recently introduced notion of quasi-tight closure for  $F$ -finite rings due to Yoshikawa [Yos24] also characterizes quasi- $F$ -rationality. In particular, an*

$F$ -finite local domain  $R$  is  $n$ -quasi- $F$ -rational if and only if for all (or equivalently, for some) system of parameters  $x_1, \dots, x_d$ , the ideal  $I = (x_1, \dots, x_d)$  is  $n$ -quasi-tightly closed (see [Yos24, Theorem 3.14]). Interestingly,  $n$ -quasi-tight closure of a parameter ideal forces  $R$  to be Cohen-Macaulay much like standard tight closure over excellent rings.

REMARK 3.76. If  $K_R = 0$  (e.g.  $R$  is quasi-Gorenstein) then comparing lemma 3.74 and theorem 3.66 implies that quasi- $F$ -rationality and quasi- $F$ -pure regularity are equivalent. Assuming  $R$  is Cohen-Macaulay, then quasi- $F$ -pure regularity implies quasi- $F$ -rationality. It is conjectured that  $R$  being quasi- $F$ -regular implies that  $R$  is Cohen-Macaulay, and this conjecture is known for dimension  $\leq 3$  provided  $K_R$  is  $\mathbb{Q}$ -Cartier ([KTT<sup>+</sup>24b, Theorem 8.9]). If this conjecture holds in general, then quasi- $F$ -regularity always implies quasi- $F$ -rationality, with equivalence if  $R$  is Gorenstein. This mirrors the relationship of strong  $F$ -regularity and  $F$ -rationality.

In the  $F$ -finite setting, [KTT<sup>+</sup>24b] defined the  $n$ -quasi-test submodule  $\tau_n(\omega_R)$  to be the inverse image of  $\tau_n(W_n\omega_R)$  under the restriction morphism  $(\mathfrak{R}^{n-1})^* : \omega_R \rightarrow W_n\omega_R$ , and defined the quasi-test submodule  $\tau^q(\omega_R)$  to be the union of these submodules.  $\tau_n(W_n\omega_R)$  was defined to be the smallest (with respect to inclusion)  $W_n(R)$ -submodule of  $W_n\omega_R$  satisfying the following two conditions:

- **$T$ -Stability:** A  $W_n(R)$ -submodule  $M$  of  $W_n\omega_R$  is  **$T$ -stable** if  $T_n(F_*M) \subseteq M$ , for  $T_n : F_*W_n\omega_R \rightarrow W_n\omega_R$  is the trace of the Witt Frobenius.
- **Co-Smallness:** A  $W_n(R)$ -submodule  $M$  of  $W_n\omega_R$  is **co-small** if there exists a  $c \in R^\circ$  for which  $[c] \cdot W_n\omega_R \subseteq M$ .

$\tau_n(\omega_R) \subseteq \omega_R$  in general, and the authors defined a ring  $R$  to be  $n$ -quasi- $F$ -rational (resp. quasi- $F$ -rational) if  $\tau_n(\omega_R) = \omega_R$  (resp.  $\tau^q(\omega_R) = \omega_R$ ). When  $R$  is not  $F$ -finite we do not have access to the Witt-dualizing complex, and hence, this definition cannot be used. Following the trend of previous sections, we define the quasi-test submodule via taking the “Matlis dual” approach.

DEFINITION 3.77. Let  $R$  be an excellent local normal domain admitting a dualizing module  $\omega_R$  and let  $\Delta$  be a  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor. Define the  $n$ -**quasi-test submodule**  $\tau_n(\omega_R) \subseteq \omega_R$  as follows:

$$\tau_n(\omega_R, \Delta) := \left( \frac{H_m^d(R(\Delta))}{0_{R,\Delta,n}^*} \right)^\vee$$

And let  $\tau^q(\omega_R, \Delta) := \bigcup_{n \in \mathbb{N}} \tau_n(\omega_R, \Delta)$  be the **quasi-test submodule**.

From proposition 3.51(b), it follows that  $\tau_1(\omega_R, \Delta) \subseteq \tau_2(\omega_R, \Delta) \subseteq \dots$ , and hence  $\tau^q(\omega_R, \Delta) = \tau_N(\omega_R, \Delta)$  for  $N \gg 0$ . This definition is equivalent to the original definition for  $F$ -finite rings (see [KTT<sup>+</sup>24b, Theorem 3.29]), and by lemma 3.74, when  $[\Delta] = 0$  this characterizes quasi- $F$ -rationality.

PROPOSITION 3.78. *Let  $R$  be an excellent local normal domain admitting a dualizing module  $\omega_R$  and let  $\Delta$  be a  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor. Then  $\tau_n(\omega_R, \Delta) = H_m^d(R(\Delta))^\vee \cong \omega_R$  if and only if  $0_{R,\Delta,n}^* = 0$ , if and only if  $(R, \Delta)$  is  $n$ -quasi- $F$ -rational. Similarly,  $\tau^q(\omega_R, \Delta) = H_m^d(R(\Delta))^\vee \cong \omega_R$  for some  $n$  if and only if  $0_{R,\Delta,n}^* = 0$  for some  $n$ , if and only if  $(R, \Delta)$  is quasi- $F$ -rational.*

Further, as  $\Delta$  is  $\mathbb{Q}$ -Cartier there exists an  $e \gg 0$  and  $c \in R^\circ$  chosen well for which  $0_{R,\Delta,n}^* = K_{\Delta,n}^{e,c}$ . Thus  $\tau_n(\omega_R, \Delta) = \left( \frac{H_m^d(R(\Delta))}{K_{\Delta,n}^{e,c}} \right)^\vee$  for such an  $e$  and  $c$ . Using this, we show that  $n$ -quasi-test submodule, and hence quasi- $F$ -rationality, are compatible with localization, completion and étale extension.

PROPOSITION 3.79. *Let  $R$  be an excellent local normal domain admitting a dualizing module  $\omega_R$  and let  $\Delta$  be a  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor. Then for any  $P \in \text{Spec}(R)$ ,*

$$\tau_n(\omega_R, \Delta) \otimes_R R_P \cong \tau_n(\omega_{R_P}, \Delta_P)$$

$$\tau^q(\omega_R, \Delta) \otimes_R R_P \cong \tau^q(\omega_{R_P}, \Delta_P)$$

$$\tau_n(\omega_R, \Delta)^\wedge \cong \tau_n(\omega_{\widehat{R}}, \Delta^\wedge)$$

$$\tau^q(\omega_R, \Delta)^\wedge \cong \tau^q(\omega_{\widehat{R}}, \Delta^\wedge)$$

PROOF. We prove the statements for  $\tau_n(\omega_R, \Delta)$ , as the analogous statements for  $\tau^q(\omega_R, \Delta)$  will follow. As Matlis duality commutes with localization, it is sufficient to check that

$$\frac{H_m^d(R(\Delta))}{K_{\Delta,n}^{e,c}} \otimes_R R_P \cong \frac{H_m^d(R_P(\Delta_P))}{K_{\Delta_P,n}^{e,c}}$$

and

$$\left( \frac{H_m^d(R(\Delta))}{K_{\Delta,n}^{e,c}} \right)^\wedge \cong \frac{H_m^d(\widehat{R}(\Delta^\wedge))}{K_{\Delta^\wedge,n}^{e,c}}$$

