

Preliminaries

These are notes from the Summer Research Institute in Algebraic Geometry bootcamp for graduate students, which ran in Colorado State University from July 8th to July 12th, 2025. More information can be found [here](#). The bootcamp was organized by İzzet Coşkun (University of Illinois Chicago), Emanuele Macrì (Université Paris-Saclay), Alexander Perry (University of Michigan), Kevin Tucker (University of Illinois Chicago), and Isabel Vogt (Brown University). These notes were (quite hastily) typed up by me during the talks, and thus are rife with typos and missing citations. I also missed a couple talks, so I'm very sorry if your talk is not included! I can provide assurance that all errors here were caused by me and not by whoever was giving the lecture. As such,

reading these notes is allowed



you submit any typos/errors to me at [vjagat2 \(at\) uic \(.edu\)](mailto:vjagat2@uic.edu).

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0.1 Kristin DeVleming - K-stability and K-moduli spaces

The goal of Moduli is to classify geometric objects up to some equivalence. For instance, one can construct the Moduli of curves \mathcal{M}_g to classify curves of genus g up to isomorphism. You can also construct moduli of other things, like sheaves, or vector bundles, or K3 surfaces. We will be discussing the moduli of (projective) varieties over \mathbb{C} . In general, one can construct moduli via these steps:

1. Fix discrete invariants. (say $\dim = n$, the genus, the Hilbert polynomial, etc.)
2. Construct something that continuously parameterizes objects with those invariants.

Our goal is to construct some space \mathcal{M} such that there are a bijection between points $p \in \mathcal{M}$ and equivalence classes of objects. In light of this, we often write $p = [X]$, where X is some object that acts as a representative of the equivalence class that is bijectively identified with p . Similarly, one would also like to construct subsets $V \subset \mathcal{M}$ such that every V is in bijective correspondence with continuously parameterized families \mathcal{X} in the moduli space. If we have a good notion of subsets then one can then ask: what is the structure of \mathcal{M} itself as a space? Is it a topological space? A scheme? A variety?

Let's do a specific example and classify the set of genus $g \geq 2$ curves C up to isomorphism. $\forall C, \omega_C^{\otimes 3}$ is very ample and gives an embedding into projective space. Further,

$|\omega_C^{\otimes 3}| : C \hookrightarrow \mathbb{P}^N$ for every C , for the same N ! Thus $\exists \text{Hilb}(\mathbb{P}^N)$, the Hilbert scheme parameterizing all subvarieties of \mathbb{P}^N . So now consider $U \subset \text{Hilb}(\mathbb{P}^N)$ the locus of all images of curves. Then, $U/\cong =: \mathcal{M}$.

If we were to generalize this construction, we will need to construct a similar thing to $\omega_C^{\otimes 3}$; the fact that the third power is always very ample is crucial to the above construction, and it doesn't work for all families of varieties. Indeed, the fact that this is always very ample at just the third power can be framed as some kind of invariant being bounded. We will also need to generalize the notion of modding U out by equivalence when that equivalence is not just being an isomorphism; being able to do this systematically is (in part) the who reason why GIT was developed!

Our 'secret' goal here is to construct a **proper** moduli space of objects, where we include the limit points at the boundary of M . We not only want these limit points to exist, but these limit points ought to parameterize a limit of the geometric objects that the points correspond to (thus just naively taking the closure is NOT the thing to do!). We need to enlarge our class of objects to include possible limits of families and perhaps also enforce separatedness; limits must be unique.

For curves, this is precisely what the Deligne-Mumford Compactification does. The moduli of smooth genus g curves lives inside the set of stable genus g curves; stable meaning that the singularities are at worst nodal and ω_C is still ample. The DM Compactification has a key property; any family of smooth curves can be completed uniquely to a family of stable curves. Further, there is an algorithm to construct this object in the limit; this is secretly the MMP. Since projective space is proper, and these moduli are projective, we can simply compactify a family V so that the limit point is filled in; this thing is arbitrarily bad if you resolve that thing you just filled in, then it ends up not being bad. In fact, as we are resolving the total space of this family, so certain fibers may still be singular but are at worst SNCs. Finally, we use the minimal model program to contract parts where ω_C is not ample, which in fact constructs a unique final object. This final family is precisely the 'larger' moduli space that we want.

This leads us to the more general case; we can follow the algorithm above to construct a proper moduli of varieties X where ω_X is ample. Can we generalize to moduli of Fano varieties? By Fano varieties we mean varieties for which ω_X is anti-ample. The short answer is no; the traditional approach has issues with being generalized. We needed this sort of boundedness discussed earlier, and we must include ilidly singular varieties if we hope to get a parameterization. We cannot do this! Consider the family $\{\mathbb{P}(1, 1, n)\}_n$; as n can get arbitrarily large this family is not bounded; you will need successively larger and larger N for which we can embed them into \mathbb{P}^N . This has unbounded volume, so we can also consider another example $\{\mathbb{P}(a, b, c) \mid a^2 + b^2 + c^2 = 3abc\}$. These are all Fano varieties with $V := (-K_X)^2 = 2$, but there are infinitely many of these!

There are more issues. Quotients are not well behaved; for a given variety of X the au-

tomorphism group need not be reductive, thus if you want to take a quotient by a group action, the quotient ring need not be finitely generated. Even simple examples exhibit this, like \mathbb{P}^2 blown up at a point. Further, MMP does not produce unique limits as they did in the standard case. For instance, take a family of \mathbb{P}^1 s. We can converge it to a copy of \mathbb{P}^1 ; this has anti-ample canonical and is in fact a limit. However, we can also limit to $\mathbb{P}^1 \cup \mathbb{P}^1$, which is still Fano and satisfies all the desired properties.

This situation looks dire, but we have a fix: K -stability. In the late 90s and early 2000s, Tian and Donaldson wanted to introduce an algebraic criterion to capture the existence of a Kähler-Einstein metric on smooth Fano manifolds. Now, we have a theorem/definition as follows: A Fano variety is K -polystable if and only if it admits a KE-Metric. Before going through definitions, here are some consequences of K -stability:

1. K -semistable Fano varieties of dimension n are bounded; in particular there is a projective space which they ALL embed into.
2. Any Fano polystable variety has reductive automorphism group. Thus, quotients 'work'.
3. K -semistability is an open condition in families. Thus, the corresponding locus $U \subset \text{Hilb}(\mathbb{P}^N)$ is 'algebraic' in a meaningful way.
4. K -moduli spaces (e.g. moduli of K -polystable things) are proper. However, we caution that there is no algorithm like in the easier case; this is a major open problem!