The first statement can be seen from exactness of localization and the following map of short exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_{\Delta,n}^{e,c} \otimes_R R_P & \longrightarrow & H_m^d(R(\Delta) \otimes_R R_P) & \longrightarrow & \frac{H_m^d(R(\Delta))}{K_{\Delta,n}^{e,c}} \otimes_R R_P \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \\ 0 & \longrightarrow & K_{\Delta_P,n}^{e,c} & \longrightarrow & H_m^d(R_P(\Delta_P)) & \longrightarrow & \frac{H_m^d(R_P(\Delta_P))}{K_{\Delta_P,n}^{e,c}} \longrightarrow 0 \end{array}$$

The top row is exact as localization is exact. The left vertical map is an isomorphism by lemma 3.62 and the middle vertical map is an isomorphism via standard facts about local cohomology. Thus the right vertical map must too be an isomorphism. To prove the second statement, we see that  $\frac{H_m^d(R(\Delta))}{K_{\Delta,n}^{e,c}}$  is an  $\mathfrak{m}$ -torsion module and hence  $\frac{H_m^d(R(\Delta))}{K_{\Delta,n}^{e,c}} \cong \left( \frac{H_m^d(R(\Delta))}{K_{\Delta,n}^{e,c}} \right)^\wedge$ . The desired isomorphism follows from an identical proof to the localization case, again utilizing basic properties of local cohomology and lemma 3.62.  $\square$

**THEOREM 3.80.** *In the context of setting 3.37, if  $\Delta$  is a  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor then*

$$\tau_n(\omega_R, \Delta) \otimes_R S \cong \tau_n(\omega_S, \Delta_S)$$

$$\tau^q(\omega_R, \Delta) \otimes_R S \cong \tau^q(\omega_S, \Delta_S)$$

PROOF. We prove the statements for  $\tau_n(\omega_R, \Delta)$ , as the analogous statements for  $\tau^q(\omega_R, \Delta)$  will follow. From lemma 3.68 we see that  $K_{\Delta,n}^{e,c} \otimes_R S \cong K_{\Delta_S,n}^{e,c}$ , yielding the following map of short exact sequences:

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_{\Delta,n}^{e,c} \otimes_R S & \longrightarrow & H_m^d(R(\Delta)) \otimes_R S & \longrightarrow & \frac{H_m^d(R(\Delta))}{K_{\Delta,n}^{e,c}} \otimes_R S \longrightarrow 0 \\
& & \downarrow \cong & & \downarrow \cong & & \downarrow \\
0 & \longrightarrow & K_{\Delta_S,n}^{e,c} & \longrightarrow & H_m^d(S(\Delta_S)) & \longrightarrow & \frac{H_m^d(S(\Delta_S))}{K_{\Delta_S,n}^{e,c}} \longrightarrow 0
\end{array}$$

Implying that

$$\tau(S, \Delta_S) \cong \left( \frac{H_m^d(S(\Delta_S))}{K_{\Delta_S,n}^{e,c}} \right)^\vee \cong \left( \frac{H_m^d(R(\Delta))}{K_{\Delta,n}^{e,c}} \otimes_R S \right)^\vee$$

after applying Matlis Duality  $(-)^\vee$  as  $S$ -modules. It is thus sufficient to check that the right hand side is isomorphic to  $\tau(\omega_R, \Delta) \otimes_R S$ . This follows from similar logic to the proof of theorem 3.69:

$$\begin{aligned}
\left( \frac{H_m^d(R(\Delta))}{K_{\Delta,n}^{e,c}} \otimes_R S \right)^\vee &= \text{Hom}_S \left( \frac{H_m^d(R(\Delta))}{K_{\Delta,n}^{e,c}} \otimes_R S, E_S \right) \\
&\cong \text{Hom}_S \left( \frac{H_m^d(R(\Delta))}{K_{\Delta,n}^{e,c}} \otimes_R S, E_R \otimes_R S \right) \\
&\cong S \otimes_R \text{Hom}_R \left( \frac{H_m^d(R(\Delta))}{K_{\Delta,n}^{e,c}}, E_R \right) \\
&\cong S \otimes_R \left( \frac{H_m^d(R(\Delta))}{K_{\Delta,n}^{e,c}} \right)^\vee \\
&\cong \tau(\omega_R, \Delta) \otimes_R S
\end{aligned}$$

□

**COROLLARY 3.80.1.** *Let  $(R, \mathfrak{m})$  be an excellent local normal domain admitting a dualizing module  $\omega_R$  and let  $\Delta$  be a  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor. Then  $(R, \Delta)$  is  $n$ -quasi- $F$ -rational (resp. quasi- $F$ -rational) if and only if  $R_P$  is  $n$ -quasi- $F$ -rational (resp. quasi- $F$ -rational) for all  $P \in \text{Spec}(R)$ , if and only if  $\widehat{R}$  is  $n$ -quasi- $F$ -rational (resp. quasi- $F$ -rational). In setting 3.37,  $(R, \Delta)$  is  $n$ -quasi- $F$ -rational if and only if  $(S, \Delta_S)$  is.*

**THEOREM 3.81.** *Let  $(R, \mathfrak{m})$  be an excellent normal local domain admitting a dualizing module  $\omega_R$  and  $\Delta$  a  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor on  $R$  such that  $[\Delta] = 0$ . Let  $\varphi : R \rightarrow S$  be a finite*

étale map to a domain  $S$ . Let  $\Delta_S = \Delta \otimes_R S$ . Then  $(R, \Delta)$  is  $n$ -quasi- $F$ -rational if and only if  $(S, \Delta_S)$  is.

PROOF. In this context, similarly to that of setting 3.37 but where  $\varphi$  is a finite map and  $S$  is not necessarily local,  $S$  is still an excellent normal domain admitting a dualizing module  $\omega_S = \omega_R \otimes_R S$ . As  $\varphi$  is finite there exists finitely many maximal ideals of  $S$  lying over  $R$ . Denote these  $\mathfrak{n}_1, \dots, \mathfrak{n}_\ell$ . As  $R \rightarrow S$  is étale the base change  $\widehat{R} \rightarrow \widehat{R} \otimes_R S$  is still étale, and by [Sta24, 04GH] one can decompose

$$\widehat{R} \otimes_R S \cong S_{\mathfrak{n}_1} \times \cdots \times S_{\mathfrak{n}_\ell}$$

Where each  $\widehat{R} \rightarrow \widehat{R} \otimes_R S \twoheadrightarrow S_{\mathfrak{n}_i}$  is a local étale map.  $\widehat{R} \otimes_R S$  has  $\ell$  maximal ideals, and localizing at each yields  $S_{\mathfrak{n}_i}$  for some  $i$ . As a consequence of lemma 3.63 we have that  $(R, \Delta)$  is  $n$ -quasi- $F$ -rational if and only if  $(\widehat{R}, \Delta^\wedge)$  is. By the previous corollary,  $(\widehat{R}, \Delta^\wedge)$  is  $n$ -quasi- $F$ -rational if and only if  $(S_{\mathfrak{n}_i}, \Delta_S|_{S_{\mathfrak{n}_i}})$  is for each  $\mathfrak{n}_i$ , which is  $n$ -quasi- $F$ -rational if and only if  $(S, \Delta_S)$  is since quasi- $F$ -rational is a local property verifiable at the level of  $\max\text{Spec}$ .  $\square$

#### 4. Applications to Birational Geometry

As a consequence of quasi- $F$ -split varieties satisfying vanishing theorems (proposition 2.9) and the distinction between quasi- $F$ -splittings and  $F$ -splittings existing purely in positive characteristic (remark 2.10), it would be natural to study the birational geometry of quasi- $F$ -split varieties. This was the focus of a series of papers of Kawakami, Takkamatsu, Tanaka, Witaszek, Yobuko, and Yoshikawa [KTT<sup>+</sup>25, KTT<sup>+</sup>24a, KTT<sup>+</sup>24b, TWY24] where they study quasi- $F$ -splittings in the context of birational geometry. We have referenced these papers quite significantly throughout this thesis, as it is in these papers where quasi- $F^e$ -splittings, quasi- $F$ -regularity, and quasi- $F$ -rationality were first defined. They were also the first to generalize the study of quasi- $F$ -singularities to divisor pairs  $(X, \Delta)$ . In this subsection we review some of their results.