Theorem 0.1.1 (Xu++). *If you fix a dimension n and V the volume. Then there exists an Artin stack $\mathcal{M}_{n,V}^{k-ss}$ mapping down to $M_{n,V}^{K-ps}$ that is projective and a good moduli space, parameterizing K -semistable and K -polystable fanos with $\dim n$ and $(-K_X)^n = V$ respectively.*

Examples:

- For $\dim = 1$, \mathbb{P}^1 is the only K -semistable (or K -polystable) Fano of dimension 1. Thus the corresponding Moduli stack is just a point.
- For $\dim = 2$, X smooth Fanos must be blowups of \mathbb{P}^2 at ≤ 8 points or $\mathbb{P}^1 \times \mathbb{P}^1$. Not all of these are stable, though; X blown up at ≤ 2 points are unstable for instance. The rest of them, however, are K -semistable and often you can construct their moduli using other notions like GIT. For instance, for cubic surfaces the K -moduli agrees with the GIT moduli.

Now let's provide a definition. Let's say X is a Fano with at worst KLT singularities. A special test configuration is defined to be a family $\mathcal{X} \rightarrow \mathbb{A}^1$ such that $\mathcal{X}_{\mathbb{G}_m} \rightarrow \mathbb{G}_m$ is \mathbb{G}_m -equivariantly isomorphic to the trivial family $X \times \mathbb{G}_m \rightarrow \mathbb{G}_m$, and at 0, $\mathcal{X}_0 \rightarrow 0$ is KLT and Fano. Given a special test configuration, we can glue to the trivial family $X \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ to form a family $\overline{\mathcal{X}} \rightarrow \mathbb{P}^1$. The Futaki invariant $\text{Fut}(\mathcal{X}) := \frac{-1}{2(-K_X)^n(n+1)} (-K_{\overline{\mathcal{X}}/\mathbb{P}^1})^{n+1}$. We say that X is K -semistable (resp. K -stable) if \forall special test configurations, $\text{Fut}(\mathcal{X}) \geq 0$ (resp. > 0). We say that X is K -polystable if it is K -semistable and for all special test configurations $\text{Fut}(X) = 0$ implies that $\mathcal{X} = X \times \mathbb{A}^1$.

0.2 Junliang Shen - Decomposition theorem of algebraic maps and applications

Just like the last talk, this will all be over \mathbb{C} .

We will begin with a brief review of spectral sequences. Suppose $M \rightarrow B$ is a topological fibration over a base B with fibers $F \hookrightarrow M$. The Leray Spectral Sequence tells us that we can recover all cohomological data of M from the data of F and B . For example, consider the Hopf Fibration where $M = S^3$, $B = S^2$, $F = S^1$. In this case, on the second page of the spectral sequence we have that

$$H^p(S^2, R^q f_* \mathbb{Q}) \Rightarrow H^{p+q}(S^3)$$

Where we can view the coefficients as the relative cohomology of the fibers f :

$$R^q f_* \mathbb{Q} = \begin{cases} \mathbb{Q} & q = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

The E_2 page is as follows:

$$\begin{array}{ccccccc} H^0(S^2) & & 0 & & H^2(S^2) & & 0 & & 0 & & \dots \\ & \searrow & & \searrow & & \searrow & & \searrow & & \searrow & \\ H^0(S^2) & & 0 & & H^2(S^2) & & 0 & & 0 & & \dots \end{array}$$

Let the first diagonal map be d_2 , which can be shown to be the identity map. Passing to the E_3 page, we see that these objects are trivial:

$$\begin{array}{ccccccc} \ker(d_2) = 0 & & 0 & & H^2(S^2) & & 0 & & 0 \\ & & & & & & & & & & \\ H^0(S^2) & & 0 & & \text{coker}(d_2) = 0 & & 0 & & 0 \end{array}$$

In particular, this shows that $H^i(S^3) = \mathbb{Q}$ for $i = 0, 3$ and vanishes otherwise. In algebraic geometry, this process is actually quite simpler due to work of Deligne.

Theorem 0.2.1 (Deligne). $d_r = 0$ for $r \geq 2$ if $f : M \rightarrow B$ is a smooth projective map. In particular,

$$H^k(M) = \bigoplus_{p+q=k} H^p(B, R^q f_* \mathbb{Q})$$

This is the baby case of the decomposition theorem and can be proved via Hodge theory.

Proof. Take $\eta \in H^0(B, R^2 f_* \underline{\mathbb{Q}})$ relatively ample for $f : M \rightarrow B$. We have the Relative hard Lefschetz:

$$R^q f_* \underline{\mathbb{Q}} = R_{\text{prim}}^q f_* \underline{\mathbb{Q}} \oplus \eta R_{\text{prim}}^{q-2} f_* \underline{\mathbb{Q}} \oplus \eta^2 \dots$$

Now set R to be the complex dimension of the fiber.

$$\begin{array}{ccc} H^p(B, R^q f_{\text{prim}} \underline{\mathbb{Q}}) & \xrightarrow{d_1 \neq 0} & H^{p+2}(B, R^{q-1} f_* \underline{\mathbb{Q}}) \\ \downarrow & & \downarrow \\ H^p(B, R^{2R-q+2} f_* \underline{\mathbb{Q}}) & \xrightarrow{d_2} & H^{p+2}(B, R^{2R-q+1} f_* \underline{\mathbb{Q}}) \end{array}$$

As the first vertical map is 0 and the second map is an isomorphism we can conclude that $d_2 = 0$. The higher order versions are similar. \square

Theorem 0.2.2 (Deligne). *If $f : M \rightarrow B$ is a smooth projective map,*

1. $Rf_* \underline{\mathbb{Q}}_M \cong \bigoplus_i R^i f_* \underline{\mathbb{Q}}_M[-i] \in D_{\text{cons}}^b(B)$
2. *Each $R^i f_* \underline{\mathbb{Q}}_M$, a local system, is in fact semisimple. In other words, it is a finite direct sum of irreducible local systems.*

The first statement show that $Rf_* \underline{\mathbb{Q}}_M$ can decompose as a product of local systems, and the second part shows that it can be further decomposed into a product of irreducible local systems which can each be viewed as irreducible $\pi_1(B)$ representations.

But what about general projective morphisms $f : M \rightarrow B$? We still have objects $Rf_* \underline{\mathbb{Q}}_M$ and $R^i f_* \underline{\mathbb{Q}}_M[-i]$, but the latter need not be local systems. In this case, however, these need not be isomorphic; the obstruction again comes from Hodge theory. For example, pick $f : X \rightarrow Y$ is a resolution of singularities where Y is a singular projective variety where $f_* \underline{\mathbb{Q}}_X \cong \underline{\mathbb{Q}}_Y$. In this setting, say there was a natural decomposition as in the theorem where $H^k(Y, \underline{\mathbb{Q}}_Y) \subset H^k(X, \underline{\mathbb{Q}})$ is a direct summand. As X is smooth, $H^k(X, \underline{\mathbb{Q}})$ is a pure hodge structure of weight k , but $H^k(Y, \underline{\mathbb{Q}}_Y)$ need not be a pure Hodge Structure; it only need be a mixed Hodge Structure. Thus such a decomposition cannot be expected to hold in general.

Fortunately, some work has done in finding a solution. Goresky and MacPherson developed a topological theory earlier on, but more recent work of Beilinson, Bernstein, Deligne, and Gabber (BBDG) generalized Deligne's work. To do this, they replaced $\underline{\mathbb{Q}}_Y$ with a nicer, but more complicated, object $IC_Y \in D_{\text{cons}}^b(Y)$ such that $H^*(Y, IC_Y) =: IH^*(Y)$ is pure. We note that the construction of IC_Y is sensitive to the singularities of Y . Generally, for Y a variety over \mathbb{C} we associate an abelian category $\text{Perv}(Y)$ inside $D_{\text{cons}}^b(Y)$ that is analogous to $\text{Sh}_{\text{cons}}(Y)$ of Perverse sheaves. This is a "nice" category, closed under \otimes that is both Artinian and Noetherian. The simple objects in this category are built from IC_Y and irreducible local systems.

We construct IC_Y as follows. Within Y lies a closed $Z \xrightarrow{i} Y$ and within Z lies an open $U \xrightarrow{j} Z$. Let \mathcal{L} denote a local system on U . We use 6-functor formalism to extend \mathcal{L} ; in particular we define $IC_Z(\mathcal{L}) := i_* ({}^p j_{!*} \mathcal{L}[du])$. All simple objects are of this form for some U, Z . Further, take if we take $U \subset Z$ to be some smooth locus and take $Z = Y$, we define $IC_Y := IC_Y(\underline{\mathbb{Q}}_U)$.

Theorem 0.2.3 (BBDG). *Suppose $f : M \rightarrow B$ is a projective morphism. For simplicity let M be smooth.*

1. f admits a decomposition $Rf_* \underline{\mathbb{Q}}_M \cong \bigoplus_i {}^p \mathcal{H}^i (Rf_* \underline{\mathbb{Q}}_M) [-i + \text{shift}]$.
2. Each ${}^p \mathcal{H}^i (Rf_* \underline{\mathbb{Q}}_M) \in \text{Perv}(B)$.

When f is smooth this recovers the prior theorem of Deligne. The initial proof of this required a reduction to characteristic p then utilized the weight theory of the Weil Conjectures. Later, Saito proved this intrinsically over \mathbb{C} by developing a weight theory purely over \mathbb{C} using Hodge Modules. This approach is transcendental, unlike the char p proof which was algebraic, and was quite complicated. Later, de Cataldo and Migliorini deduced the result with simpler methods using just Deligne's theorem in the smooth case and classical Hodge theory.

0.3 Emelie Arvidsson - A Gentle Introduction to Characteristic p Geometry

In 1882, Weber and Dedkind published a German paper which had the English title "Theory of Algebraic functions in one variable". This paper was motivated by Riemann's theory of surfaces around 30 years prior, and the corresponding analogy:

$$\{\text{Number fields } \mathbb{Q} \subset K\} \leftrightarrow \{\text{function fields}\}$$

The authors noticed that this correspondence holds even for finite fields. This correspondence is still useful in relatively modern times and inspired work of Zariski and Weil. Grothendieck was motivated by this correspondence to introduce etale cohomology, and is even reinvented in a contemporary flavor via Scholze.

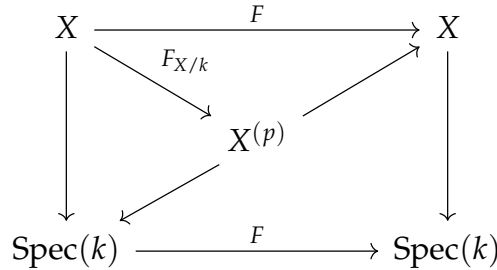
In 1961, Serre constructed an example of a characteristic $p > 0$ variety that does not lift to characteristic 0; this is in some sense the first example of positive characteristic geometry that didn't just come from the characteristic 0 setting. In 1978, Raynaud showed that Kodaira Vanishing fails for smooth varieties in positive characteristic, suggesting that the story is quite different.

Along these lines of contrasting with the characteristic 0 picture, Generic smoothness fails. Let $X \subset \mathbb{A}^3 = \text{Spec}(k[x, y, t])$. be defined by the equation $y^2 + x^p + t$. Now consider the projection of X down to $\mathbb{A}^1 = \text{Spec}(k[t])$. What do the fibers of this map look like?

Take $c \in \mathbb{A}^1$. $f^{-1}(c) = V(\varphi_c = y^2 + x^p + c)$. We see that if $y = 0$, the fiber is singular, the x, y partials of this equation are of the form $0, 2y$ respectively. Thus, we have a singular point on every fiber.

As another example, varieties can have non-reduced fibers. For instance, take the k -Relative Frobenius map on $k[t]$. Viewing this as a map $\mathbb{A}^1 \rightarrow \mathbb{A}^1$, consider the fiber over $a \in \mathbb{A}^1$ that is a p th root. $a - t \mapsto a - t^p$, so the preimage of $a = b^p$ is defined by $b^p - t^p = (b - t)^p$.

But how do we fix this. Fortunately the cause of many of these problems is also (very often) the solution. On a given variety of positive characteristic X/k , we have associated to it an absolute Frobenius $F : X \rightarrow X$ sending a section $t \mapsto t^p$. To make this a morphism of \mathcal{O}_X -Modules, that is defined over the Frobenius on k . We integrate $X^{(p)} := X \times_k F_*k$ as follows:



Frobenius also has a natural action on line bundles $F^* \mathcal{L} = \mathcal{L}^p$, so if D is effective Cartier then $F^*D = pD$ is a thickening of $D \subset X$. This fact is key to fixing the earlier problem where Kodaira Vanishing fails. Recall; Kodaira Vanishing says that $H^i(X, \omega_X \otimes \mathcal{L}) = 0$ for all \mathcal{L} ample and $i > 0$. This fails in characteristic p , but if we replace \mathcal{L} with an arbitrarily large power \mathcal{L}^{p^e} for $e \gg 0$, then \mathcal{L}^{p^e} is very ample and the cohomology vanishes due to Serre Vanishing. But why is this useful? Well,

$$0 = H^i(X, \omega_X \otimes \mathcal{L}^{p^e}) = H^i(X, \omega_X \otimes F^{e*} \mathcal{L})$$