As we have discussed in chapter 1 sections 1.3 and 1.5, global  $F$ -regularity and  $F$ -splitting form a key role in the positive characteristic minimal model program, providing suitable analogues to KLT and log canonical singularity conditions arising from characteristic 0. There are two central issues preventing results on the existence of minimal models from characteristic 0 [BCHM10] to characteristic  $p$ : first, resolutions of singularities need not exist in dimensions larger than 3, and second, smooth projective varieties in characteristic  $p$  do not necessarily satisfy key vanishing theorems, such as Kawamata-Viehweg Vanishing.

Hacon and Xu [HX15] were able to overcome these hurdles to prove the existence of minimal models for threefolds in characteristic  $p > 5$ . Threefolds admit a resolution of singularities, and further, vanishing theorems are known to hold for globally  $F$ -regular and globally  $F$ -split varieties. This allowed the authors to prove the existence of a certain kind of “flip”, which implies the existence of minimal models due to work of Shokurov [Sho92].

The characteristic bound comes from work of Hara [Har98], where he shows that KLT surfaces in characteristic  $p > 5$  are strongly  $F$ -regular. These low characteristic counterexamples are well known, consisting of Rational Double Points or Du Val singularities. In [KTT<sup>+</sup>24b], the authors show that these low characteristic counter examples are quasi- $F$ -regular.

**THEOREM 3.82 ([KTT<sup>+</sup>24b]).** *Let  $k$  be a perfect field of characteristic  $p > 0$  and  $(X, \Delta)$  be an affine KLT surface of finite type over  $k$ . Then  $(X, \Delta)$  is quasi- $F$ -regular.*

For Del Pezzo surfaces in sufficiently large characteristic they obtain a global singularity condition:

**THEOREM 3.83 ([KTT<sup>+</sup>25]).** *Let  $(X, \Delta)$  be a log Del Pezzo pair over a perfect field  $k$  of characteristic  $p \gg 0$ . Then  $(X, \Delta)$  is 2-quasi- $F$ -split.*

These statements can fail in higher dimensions (see [KTT<sup>+</sup>24a], section 8) but the converse of theorem 3.82 holds for any  $\mathbb{Q}$ -Gorenstein ring.

**THEOREM 3.84 ([KTT<sup>+</sup>24b]).** *Any quasi- $F$ -regular  $\mathbb{Q}$ -Gorenstein  $F$ -finite normal domain over  $\mathbb{F}_p$  has KLT singularities.*

We recall that KLT threefolds in positive characteristic can be spectacularly poorly behaved: [CTW18] constructed examples of KLT threefolds in all characteristics which fail to be  $F$ -pure. Remarkably, however,  $\mathbb{Q}$ -factorial KLT threefolds in sufficiently large characteristic are necessarily (locally) quasi- $F$ -regular.

**THEOREM 3.85 ([KTT<sup>+</sup>24b]).** *Let  $k$  be a perfect field of characteristic  $p > 42$  and  $X$  an affine  $\mathbb{Q}$ -factorial KLT threefold of finite type over  $k$ . Then  $X$  is quasi- $F$ -regular.*

We note that this bound is sharp; in [KTT<sup>+</sup>24a] authors construct a  $\mathbb{Q}$ -factorial threefold in characteristic 41 that fails to be locally quasi- $F$ -split. A significant part of proving these results is showing that quasi- $F$ -split varieties, much like  $F$ -split varieties, have non-positive Kodaira dimension and satisfy Kawamata Viehweg vanishing.

**PROPOSITION 3.86 (Kodaira Dimension, [KTT<sup>+</sup>25]).** *Let  $X$  be an integral normal projective scheme over a perfect field  $k$  of characteristic  $p$ . Let  $\Delta$  be a  $\mathbb{Q}$ -divisor on  $X$  such that  $[\Delta] = 0$  and suppose that  $(X, \Delta)$  is quasi- $F$ -split of height  $n$ . Then*

$$\kappa \left( X, - \left( K_X + \frac{1}{p^n - 1} [p^n \Delta] \right) \right) \geq 0$$

This implies that if  $X$  is quasi- $F$ -split then  $\kappa(X) \leq 0$ .

**THEOREM 3.87 (Kawamata Viehweg Vanishing, [KTT<sup>+</sup>25] (cf. [NY21])).** *Let  $X$  be an integral normal projective scheme over an  $F$ -finite field  $k$  of characteristic  $p$ . Let  $\Delta$  be a  $p$ -compatible  $\mathbb{Q}$ -divisor on  $X$  such that  $[\Delta] = 0$  and  $K_X + \Delta$  is  $\mathbb{Q}$ -Cartier. If  $X$  is divisorially Cohen-Macaulay and  $(X, \Delta)$  is quasi- $F$ -split and (locally) naively keenly  $F$ -split, then for any  $\mathbb{Q}$ -Cartier Weil divisor  $D$  such that  $D - (K_X + \Delta)$  is ample,*

$$H^j(X, \mathcal{O}_X(D)) = 0$$

for all  $j > 0$ .

## 5. Applying the Gamma Construction to Quasi- $F$ -Singularities

Let  $(R, \mathfrak{m})$  be any complete local ring. Recall from chapter 1 section 2.3 that Hochster and Huneke [HH94a] developed the  $\Gamma$ -construction: By picking a co-finite subset  $\Gamma$  of the  $p$ -basis of a coefficient field  $k \subseteq R$ , one can construct a faithfully flat  $F$ -finite  $R$ -algebra denoted  $R^\Gamma$ . As discussed in 1 section 2.3, by choosing  $\Gamma$  small enough, one can have  $R^\Gamma$  inherit many properties of  $R$  (theorem 1.95). In this section, we highlight apparent difficulties of applying the  $\Gamma$ -construction to quasi- $F$ -singularities, and discuss different approaches. We will also reduce checking that the  $\Gamma$ -construction preserves all the quasi- $F$ -singularities in this paper to checking that it preserves just quasi- $F^e$ -pure singularities for any  $e$ .