If X is F -split, Then $F^e : \mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X$ splits for some e , tensoring with \mathcal{L}^{-1} gives us an injection $\mathcal{O}_X \otimes \mathcal{L}^{-1} \hookrightarrow F_*^e \mathcal{O}_X \otimes \mathcal{L}^{-1} \cong F_*^e(\mathcal{O}_X \otimes F^{e*} \mathcal{L})$. This induces an injective map on local cohomology

$$H^{n-i}(X, \omega_X \otimes \mathcal{L}) \cong H^i(X, \mathcal{L}^{-1}) \hookrightarrow H^i(X, \mathcal{L}^{-p^e}) \cong H^{n-i}(X, \mathcal{L}^{p^e} \otimes \omega_X) = 0$$

Where the isomorphisms follows from X being smooth, and hence, CM. we have an injection $H^i(X, \omega_X \otimes \mathcal{L}) \hookrightarrow H^i(X, \omega_X \otimes F^{e*} \mathcal{L})$, so it follows that the source must vanish. Thus for smooth F -split varieties, Kodaira Vanishing holds! Thus, F -splitting is a very useful invariant. Toric varieties are F split, but varieties of general type are never F -split. As a generalization of the above reasoning, Frobenius split smooth varieties always lift to $W_2(k)$. Deleigne and Illusie, in 1987, proved that KV holds on X if it admits such a lift and $\dim(X) < p$. Further, if it can lift a SNC divisor, then Kawamata-Viehweg vanishing

holds as well.

What varieties admit such a lifting? Abelian varieties lift and many Fano varieties lift. As an example of the latter, all KLT log del Pezzo surfaces admit a log resolution that lift (when $p > 5$), along with a SNC divisor on it. Very recently (this year!), Kawakami and Tanaka proved that all smooth Fano 3-folds lift.

0.4 Jakub Witaszek - Singularities in positive and mixed characteristic

Let R be a Noetherian domain. We say that R is a splinter if every finite inclusion of domains $R \hookrightarrow S$ splits as an R -module map. The direct summand conjecture (now theorem) states that every regular ring is a splinter. In characteristic 0 the proof that every normal ring is simple.

Proof. Let $R \rightarrow S$ be a finite extension. We can grow S enough to reduce to the case where the extension on fraction fields is Galois of degree d . Let G denote the Galois Group. $\psi : S \rightarrow R$ can be defined by averaging over the Galois group: $\frac{1}{d} \sum_{\delta \in G} \delta(s) \in R$

□

In positive characteristic, the statement is a bit harder to show, but was proved by Hochster in the 70s. Recall that R is strongly F -regular (SFR) if $\forall d \neq 0, \exists e \gg 0$ such that $R \xrightarrow{F^e} F_*^e R \xrightarrow{\cdot F_*^e d} F_*^e R$ splits. For instance, the polynomial ring $R = k[x_1, \dots, x_n]$ is SFR.

Proof. Fix $d \neq 0$. We want to find $\psi : F_*^e R \rightarrow R$ that assigns $F_*^e d \mapsto 1$. Pick $e \gg 0$ such that all monomials $x_1^{b_1} \dots x_n^{b_n}$ of d satisfy $0 \leq b_1, \dots, b_n < p^e$. Since $F_*^e R$ is a free R -Module, we only need to assign where each monomial is sent to define a map ψ . suppose $d = \sum c \cdot x_1 x_2 \dots x_n$, We can define ψ such that $\psi(x_1^{b_1} \dots x_n^{b_n})$ is c^{-1} if $b_i = a_i$ for all i , and 0 otherwise. It follows from an easy check that $\psi(F_*^e d) = 1$.

□

Indeed, similar arguments can show that all regular rings are SFR. This motivates the following theorem of Hochster, proving the Direct Summand Conjecture in characteristic $p > 0$:

Theorem 0.4.1 (Hochster). *If R is SFR, then it is a splinter.*

Proof. Let $R \rightarrow S$ be a finite extension. As $K(R) \subset K(S)$ splits, $\exists \psi : S \rightarrow R$ such that $\psi(1) = f \in R$. Consider the composition

$$R \xrightarrow{F^e} F_*^e R \rightarrow F_*^e S \xrightarrow{F_*^e \psi} F_*^e R$$

Assigning $1 \mapsto F_*^e f$. For e sufficiently large, this splits, so the map $R \rightarrow S$ must split. □

Being SFR is easy to check.

Theorem 0.4.2 (Glassbrenner’s Criterion). $\mathbb{F}_p[x_1, \dots, x_n]/(f)$ is SFR $\iff \exists e > 0$ such that $f^{p^e-1} \cdot \text{Jac}(f) \notin \mathfrak{m}^{[p^e]}$.

Theorem 0.4.3 (Hochster-Huneke). If R is Gorenstein, then R is SFR $\iff R$ is a splinter.

But what about mixed characteristic? This proof required perfectoid techniques, and was only proven by André in 2018.

0.5 Alexander Petrov - Arithmetic structures on cohomology of algebraic varieties

Let X be an algebraic variety, say over \mathbb{C} . Associated to this is a useful invariant $H^*(X(\mathbb{C}), \mathbb{Z})$: the singular cohomology. We can consider the relative perspective; for a map $f : X \rightarrow Y$, what can the map $f^* : H^n(Y(\mathbb{C}), \mathbb{Z}) \rightarrow H^n(X(\mathbb{C}), \mathbb{Z})$ be? For example, consider Y/\mathbb{C} a smooth proper variety and $X = \text{Spec} \left(\frac{\mathbb{C}[x,y]}{y^2-x^3-x} \right)$ a nodal curve. For $n = 1$ we have a map $H^1(Y(\mathbb{C}), \mathbb{Z}) \rightarrow H^1(X(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}$, and we can study maps of abelian groups onto \mathbb{Z} .

Now fix a prime ℓ . Let $F \subset \mathbb{C}$ be a field and X a variety over F . We can also consider the étale cohomology groups $H_{\text{ét}}^n(X_{\overline{F}}, \mathbb{Z}_\ell) \cong H^n(X(\mathbb{C}), \mathbb{Z}_\ell)$. The étale cohomology group admits a natural action via $\text{Gal}(\overline{F}/F) = G_F$, which we can use to ascertain some structural properties. For instance, consider the case of the smooth curve over F . The interesting cohomology exists in H^1 , so we analyze $H^1(X(\mathbb{C}), \mathbb{Z}_\ell)$. $\text{Aut}(\mathbb{C}/F)$ acts on $X(\mathbb{C})$ in a natural way, but it doesn’t seem to be useful for studying topology; the action is quite discontinuous. However, this action can be made quite explicit, even outside of the curve case.