**5.1. Invariance under Inseparable Extensions.** In devising Fedder's Criterion, Fedder [Fed83] showed that for a  $F$ -pure  $k$ -algebra  $R$ ,  $L \otimes_k R$  is  $F$ -pure for any extension  $L$  of  $k$ . The author has shown that such a statement holds for quasi- $F$ -purity provided that  $L$  is a separable extension (corollary 2.43.2), but unfortunately, this is a required assumption:

REMARK 3.88. Consider an augmentation of [KTY22, Example 7.13]:

$$R = \frac{\mathbb{F}_2(S, T)[[x, y, z]]}{(Sx^2 + Ty^2 + z^2)}$$

A computation in [KTY22] shows that this can be realized as the general fiber of a quasi- $F$ -split height 2 fibration, and is hence also quasi- $F$ -split. A Fedder's criterion computation shows this is not  $F$ -split, so  $2 \leq \text{ht}(R) < \infty$ . However, the base change to the algebraic (or even perfect) closure  $\overline{\mathbb{F}_2(S, T)} \otimes_{\mathbb{F}_2(S, T)} R$  is non-reduced, as  $(\sqrt{S}x + \sqrt{T}y + z)^2 = 0$ . Thus, by corollary 2.33.1,  $\overline{\mathbb{F}_2(S, T)} \otimes_{\mathbb{F}_2(S, T)} R$  is not quasi- $F$ -split.

Thus, base change under inseparable extensions can fail spectacularly, even over  $F$ -finite irreducible hypersurfaces. That is not to say that quasi- $F$ -pure height is not stable under any inseparable base change. In the example above, the  $p$ -base of  $\mathbb{F}_2(S, T)$  is  $\Lambda := \{1, S, T\}$ . If we only adjoin roots for  $S$  but not  $T$ , then the extension  $L = \bigcup_{i \in \mathbb{N}} \mathbb{F}_2(S^{1/2^i}, T)$  is a purely inseparable extension of  $\mathbb{F}_2(S, T)$ , but  $L \otimes_{\mathbb{F}_2(S, T)} R$  remains reduced and quasi- $F$ -split. This suggests that more careful approaches when base changing to an  $F$ -finite field may be useful;

in particular, the  $\Gamma$ -construction would rectify this problem by “tossing out” bad elements of the  $p$ -basis before their roots are adjoined.

We note that quasi- $F$ -purity is stable under arbitrary inseparable extension in specific cases, see [KTY25, lemma 4.7]. However, this result assumes that the base field is  $F$ -finite to begin with, and hence does not provide a way to reduce to the  $F$ -finite setting.

**5.2. Reduction of Conjecture to the Quasi- $F^e$ -Pure Case.** Recall that  $R \rightarrow R^\Gamma$  is a faithfully flat, and hence pure, extension. Thus if  $R^\Gamma$  is  $n$ -quasi- $F$ -pure, then so is  $R$  (lemma 2.35). In [Jag26], the author conjectured that the converse holds, i.e. for any complete local  $n$ -quasi- $F$ -pure ring  $(R, \mathfrak{m})$ , there exists a  $\Gamma$  for which  $R^\Gamma$  is  $n$ -quasi- $F$ -pure. One can naturally extend this conjecture to all quasi- $F$ -singularity types discussed in this thesis.

CONJECTURE. Let  $(R, \mathfrak{m})$  be a complete local ring and  $n, e \in \mathbb{N}$ .

- (a) If  $R$  is  $n$ -quasi- $F$ -pure, then there exists a  $\Gamma$  for which  $R^\Gamma$  is  $n$ -quasi- $F$ -pure.
- (b) If  $R$  is  $n$ -quasi- $F^e$ -pure, then there exists a  $\Gamma$  for which  $R^\Gamma$  is  $n$ -quasi- $F^e$ -pure.
- (c) If  $R$  is  $n$ -quasi- $F^\infty$ -pure, then there exists a  $\Gamma$  for which  $R^\Gamma$  is  $n$ -quasi- $F^\infty$ -pure.
- (d) If  $R$  is  $n$ -quasi- $F$ -pure regular, then there exists a  $\Gamma$  for which  $R^\Gamma$  is  $n$ -quasi- $F$ -pure regular.
- (e) If  $R$  is  $n$ -quasi- $F$  rational, then there exists a  $\Gamma$  for which  $R^\Gamma$  is  $n$ -quasi- $F$  rational.

These parts all follow (with mild restrictions) from a single verification on local cohomology:

PROPOSITION 3.89. *Let  $(R, \mathfrak{m})$  be a complete local ring and fix  $n, e \in \mathbb{N}$ . Let  $\Delta$  be a  $\mathbb{Q}$ -divisor,  $c \in R^\circ$  either be equal to 1 or chosen well when  $\Delta$  is  $\mathbb{Q}$ -Cartier (in the sense of remark 3.61), and suppose*

$$H_{\mathfrak{m}}^d(R(\Delta)) \rightarrow H_{[\mathfrak{m}]}^d(Q_{R, \Delta, n}^{e, c})$$

is injective as a map of  $W_n(R)$ -modules. If there exists a  $\Gamma$ , possibly dependent on  $n, e, c$ , and  $\Delta$ , such that

$$H_{\mathfrak{m}}^d(R^\Gamma(\Delta^\Gamma)) \rightarrow H_{[\mathfrak{m}]}^d(Q_{R^\Gamma, \Delta^\Gamma, n}^{e,c})$$

is injective as a map of  $W_n(R^\Gamma)$ -modules for  $\Delta^\Gamma := \Delta \otimes_R R^\Gamma$ , then the part (c) of the conjecture holds when  $R$  is divisorially Cohen-Macaulay, part (d) of the conjecture holds when  $R$  is  $\mathbb{Q}$ -Gorenstein, and parts (a), (b), and (e) hold in general.

PROOF. The above holding for  $(n, e, c, \Delta) = (n, 1, 1, K_R)$  is part (a) of the conjecture and  $(n, e, c, \Delta) = (n, e, 1, K_R)$  is part (b). For part (c), recall that

$$\begin{aligned} q^e S_n^0(R, 0) = R &\iff R \text{ is } n\text{-quasi-}F^e\text{-pure} \\ q^\infty S_n^0(R, 0) = R &\iff R \text{ is } n\text{-quasi-}F^\infty\text{-pure} \end{aligned}$$

Where  $q^e S_n^0(R, 0)$  and  $q^\infty S_n^0(R, 0)$  are the quasi- $F^e$ -stable sections defined in section 2.4. We recall from corollary 3.33.1 that

$$\forall_{n>0} \exists_{e_0>0} \forall_{e \geq e_0} q^\infty S_n^0(R, 0) = q^e S_n^0(R, 0)$$

Thus for  $n$  as above and  $e$  sufficiently large, choose  $\Gamma$  such that  $R^\Gamma$  is  $n$ -quasi- $F^e$ -pure by the statement of part (b) of the conjecture, which we have already checked is true. It follows then that  $q^\infty S_n^0(R^\Gamma, 0) = q^e S_n^0(R^\Gamma, 0) = R^\Gamma$ , and hence,  $R^\Gamma$  must be  $n$ -quasi- $F^\infty$ -pure.

Part (d) and (e) follow from  $c$  being a quasi-test element of  $R$ . Recall that the  $\Gamma$  construction identifies regular loci: that is,  $\Gamma$  can be chosen small enough so that  $\text{Reg}(\text{Spec}(R)) = \text{Reg}(\text{Spec}(R^\Gamma))$ . In the notation of remark 3.61, as  $R$  is  $\mathbb{Q}$ -Gorenstein we can pick  $c \in R^\circ \cap (g^2 t^4)$ . Depending on this choice of  $c$ , one can choose  $\Gamma$  small enough so that  $c \in (R^\Gamma)^\circ \cap (g^2 t^4)R^\Gamma$  and  $(g^2 t^4)R^\Gamma = ((g^\Gamma)^2 (t^\Gamma)^4)$  for  $K_{R^\Gamma} = a \cdot \text{div}(g^\Gamma)$  and  $t^\Gamma \in \tau(R^\Gamma) \cap (R^\Gamma)^\circ$  by identification of test ideals and regular loci under the  $\Gamma$ -construction. It follows then that any quasi-test element of  $R$  will be a quasi-test element  $c \in R^\Gamma$ . (d) follows from checking condition (3) of theorem 3.66 and (e) follows from check condition (d) of lemma 3.74.  $\square$

**5.3. The Gamma Construction and Witt Vectors.** For  $n = 1$ , checking this condition simply means checking that the if the composition

$$H_m^d(R(\Delta)) \xrightarrow{F^e} H_m^d(F_*^e R(p^e \Delta)) \xrightarrow{F_*^e c} H_m^d(F_*^e R(p^e \Delta))$$

is injective, then the corresponding sequence on  $(R^\Gamma, \Delta^\Gamma)$  is injective. For classical  $F$ -singularities, the following result shows how compatible the  $\Gamma$  construction is with checking injectivity of Frobenius on local cohomology:

LEMMA 3.90 ([MP25] lemma 6.13 (cf. [HH94a])). *Let  $R$  be a complete local ring of characteristic  $p$ ,  $P \in \text{Spec}(R)$  and  $M$  an Artinian  $R_P$ -Module with an injective Frobenius action. Then for all sufficiently small choices of  $\Gamma$ ,  $M \otimes_{R_P} R_{P^\Gamma}^\Gamma$  has an injective Frobenius action.*

Applying this to  $M = H_m^d(R(\Delta))$  shows that proposition 3.89 holds for  $n = 1$ . The  $n > 1$  case is more difficult in practice. Even when  $e, c = 1$  and the morphism is an honest map of  $R$ -Modules, it is not true that  $Q_{R,n} \otimes_R R^\Gamma \cong Q_{R^\Gamma,n}$ , unlike the target of the Frobenius map in the classical setting where  $F_* R \otimes_R R^\Gamma \cong F_* R^\Gamma$ . Even worse, the induced map  $Q_{R,n} \otimes_R R^\Gamma \rightarrow Q_{R^\Gamma,n}$  need not be injective.

LEMMA 3.91. *Let  $R$  be a complete local ring of characteristic  $p$ . For any  $\Gamma \neq 0$  and  $n \geq 2$ , the map  $Q_{R,n} \otimes_R R^\Gamma \rightarrow Q_{R^\Gamma,n}$  is not injective.*

PROOF. We check this when  $n = 2$  for simplicity, the general argument is analogous. As  $\Gamma \neq 0$ , we choose  $\lambda \in R$  such that  $\lambda^{1/p} \notin R$  but  $\lambda^{1/p} \in R^\Gamma$ . Choose  $F_* \overline{(0, \lambda)} \in Q_{R,2} = F_* W_n(R)/p$ ; where  $\overline{(-)}$  denotes the reduction mod  $p$ : this is non-zero by construction. However, its image in  $Q_{R^\Gamma,2} = F_* W_n(R^\Gamma)/p$  can be expressed as  $F_* \overline{(0, \lambda)} = F_* \overline{p \cdot (\lambda^{1/p}, 0)} = 0$ . Applying this argument for general  $n$ ,  $F_* \overline{(0, \lambda, 0, \dots, 0)} \neq 0$  in  $Q_{R,n}$  but

$$F_* \overline{(0, \lambda, 0, \dots, 0)} \otimes 1 \in \ker (Q_{R,n} \otimes_R R^\Gamma \rightarrow Q_{R^\Gamma,n})$$

□

There are additional difficulties when  $e > 1$  as  $Q_{R,n}^e$  is no longer an  $R$ -module, and one cannot check this condition via base change at all.

**REMARK 3.92.** *Both the  $\Gamma$ -construction and the Witt vector construction thicken  $R$  by adding in  $p$ th roots, though often in conflicting ways. Take for instance  $R = Q_{\mathbb{F}_p(t),2} = W_2(\mathbb{F}_p(t))/p$ .  $(0,t) \in R$  and  $t^{1/p} \notin \mathbb{F}_p(t)$ , but  $p \cdot (t^{1/p}, 0) = (0,t)$  inside  $W_2(\mathbb{F}_p(t^{1/p}))$ . Thus  $W_2(\mathbb{F}_p(t))/p$  essentially contains a representative of  $t^{1/p}$  as the element  $(0,t)$ , under this mod  $p$  equivalence. More generally,  $Q_{\mathbb{F}_p(t),n}$  contains elements representing the  $p^n$ -th roots of elements of  $\mathbb{F}_p(t)$ , though it explicitly doesn't contain elements like  $t^{(1/p^i)}$  for  $1 \leq i \leq n$ . Elements such as  $V^i([\lambda])$  for  $\lambda^{1/p} \in R^\Gamma \setminus R$  seem to generate the kernel of the map  $Q_{R,n} \otimes_R R^\Gamma \rightarrow Q_{R^\Gamma,n}$  in the previous lemma.*

**5.4. Applications.** As a consequence of prior results in this thesis (corollary 3.27.1, lemma 3.63, lemma 1.61, lemma 1.54, proposition 3.79, corollary 2.39.1), all singularity types in the conjecture on the Gamma construction complete. Hence, If the conjecture holds, then one can systematically reduce the study of quasi- $F$ -singularities of excellent rings to the study of  $F$ -finite rings, and can apply known results in the  $F$ -finite setting to state results in higher generality. For instance, the Fedder type criterion (theorem 2.51) of [KTY22] used to test for  $n$ -quasi- $F$ -splitting will naturally generalize outside of the  $F$ -finite setting:

**REMARK 3.93.** *Let  $(A, \mathfrak{m})$  be an excellent regular local ring and  $I = (f_1, \dots, f_c) \subseteq A$  an ideal generated by a regular sequence. Then one can compute the quasi- $F$ -pure height of  $R = A/I$  as follows. First, it is sufficient to consider the completion  $\widehat{R} \cong \frac{k[[x_1, \dots, x_d]]}{I}$  by corollary 2.39.1. Then one can choose  $\Gamma$  small enough so that  $R^\Gamma := \frac{k^\Gamma[[x_1, \dots, x_d]]}{I}$  is a complete intersection and  $\text{ht}(R) = \text{ht}(\widehat{R}) = \text{ht}(R^\Gamma)$ . Then one runs the Fedder type criterion [KTY22, Theorem 4.11] on  $R^\Gamma$ .*

The Fedder type criterion for quasi- $F^e$ -purity and quasi- $F$ -pure regularity [Yos25] can be used to compute the quasi- $F^e$ -pure height and detect quasi- $F$ -regularity by a similar approach. Thus, many of the quasi- $F$ -pure height computations in [KTY22, section 7], such as that of Fermat hypersurfaces and rational double points, hold without the  $F$ -finite

assumption.

In suitable settings, one can reduce the study of quasi- $F$ -purity of varieties  $X$  over an arbitrary field of positive characteristic  $k$  to the case where  $k$  is perfect.