Let’s consider another example, this time a variety X/\mathbb{Q} . $G_{\mathbb{Q}}$ acts on $H_{\text{ét}}^n(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_\ell)$. Now pick p a prime unrelated to ℓ , and localize $G_{\mathbb{Q}}$ (viewed as a \mathbb{Z} -module) at this p . $G_{\mathbb{Q}_p} \subset G_{\mathbb{Q}}$ up to conjugacy, and we can understand this subgroup action a bit better. For instance, $G_{\mathbb{Q}_p} \twoheadrightarrow G_{\mathbb{F}_p} = \widehat{\mathbb{Z}} \cdot \text{Fr}_p$, where $\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}/n$ and Fr_p denotes the Frobenius with respect to p . It behooves us to understand the eigenvalues of the action of Fr_p , which was framed 70 years ago as the Weil conjectures:

Theorem 0.5.1 (Weil, Deligne). *Eigenvalues of Fr_p on $H_{\text{ét}}^n(X_{\overline{F}}, \mathbb{Q}_\ell)$ are in $\overline{\mathbb{Q}}$ and are independent of ℓ .*

If X/\mathbb{Q} is smooth and proper, you can say more. Indeed, any eigenvalue α of the Fr_p action on étale cohomology is a Weil number of weight n , meaning that $\forall \overline{\mathbb{Q}} \xrightarrow{\tau} \mathbb{C}$, $|\tau(\alpha)| = p^{n/2}$. This fact seems somewhat miraculous, so how do you prove it. If we take a model of X , denoted $\mathcal{X}/\mathbb{Z}[1/N]$, which is smooth and proper such that $\mathcal{X}_{\mathbb{Q}} = X$, Then one can show that $\forall p$ not dividing N , $\#\mathcal{X}(\mathbb{F}_{p^n}) = \sum (-1)^i \text{Tr} \text{Fr}_p^n |_{H_{\text{ét}}^i(X_{\overline{F}}, \mathbb{Q}_\ell)}$, giving us control over the \mathbb{F}_{p^n} points.

Let X/\mathbf{C} be a variety with descent X_0/F , with a G_F action on $H^n(X(\mathbf{C}), \mathbf{Z}_\ell)$. For any other choice of X_0 , the action would be the same after passing to a finite index subgroup of G_F .

Theorem 0.5.2 (Tate, Serre, Faltings). *Let A, B be Abelian Varieties over F , a finitely generated field of characteristic 0. Then*

$$H_{et}^1(A_{\bar{F}}, \mathbf{Q}_\ell) \cong H_{et}^1(B_{\bar{F}}, \mathbf{Q}_\ell) \iff \exists \text{ an isogeny } f : A \rightarrow B$$

But what about other varieties? This is the Tate Conjecture: For F a finitely generated field of characteristic 0, and X, Y smooth and proper over F such that the corresponding top etale cohomology groups are isomorphic. Then, the conjecture asserts that there must exist $c \in Z^{\dim X}(X \times Y)$ such that the map of multiplying by c , sending the n th cohomology group of $X_{\bar{F}}$ to the corresponding one over $Y_{\bar{F}}$, is an isomorphism. Related to the Tate conjecture are the following questions:

- Which \mathbf{Q}_ℓ vector spaces V which are acted on by G_F arise as $H_{et}^n(X_{\bar{F}}, \mathbf{Q}_\ell)$ for some X ?
- How do we recover the Hodge structure on the singular cohomology of X from G_F , \mathcal{X} and H_{et}^n ?

We can frame ideas that conjecturally solve these questions via p -adic Hodge Theory. Suppose that X/\mathbf{Q} is a variety and $G_{\mathbf{Q}}$ acts on $H_{et}^n(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_p)$.

Theorem 0.5.3 (Fontaine, Faltings). *There exists a functor*

$$D_{dR} : \{ \mathbf{Q}_p \text{ vector spaces with continuous } G_{\mathbf{Q}_p} \text{ action} \} \rightarrow \{ \text{filtered vector spaces} / \mathbf{Q}_p \}$$

Given a smooth proper variety X/\mathbf{Q}_p , $D_{dR} \left(H_{et}^n(X_{\bar{\mathbf{Q}}_p}, \mathbf{Q}_p) \right) = H_{dR}^n(X/\mathbf{Q}_p)$ where the filtration is the Hodge Filtration. In general we say that a vector space V with $G_{\mathbf{Q}_p}$ action is de Rham if $\dim_{\mathbf{Q}_p} D_{dR}(V) = \dim_{\mathbf{Q}_p}(V)$. Fontaine and Mazur conjecture that the $G_{\mathbf{Q}_p}$ action is irreducible and V is a subquotient of $H_{et}^n(X/\mathbf{Q}_p, \mathbf{Q}_p)(i)$ if and only if V is almost everywhere unramified and $V|_{G_{\mathbf{Q}_p}}$ is de Rham.

0.6 Hannah Larson - Intersection theory on M_g

Let X be a smooth variety. Let the i th Chow Group of X , denoted $A^i(X)$, be the free group generated by irreducible codimension i subvarieties $Y \subset X$, modulo rational equivalence. From this we define the Chow Ring $A^*(X) := \bigoplus_{i \geq 0} A^i(X)$. "+" is defined by taking the union of subvarieties and "." corresponds to taking the intersection of subvarieties. $A^*(X)$ is actually graded by codimension, but the fact that this multiplication operator is compatible with the grading is non-trivial at first glance. Indeed, for $Y, Z \in A^*(X)$ of codimension i, j , the fact that $[Y] \cdot [Z] = [Y \cap Z]$ represents a class of codimension $i + j$ subvarieties is nontrivial; $Y \cap Z$ is only of the correct codimension when $Y \pitchfork Z$ (i.e. when

they intersect transversely).

Further, this construction is functorial. If $f : X \rightarrow Y$ is proper, then there is a pushforward map $f_* A^*(X) \rightarrow A^*(Y)[\dim(Y) - \dim(X)]$ assigns

$$[Z] \mapsto \begin{cases} [f(Z)] \cdot \deg(f|_Z) & f|_Z \text{ is generically finite} \\ 0 & \text{else} \end{cases}$$

And if $f : X \rightarrow Y$ is flat, there is a pullback map $f^* : A^*(Y) \rightarrow A^*(X)$ assigning

$$[Z] \mapsto [f^{-1}(Z)]$$

If X is the total space of a vector bundle, for instance, this pullback map is an isomorphism on the bundle. In particular, $A^*(\mathbb{A}^n) = A^*(\{\text{a point}\}) = \mathbb{Z}$. We also have a notion of excision for Chow rings, i.e. If $Z \hookrightarrow X$ is closed of codimension c ,

$$A^{i-c}(Z) \rightarrow A^i(X) \rightarrow A^i(X \setminus Z) \rightarrow 0$$

Let's consider an explicit example of Chow rings. Let $G(k, n)$ be the Grassmannian, i.e. the moduli space of rank k quotients of an n -dimensional vector space. This is a variety, and we have a tautological sequence:

$$0 \rightarrow S \rightarrow \mathcal{O}^{\oplus n} \rightarrow Q \rightarrow 0$$