LEMMA 3.94 ([KTT<sup>+</sup>24b], lemma 7.1). *Let  $k \subseteq k'$  be a field extension of  $F$ -finite fields of characteristic  $p > 0$ . Let  $X$  be a projective normal variety over  $k$  such that  $H^0(X, \mathcal{O}_X) = k$  and  $X \times_k k'$  is normal. Take an effective  $\mathbb{Q}$ -divisor  $\Delta$  on  $X$ . Then the following hold:*

- (a) *If  $(X \times_k k', \Delta \times_k k')$  is  $n$ -quasi- $F^e$ -split, then so is  $(X, \Delta)$ .*
- (b) *If  $(X \times_k k', \Delta \times_k k')$  is  $n$ -quasi- $F$ -regular, then so is  $(X, \Delta)$ .*

Now suppose  $k$  as above were not  $F$ -finite. By the conjecture, there would exist a  $\Gamma$  for which  $X^\Gamma := X \times_k k^\Gamma$  is projective, normal, and  $H^0(X^\Gamma, \mathcal{O}_{X^\Gamma}) = k^\Gamma$ . Letting  $\bar{k}$  be the perfect, or even algebraic, closure of  $k$ , one sees that  $k^\Gamma \subseteq \bar{k}$  is an extension of  $F$ -finite fields. Hence, we have the following corollary:

COROLLARY 3.94.1. *Let  $k$  be any field of positive characteristic and  $X$  be a projective normal variety over  $k$  such that  $H^0(X, \mathcal{O}_X) = k$  and  $X \times_k \bar{k}$  is normal. Let  $\Delta$  be a  $\mathbb{Q}$ -divisor on  $X$ . Then if  $(X \times_k \bar{k}, \Delta \times_k \bar{k})$  is  $n$ -quasi- $F^e$ -pure (resp. quasi- $F$ -pure regular) then so is  $(X, \Delta)$ .*

This allows for quasi- $F^e$ -purity and quasi- $F$ -pure regularity of such a variety to be verified over the base change of it to a perfect field. This immediately allows us to remove the assumption about being over a perfect field of many of the results in section 4: We state some of these consequences below:

- If  $(X, \Delta)$  is a KLT surface of finite type over any field of positive characteristic and  $H^0(X, \mathcal{O}_X) = k$ , then  $(X, \Delta)$  is quasi- $F$ -pure regular.
- If  $(X, \Delta)$  is a normal log Del Pezzo pair over any field of (sufficiently large) positive characteristic and  $H^0(X, \mathcal{O}_X) = k$ , then  $(X, \Delta)$  is 2-quasi- $F$ -pure.
- If  $X$  is a  $\mathbb{Q}$ -factorial KLT threefold over any field of characteristic  $p > 42$  and  $H^0(X, \mathcal{O}_X) = k$ , then  $X$  is quasi- $F$ -pure regular.

These follow from passing to  $X^\Gamma$ , choosing  $\Gamma$  small enough for  $X^\Gamma$  to still satisfy the given hypotheses (see [Mur21, Corollary 3.7]), then citing the prior corollary.

The  $\Gamma$ -construction also extends results of Garzella and the author [GJ25] from chapter 2 section 4 to holding over arbitrary base fields.

Let  $(A, \mathfrak{m})$  be an excellent regular local ring with  $f \in A$  an isolated singularity (which up to a change of coordinates, can be assumed to be at  $\mathfrak{m}$ ) such that  $A/(f)$  is quasi- $F$ -pure. Quasi- $F$ -purity completes, so it is sufficient to assume that  $A = k[[x_1, \dots, x_d]]$  and that  $R := \frac{k[[x_1, \dots, x_d]]}{f}$  is quasi- $F$ -pure. If the conjecture holds, then one can find a  $\Gamma$  for which  $R^\Gamma = \frac{k^\Gamma[[x_1, \dots, x_d]]}{f}$  is quasi- $F$ -pure and  $F$ -finite. We then cite theorem 2.52 to compute the  $F$ -pure threshold of  $f$  in  $R^\Gamma$ . But the  $F$ -pure threshold is agnostic of the base field and completion in this setting, allowing us to conclude that theorem 2.52 holds without the  $F$ -finite hypothesis.

## APPENDIX A

**Local Cohomology for non-Noetherian Rings**

Though Local cohomology modules  $H_I^i(R)$  are well defined over non-Noetherian rings, they may fail to satisfy expected properties when  $I$  is not finitely generated. In this section we will reprove a few standard results about local cohomology over non-Noetherian rings, but taken over a finitely generated ideal. Much of this information is known to experts, but we reprove them for the benefit for the reader.

Here,  $R$  will be a ring (not necessarily Noetherian) of characteristic  $p > 0$ .

LEMMA A.1. *Let  $R$  be any commutative ring, and  $M$  any  $R$ -module. Choose  $I, J \subset R$  ideals and  $n, m \in \mathbb{N}$  such that  $I^n \subset J$  and  $J^m \subset I$ . Then,*

$$(1) H_J^i(M) = H_I^i(M)$$

$$(2) \widehat{R}_I = \widehat{R}_J$$

PROOF. We utilize the definition

$$H_I^i(M) := \varinjlim_t \text{Ext}_R^i(R/I^t, M)$$

As  $I^n \subset J$  and  $J^m \subset I$  for finite  $n, m$ , it follows that the directed systems  $\{I^t\}_t$  and  $\{J^t\}_t$  are cofinal with each other. Thus,  $\varinjlim_t \text{Ext}_R^i(R/I^t, M) = \varinjlim_t \text{Ext}_R^i(R/J^t, M)$ , and the first result follows. Similarly, the second result follows from the fact that  $\{I^t\}_t$  and  $\{J^t\}_t$  are cofinal with each other.

□

LEMMA A.2. *Let  $(R, \mathfrak{m})$  be a local ring and  $I = (x_1, \dots, x_\ell) \subset R$  a finitely generated ideal. Then  $H_I^\bullet(-)$ ,  $\check{H}_I^\bullet(-)$  and  $H^\bullet(\underline{x}^\infty; -) := \varinjlim_d H^\bullet(x_1^i, \dots, x_d^i; -)$  determine the same cohomology theory.*

PROOF. This can be seen from the following facts:

- (a)  $H_I^\bullet(-)$ ,  $\check{H}_I^\bullet(-)$  and  $H^\bullet(\underline{x}^\infty; -)$  are all universal  $\delta$ -functors.
- (b) If  $M$  is injective,  $H_I^i(M) = \check{H}_I^i(M) = H^i(\underline{x}^\infty; M) = 0$  for all  $i > 0$ .
- (c)  $H_I^0(M)$ ,  $\check{H}_I^0(M)$ ,  $H^0(\underline{x}^\infty; M)$  are all canonically isomorphic, as they can all be identified with the set

$$\bigcup_{n \in \mathbb{N}} \text{Ann}_M(I^n) = \{m \in M \mid \exists n \text{ such that } I^n m = 0\} \subset M$$

Note that this equivalence requires that  $I$  be finitely generated; otherwise not all of these modules necessarily have this description.

- (d)  $H_I^\bullet(-)$ ,  $\check{H}_I^\bullet(-)$  and  $H^\bullet(\underline{x}^\infty; -)$  have functorial long exact sequences induced via acting on a short exact sequence of  $R$ -modules.

(a) implies that functors which satisfy (b),(c), and (d) are unique up to unique isomorphism [Gro57]. □

LEMMA A.3. *Suppose that  $R \rightarrow S$  is a morphism of rings and  $M$  is an  $S$ -module (and thus, also an  $R$ -module). Let  $I = (x_1, \dots, x_\ell) \subset R$  be a finitely generated ideal. Then  $H_I^i(M) \cong H_{IS}^i(M)$  as  $R$ -modules.*

PROOF. Let  $y_1 \dots y_\ell$  be the images of the  $x_1, \dots, x_\ell$  in  $S$ . Via the previous lemma, we have that

$$H_I^i(M) \cong H^i(\underline{x}^\infty; M) \quad H_{IS}^i(M) \cong H^i(\underline{y}^\infty; M)$$

As the  $x_i$  and  $y_i$  act on  $M$  identically by definition of the  $y_i$  and the inherited  $R$ -module structure on  $M$ , we see that  $H^i(\underline{x}^\infty; M) \cong H^i(\underline{y}^\infty; M)$  as  $R$ -modules. The result follows. □

LEMMA A.4. *Suppose  $R$  is a ring of dimension  $d$ ,  $M$  an  $R$ -Module,  $S$  an  $R$ -algebra, and  $I = (x_1, \dots, x_\ell) \subset R$  a finitely generated ideal. Then  $S \otimes_R H_I^d(M) \cong H_I^d(S \otimes_R M) \cong H_{IS}^d(S \otimes_R M)$ .*

PROOF. The second isomorphism follows from Lemma A.3. Thus we focus on the first isomorphism. Utilizing the equivalence  $H_I^d(M) \cong H^d(\underline{x}^\infty; M) \cong H^d(K_\bullet(\underline{x}^\infty; M))$ , we want to

show that

$$S \otimes_R H_I^d(M) \cong S \otimes_R H^d(K_\bullet(\underline{x}^\infty; M)) \cong H^d(K_\bullet(\underline{x}^\infty; M) \otimes_R S) \cong H_I^d(S \otimes_R M)$$

It suffices to show the second  $\cong$  holds. Indeed,

$$H^d(K_\bullet(\underline{x}^\infty; M)) := \varinjlim H^d(K_\bullet(\underline{x}^t; M))$$

As  $\dim(R) = d$ , this is a colimit of top cohomology modules, and hence a colimit of cokernels. Cokernels are indeed colimits themselves, and tensor products commute with colimits [AM69, Exercise 2.20]. The result follows.  $\square$

LEMMA A.5. *Suppose  $R$  is a ring of dimension  $d$ ,  $I \subset R$  a finitely generated ideal, and  $M$  an  $R$ -Module. Then,  $H_I^d(M) \cong H_{\widehat{R}_I}^d(\widehat{R}_I \otimes_R M)$ .*

PROOF.  $H_I^d(M)$  is an  $I$ -torsion module, so one immediately sees that

$$H_I^d(M) \cong \widehat{R}_I \otimes_R H_I^d(M)$$

Via the previous lemma,

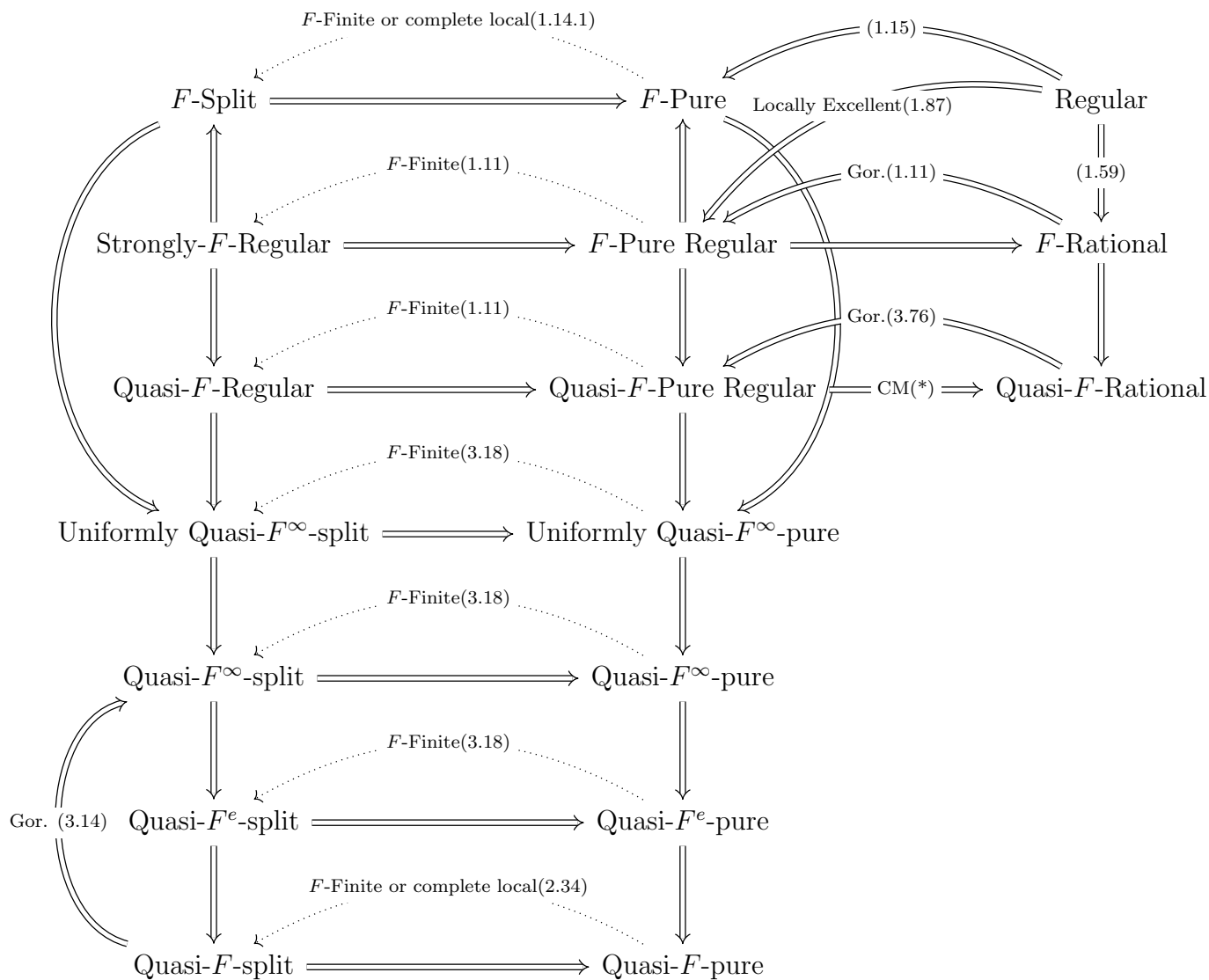
$$\widehat{R}_I \otimes_R H_I^d(M) \cong H_{\widehat{R}_I}^d(\widehat{R}_I \otimes_R M)$$

$\square$

APPENDIX B

**Relations between Singularity Types**

The following implications hold for any Noetherian ring of characteristic  $p > 0$  for which the corresponding singularity classes are defined.



We make the following remarks about when certain arrows above can be reversed or when hypotheses can (or can not) be removed.