Where Q is rank k and S is rank $n - k$. We can use this to make statements about $A^*(G(k, n))$. Let $c_i := c_i(Q)$ be the locus where $k + 1 - i$ sections become dependent. This shift is there to make sure that these things occur in codimension i . This is the " i th Chern class", and can be thought of as a "tautological class" which is of the form

$$c_i = [\{[S] : \langle e_1, \dots, e_{k+1-i} \rangle \cap [S] \neq \emptyset\}] \in A^i(G(k, n))$$

In fact, $1 + c_1(S) + \dots + c_{n-k}(S) = \frac{1}{1+c_1+\dots+c_k}$, and in fact these tautological classes generate the Chow ring:

$$A^*(G(k, n)) = \frac{\mathbb{Z}[c_1, \dots, c_k]}{\left\langle \left\{ \frac{1}{1+c_1+\dots+c_k} \right\}^j : j > n - k \right\rangle}$$

We can do another (more complicated) example. \mathcal{M}_g the moduli of curves of genus g . As a matter of bookkeeping, the Chow ring will be viewed in rational coefficients, i.e. $A^* = A^* \otimes \mathbb{Q}$. The universal family of curves \mathcal{C} lies over \mathcal{M}_g , but is highly nonlinear. Let the map defining this laying over be $f : \mathcal{C} \rightarrow \mathcal{M}_g$. For a given curve $C \in \mathcal{M}_g$, the $(T_p C)^\vee$ for a point $p \in C$ fit together into a line bundle ω_f , and $c_1(\omega_f) \in A^1(\mathcal{C})$. Well, this map f is proper, so there exists a morphism $f_* : A^1(\mathcal{C}) \rightarrow A^0(\mathcal{M}_g)$. Under this assignment, $c_1(\omega_f) \mapsto (2g - 2) \cdot [1]$. Even better $c_1(\omega_f)^{i+1} \in A^{i+1}(\mathcal{C})$ is mapped to $\kappa_i \in A^i(\mathcal{M}_g)$ in higher degrees, where κ_i corresponds to the tautological class.

We may wonder: Like the Grassmannian, is $A^*(\mathcal{M}_g)$ generated by these tautological classes κ_i ? Also, what are the relations between the κ_i ? We would like something like the following to hold:

$$A^*(\mathcal{M}_g) = \frac{\mathbb{Q}[\kappa_1, \kappa_2, \dots]}{\langle \text{relations} \rangle}$$

We define the tautological ring $R^*(\mathcal{M}_g) \subset A^*(\mathcal{M}_g)$ to be the subring generated by the tautological classes κ_i . If $g \leq 9$, due to Mumford, Faber, Izadi, Penev-Vakil, and Canning-Larson for various specific g , it holds that $A^*(\mathcal{M}_g) = R^*(\mathcal{M}_g)$. Conversely, for $g = 12$ or $g \geq 16$, $A^*(\mathcal{M}_g) \neq R^*(\mathcal{M}_g)$ via work of Van Zelm and Arena et al. These were proved by constructing an explicit example that is not tautological: for $g = 12$ for instance we can consider the class $[\{[C] \in \mathcal{M}_{12} : C \xrightarrow{2:1} E\}] \notin R^*(\mathcal{M}_g)$.

But what about the relations? For starters, we know that $\kappa_1, \kappa_2, \dots, \kappa_{\lfloor g/3 \rfloor}$ are enough to generate $R^*(\mathcal{M}_g)$ and they satisfy no relations in degrees $\leq \lfloor g/3 \rfloor$ (due to Ionel, Boldsen). We also know that $R^i(\mathcal{M}_g) = 0$ for $i > g - 2$ due to work of Looijenga. Further, $R^{g-2}(\mathcal{M}_g)$ is one-dimensional/ \mathbb{Q} , and further, spanned by the fundamental class of curves which are hyperelliptic. As a more general case of this phenomenon, $R^i(\mathcal{M}_g) \times R^{g-2-i}(\mathcal{M}_g) \rightarrow R^{g-2}(\mathcal{M}_g) = \mathbb{Q}$ is a perfect pairing for $g \leq 23$ due to Faber. It is open whether or not this is true in higher genus, but there has been doubt that such a statement holds for all g . Perfect pairings do not work for arbitrary products: indeed, a theorem of Canning shows that if $g \geq 12$, the pairing $R^i(\overline{\mathcal{M}}_g) \times R^{3g-3-i}(\overline{\mathcal{M}}_g) \rightarrow \mathbb{Q}$ is NOT a perfect pairing.

Rather than using Tautological classes for the basis of our generation, we can also consider λ classes. In the same set up $f : \mathcal{C} \rightarrow \mathcal{M}_g$ as before, we can take $\mathbb{E} := f_*\omega_f$, which is a rank g vector bundle called the Hodge bundle. We define $\lambda_i := c_i(\mathbb{E})$. By Grothendieck-Riemann-Roch, we can see that $\lambda_i \in R^*(\mathcal{M}_g)$.

0.7 Phil Engel - Compactifying moduli spaces of K3 surfaces

Consider \mathcal{M}_g the moduli of smooth projective curves of genus g . This space is not compact; given a genus g curve one can contract a cycle to obtain a "pinched" Riemann surface, which is no longer smooth. Thus, it's missing limit points of this form. This can be rectified by considering $\overline{\mathcal{M}}_g$, the moduli of stable genus curves, constructed by Deligne-Mumford, Bers, and Knudsen. There are some advantages to this perspective over just considering \mathcal{M}_g . For starters, $\overline{\mathcal{M}}_g$ is compact, but it is also smooth! This nice structure allows to ask interesting questions about the collections of points in $\overline{\mathcal{M}}_g$ (e.g. classes of stable curves).

We can also construct moduli of higher dimensional varieties, such as Abelian varieties for instance. We define (X, L) to be a principally polarized Abelian Variety if $X = \mathbb{C}^g/\Lambda$ is a quotient of a vector space by a lattice and $L \in H^2(X, \mathbb{Z})$ is a class of an

ample line bundle. We note that X is diffeomorphic to $(S^1)^{2g}$, so L defines a symplectic form on $H_1(X, \mathbb{Z})$. We require that this form is 'standard', in the sense that L corresponds to a symplectic form defined by basis e_i, f_i such that $e_i f_j = \delta_{ij}$.