- The (\*) next to the Cohen Macaulay requirement for Quasi- $F$ -pure regularity implying quasi- $F$ -rationality denotes that it (conjecturally) can be removed. Quasi- $F$ -pure regularity implies Quasi- $F$ -rationality in dimensions  $\leq 3$  and when  $R$  is  $F$ -finite (see remark 3.76 and [KTT<sup>+</sup>24b, Theorem 8.9]) without assuming  $R$  is Cohen-Macaulay. It is conjectured that all quasi- $F$ -regular rings are Cohen-Macaulay, much like strongly  $F$ -regular rings.
- There exists rings that are  $F$ -pure regular (and hence, every singularity type in the second column) but not quasi- $F$  split (and hence, fails to be every singularity type in the first column) as a consequence of theorem 2.28. This ring is neither  $F$ -finite nor complete local, and hence, the  $F$ -finite assumptions (or complete and local assumptions) for the dotted lines to hold are necessary.
- Regular only implies  $F$ -pure regular when  $R$  is locally excellent, see [HY23, example 6.4, 6.8] (cf. 1.86) for a class of dimension 2 regular rings that are not  $F$ -pure regular.
- As a consequence of Theorem 3.15, any quasi- $F$ -split but not  $F$ -split abelian variety (such as a supersingular elliptic curve) is quasi- $F^\infty$ -split but not uniformly quasi- $F^\infty$ -split. Similarly, an ordinary elliptic curve is  $F$ -split, hence uniformly quasi- $F^\infty$ -split, but not quasi- $F$ -regular. Looking at the cone point of these varieties, one can similarly construct examples of quasi- $F^\infty$ -pure but not uniformly quasi- $F^\infty$ -pure and uniformly quasi- $F^\infty$ -pure but not quasi- $F$ -pure regular rings.
- For any smooth projective variety  $X$  over a perfect field such that  $K_X$  is pseudo-effective,  $F$ -splitting and uniform quasi- $F^\infty$ -splitting are equivalent [TWY24, proposition 7.4]. This need not hold in general however; consider

$$R = \frac{\mathbb{F}_2[[x, y, z]]}{z^2 + x^3 + y^2z}$$

This is 2-quasi- $F$ -regular [Yos25, example 5.12] and hence uniformly quasi- $F^\infty$ -split, but  $z^2 + x^3 + y^2z \in (x^2, y^2, z^2)$  so  $R$  is not  $F$ -split by Fedder's criterion.

- Every singularity type in the middle and right column are local notions; that is,  $R$  satisfies property  $\mathcal{P}$  if and only if  $R_P$  does for every  $P \in \text{Spec}(R)$ . Every singularity

type in the left column localizes (i.e  $R$  satisfying property  $\mathcal{P}$  implies  $R_P$  does for every  $P \in \text{Spec}(R)$ ) but the converse need not hold in every case. Along these lines, every singularity type in the left column is a fundamentally different (and stronger) invariant when viewed as a global invariant on  $(X, \mathcal{O}_X)$ , whereas the invariants in the middle and right columns can be verified at the level of stalks. Thus for ( $F$ -finite) non-affine schemes, invariants in the middle column do not imply their counterparts in the left column.

## APPENDIX C

**Permissions**

This thesis contains material from a published paper of the author [**Jag26**]. The author is permitted to reference material from this paper in this thesis. In addition, some of the material in chapter 2 section 4 in this thesis is reproduced from a (submitted, but not yet published) paper coauthored by Jack J Garzella and the author [**GJ25**]. Jack J Garzella has graciously consented to the reproduction of this material in the author's thesis.

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- [Yos25] ———, *Fedder-type criterion for quasi- $f^e$ -splitting and quasi- $f$ -regularity*, 2025.
- [Zda18] Maciej Zdanowicz, *Liftability of singularities and their Frobenius morphism modulo  $p^2$* , Int. Math. Res. Not. IMRN (2018), no. 14, 4513–4577. MR 3830576

## Vita

For an up-to-date copy of the author's CV, please visit [vigj.net/research](http://vigj.net/research).

# Vignesh Jagathese

## Mathematics PhD, Candidate, University of Illinois, Chicago

### Education

May 2026 PhD, Mathematics *University of Illinois Chicago (UIC)*

Advisor: Kevin Tucker

April 2020 Masters of Science, Mathematics *University of Michigan (UofM)*

April 2020 Bachelors of Science, Honors Mathematics *University of Michigan (UofM)*

### Mathematical Interests

Commutative Algebra, Algebraic Geometry,  $F$ -Singularities, Quasi- $F$ -Splittings,  $F$ -Pure and Plus-Pure Thresholds, Singularities via Cartier Operators

### Work Experience

2020-2021 Quantitative Trader and Researcher, *VECO Power Trading*

Summer 2019 Quantitative Trading Intern, *Chicago Trading Company*

2017-2018 Operations Director, *University of Michigan Solar Car Team*

### Publications and Preprints

- 1) **Bounds on the plus-pure thresholds of some hypersurfaces in (ramified) regular rings** (2026), with Marta Benozzo, Vaibhav Pandey, Pedro Ramirez Moreno, Karl Schwede, and Prashanth Sridhar (*Mathematische Zeitschrift*. to appear)
- 2) **On quasi- $F$ -purity of excellent rings (2026)**, (*Communications in Algebra* 54 (4). 1760-1771 DOI: 10.1080/00927872.2025.2565687)
- 3) **On  $F$ -pure thresholds and quasi- $F$ -purity of hypersurfaces** (2025), with Jack J Garzella (Submitted)

### Awards and Grants

2025 UIC Award for Graduate Research

2023 UIC Professor Herbert Alexander Award

2022-2016 NSF RTG Fellowship

Spring 2019 NSF REU

### Teaching

Instructor of Record (UIC)	TA Appointments (UIC)	Grading Positions (UofM)
Math 109 (1 Cr, F-2022, S-2025)	Math 110 (F-2021, F-2022, S-2025)	Math 417 (W-2020)
Math 090 (3 Cr, F-2023)	Math 105 (S-2023)	Math 452 (W-2019)
Math 104 (1 Cr, S-2023)	Math 125 (F-2021)	

Fall 2025 Teaching Assistant Coordinator

### Invited Talks

04/2026 AMS Sectional at North Dakota State University: Topics in Commutative Algebra  
 02/2026 University of Utah Commutative Algebra Seminar  
 01/2026 JMM special session: Advances in Commutative Algebra  
 11/2025 University of Missouri Algebra Seminar  
 10/2025 University of Nebraska Commutative Algebra Seminar  
 09/2025 Purdue Commutative Algebra Seminar  
 05/2025 UC San Diego Algebraic Geometry Seminar  
 03/2024 UIC Incoming Students Colloquium (*Also 03/2025*)  
 01/2023 UIC Graduate Student Colloquium (*Also 04/2024, 01/2025*)  
 04/2022 UIC Undergraduate Math Club (*Also 08/2023*)

### Service

- Journal Referee (*Refereed for Mathematische Zeitschrift.*)

2024-2025 UIC Directed Reading Program Advisor  
 (*Spring 2025*) Advised statistics undergraduate Juan Garcia on introductory abstract algebra.  
 (*Fall 2024*) Advised electrical engineering and physics undergraduate Caleb Williams on applications of category theory.  
 (*Spring 2024*) Advised mathematics undergraduate Max Nguyen on sheaf cohomology, specifically on computations via the Čech complex.

2023-2024 UIC Commutative and Homological Algebra Graduate Seminar Organizer  
 2022-2023 UIC Mathematics Graduate Student Association Secretary

### Outreach

2022-2026 Math Circles of Chicago Teacher  
 11/2024 Sonia Kowalevsky Math Day Organizer  
 Summer 2024 Young Scholars Program Instructor  
 2018-2020 Wolverine Pathways Instructor  
 4/2019 MMaths Math Competition Organizer