Given that we are working over Abelian varieties, we can consider the Hodge decomposition $H^1(X, \mathbb{Z}) \otimes \mathbb{C} \cong H^{1,0} \oplus H^{0,1}$. $H^{1,0}$ are the classes of 1 forms that are holomorphic; this is a key fact about an Abelian variety. As it turns out, (X, L) is (kind of) equivalent to the data of $[H^{1,0}] \in \text{LGr}(\mathbb{Z}^{2g} \otimes \mathbb{C}, L)$, where LGr denotes the Lagrangian. But why is this only sort of an equivalence? Well, we first note that $[H^{1,0}]$ sits inside the period domain \mathbb{D} , which is an open subset of $\text{LGr}(\mathbb{Z}^{2g}, \mathbb{C})$. To construct the equivalence on the lose, we need to forget our choice of basis. Thus, this data equivalence can be precisely formed via the following equality:

$$\mathcal{A}_g = \text{Sp}(2g, \mathbb{R}) \backslash \mathbb{D}$$

Where the left side is the moduli of principally polarized Abelian varieties (PPAVs). The right hand side is (loosely) a moduli of linear algebraic objects, an a priori very different collection of things. This equality is an example of a "Torelli Theorem", and there are many more analogous statements in Hodge theory and Moduli theory. Such relations can be generalized in the form of the Hodge Conjecture, which states that

$$\{\text{Varieties over } \mathbb{C}\} / \sim \xrightarrow{H^*} \{\text{Hodge Structures}\}$$

is a fully faithful functor that relates non-linear objects with linear objects, respectively. The equivalence \sim is a mild condition relating numerical Chow motives, which we won't go into now.

We now move on to the key example of our talk; K3 surfaces. A pair (X, L) is a polarized K3 surface provided that X is a smooth projective surface with $K_X \sim 0$ and $H^1(X, \mathcal{O}_X) = 0$, and that L is an ample line bundle on X . To an ample line bundle we can associate an intersection number $L \cdot L$, which for K3 surfaces is always $2d > 0$. Here are some examples of polarized K3 surfaces:

- $X \xrightarrow{2:1} \mathbb{P}^2$ a cover
- $X = V(f_4) \subset \mathbb{C}\mathbb{P}^3$, and $L \cdot L = 4$.
- $X = V(f_2, f_3) \subset \mathbb{C}\mathbb{P}^4$, and $L \cdot L = 6$.

Where in all cases we are just taking $L = \mathcal{O}_X(1)$.

Theorem 0.7.1 (Enriques, Sevri 1909). $\forall d > 0, \exists X \subset \mathbb{P}^{d+1}$ (known as an Enriques Surface) which is a K3 surface of degree $2d$ such that X depends only on exactly 19 moduli.

The "19" seems random, but it comes from the fact that the rank of H^2 is 22, so $\mathbb{P}(V^\perp \otimes \mathbb{C})$ is a vector space of dimension 21, i.e. \mathbb{P}^{20} . However, there is a nontrivial condition imposed on these, namely that $x \cdot x = 0$, defining an open subset of the vanishing locus of a quadric in \mathbb{P}^{20} ; hence there are 19 conditions that are imposed. Just like for Abelian varieties, we have a Torelli Theorem (due to Pratski-Shapiro, Shafarevich, 1973):

Theorem 0.7.2.

$$F_{2d} := \{(X, L) \text{ polarized K3 surfaces of degree } 2d\} \xrightarrow{\sim} \Gamma \backslash \mathbb{D}$$

Where $\Gamma := \{\gamma \in O(H^2(X, \mathbb{Z})) \mid \gamma(c_1(L)) = c_1(L)\}$ is the group of orthogonal actions that preserve cup products and $c_1(L)$, and $\mathbb{D} := \mathbb{P}\{x \in V^\perp \otimes \mathbb{C} \mid x \cdot x = 0, x \cdot \bar{x} > 0\}$.

This brings us to our central question: Can we construct an analogue of \overline{cM}_g for F_{2d} ? Namely, can we construct a compact variant of a moduli of K3 Surfaces? First we should mention the following theorem:

Theorem 0.7.3 (Kollár-Shepherd-Barron-Alexaev, (1988, 1996)). *The set of mildly singular (slc) pairs (X, R) (i.e. X is a projective variety and R a \mathbb{Q} -divisor with $K_X + R$ ample) form a compact moduli.*

For $R = 0$ and X a genus $g \geq 2$ curves, K_X is ample which gives us the correct stability condition; thus this construction can be thought of as a higher dimensional generalization of the construction of \overline{M}_g . For K3 surfaces X , this does not work if we take $R = 0$, by definition K_X is trivial. Thus to have any hope of forming a compact moduli, we need a choice of divisor $R \in |nL|$ for some n , for all $(X, L) \in F_{2d}$. We can take pairs of the form $(X, \varepsilon \cdot R)$, where $\varepsilon > 0$ is small. For ε small we have that $(X, \varepsilon \cdot R)$ is an slc pair, and further, we can take ε to be arbitrarily small and still ensure that $K_X + \varepsilon \cdot R \sim \varepsilon \cdot R$ is ample. This then determines the compactification

$$F_{2d} \rightarrow \overline{F}_{2d}^R = \overline{\{(X, \varepsilon \cdot R)\}}$$

where $\overline{\{\cdot\}}$ denotes the KSBA compactification. For instance, for our $2 : 1$ cover example $X \rightarrow \mathbb{P}^2$ we can take R to be the ramification divisor; (X, R) lies inside this compact moduli. For our other example $X = V(f_4) \subset \mathbb{P}^3$, the Hessian is defined to be $\text{Hess}(f_4) := \det\left(\frac{\partial^2 f_4}{\partial x_i \partial x_j}\right)$. We use this to define the divisor $V_X(\text{Hess}(f_4)) \subset X$ which is a divisor in $|8L|$.

There is a Hodge Theory side to this story. For any object of the form $\Gamma \backslash \mathbb{D}$, Baily and Borel showed that such an object admits a "natural" compactification:

Theorem 0.7.4 (Baily-Borel 1966).

$$\exists \Gamma \backslash \mathbb{D} \hookrightarrow \overline{\Gamma \backslash \mathbb{D}}^{BB}$$

For the K3 example, the boundary generated by this process yields a stratified boundary; the 1 dimensional component being the moduli of elliptic curves, and the points being moduli of K -trivial 0-dimensional varieties. We note that this boundary is small; the moduli space is of dimension 19 and the boundary is dimension 1. On a similar note, Ash-Mumford-Rapaport-Tai developed a notion of a toroidal compactification in 1975. This compactification is not completely canonical; it depends on a choice of fan. It turns out, there are choices of fan for which the toroidal compactification agrees with other notions of compactifying the moduli of K3s.

Theorem 0.7.5 (Alexaev-Engel). *For an appropriate choice of $R \in |nL|$, \exists a unique choice of fan such that the toroidal compactification defined with respect to it agrees with $(\overline{F}_{2d}^R)^v$.*

This is quite interesting, since it relates a geometric/modular approach to compactification with a combinatorial/hodge theoretic one.

0.8 Lucy Yang - Derived methods and trace theories via the infinitesimal Hodge conjecture.

Let X be a quasi-projective scheme. Recall that

$$K_0(X) := \mathbb{Z}\{\text{vector bundles on } X \text{ of finite rank}\} / \{[V] = [U] + [W] \mid 0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0\}$$

This is the Grothendieck group, but there are other ways to define this that may be useful to us.

$$K_0(X) := \{E \in \text{Perf}_X\} / \{[F] = [E] + [G] \mid E \rightarrow F \rightarrow G \xrightarrow{+1}\}$$

Related to the construction of K -groups is the Grothendieck-Riemann-Roch Theorem:

Theorem 0.8.1 (GRR). *Suppose X is smooth. Then*

$$\bigoplus_{i=0}^{\dim X} \text{CH}^i(X) \rightarrow K_0(X)$$

defined by sending $[Z] \mapsto [i_{Z} \mathcal{O}_Z]$ is an isomorphism after applying $-\otimes_{\mathbb{Z}} \mathbb{Q}$.*

Let $F \supset \mathbb{Q}$ be an algebraic extension and $f : Y \rightarrow S := \text{Spec}(F[[t]])$ is smooth and proper. Define $X := Y \times_S \mathbb{V}(t=0)$. Let $Y_n := Y \times_S \text{Spec}(F[[t]]/t^n)$ correspond to the infinitesimal thickenings of Y . Given $\eta \in K_0(X)$, if \exists a compatible family $(\eta_n) \in (K_0(Y_n))$ extending the class η , then $\text{ch}(\eta) \in \bigoplus_{p \geq 0} F_{\text{Hdg}}^p H_{dR}^{2p}(X/F)$ belongs to some subgroup $\bigoplus_{p \geq 0} F_{\text{flat}}^p H_{dR}^{2p}(X/F)$. Each $F_{\text{flat}}^p H_{dR}^{2p}(X/F)$ corresponds to the subgroup of $H_{dR}^{2p}(X/F)$ who's flat lift to $H_{dR}^{2p}(Y/F)$ lands in F_{Hdg}^p . Further a theorem of Morrow shows that these conditions are equivalent.

We now introduce some concepts from Derived Algebraic Geometry. Let R be a commutative \mathbb{Q} -algebra. Define the category of derived R -algebras (DAlg_R) to be the category of commutative DGAs over R where we invert all quasi-isomorphisms. Let DAlg_R^{cn} denote the subcategory whos underlying object in $D(R)$ has $H_{<0}(-) = 0$. DAlg_R has pushouts and has a uniquely determined version of a cotangent complex $\mathbb{L}_{(-)/R}$ determined by

$$\mathbb{L}_{\text{LSym}_R(M)/R} = \text{LSym}_R(M) \otimes_R M$$

for any $M \in D(R)$. In particular, \mathbb{L} sends resolutions in DAlg_R to colimit diagrams. When working over smooth ordinary R -algebras (viewed as a cDGA in a trivial way), $\mathbb{L}(-)$

agrees with $\Omega_{(-)/R}^1$ in the usual sense. We can also define a "derived" DeRham complex $\mathbb{L}\Omega_{(-)/R}^*$ in a similar way to the above.

We define a derived stack to be a functor $F : \mathrm{DAlg}_R^{cn} \rightarrow \mathrm{Spc}$ such that, if $A \rightarrow B$ is a morphism so that $H_0 A \rightarrow H_0 B$ is an etale cover and $H_n A \otimes_{H_0 A} H_0 B \cong H_n(B)$, then $F(A) \cong \lim^* (F(B) \rightrightarrows F(B \otimes_A B) \dots)$. Let DerStk define the category of such objects. For a given derived stack we have the diagram

$$\begin{array}{ccc}
 & & \mathrm{Set} \\
 & \nearrow & \updownarrow \\
 \mathrm{CAlg}_R & \dashrightarrow & \mathrm{Gpd} \\
 (-)[0] \downarrow & & \downarrow \uparrow \pi \\
 \mathrm{DAlg}_R & \xrightarrow{F} & \mathrm{Spc}
 \end{array}$$

In this context we define, for $A \in \mathrm{DAlg}_R$, $\mathrm{Spec}(A) := \mathrm{Hom}_{\mathrm{DAlg}_R}(A, -)$. For $X \in \mathrm{DerStk}$, we define the loop stack $\mathcal{L}X := \mathrm{Map}(S^1, X)$, then define the Hochschild homology $\mathrm{HH}(X/R) := R\Gamma(\mathcal{L}X, \mathcal{O}) := \lim_{A \rightarrow \mathcal{L}X} A \in \mathrm{DAlg}_R$. We note that maps $S^1 \rightarrow X$ can be reframed as a colimit of the collection of maps from pushouts to X ; therefore we can view this as the limit of diagonal maps $X \xrightarrow{\Delta} X \times X \xleftarrow{\Delta} X$.

We define $\mathrm{HC}^-(X/F)$, the negative cyclic homology of X/F , to essentially be the derived fixed points of Hochschild cohomology. These satisfy the following powerful decomposition theorem:

Theorem 0.8.2 (Hochschild-Kostant-Rosenberg). *Let F be a characteristic 0 field. Then,*

1. *There exists a functorial splitting $\mathrm{HH}(X/F) = \bigoplus_{n \geq 0} R\Gamma(X, \mathbb{L}\Omega_{(*)/F}^n)[*]$.*
2. *There exists a decomposition $\mathrm{H}^-(X/F) \cong \prod_{* \in \mathbb{Z}} R\Gamma(X, \mathbb{L}\Omega_{X/F}^{\geq *})[\geq *]$.*

0.9 Rachel Webb - What is a virtual class?

String theory associates spacetime to $\mathbb{A}^4 \times X$, where \mathbb{A}^4 denotes (x, y, z, t) coordinates and X is a CY-manifold that determines "correlation functions" that tracks particle interactions. At $g = n = 0$ these are functions of $\beta \in H_2(X)$. For instance, we can take $Q = V(x_0^5 + \dots + x_4^5) \subset \mathbb{P}^4$. Q and β can be plugged into a string theory black box, which outputs so called "correlators" $\in \mathbb{Q}$. How can we frame this mathematically?

Ideally, we could consider $\mathcal{M}_{0,0}(Q, d)$: the moduli stack that parameterizes maps $C \rightarrow X = Q$ of degree d . In other words, this can be viewed as a moduli of (at worst, nodal) curves of degree d that admit a mapping into my quintic. This moduli stack contains finitely many points; hence there are finitely many curves. I can use this fact to

define my correlator := # of points in $\mathcal{M}_{0,0}(Q, d)$. It would be really cool if this naive approach worked, but it doesn't.

Unfortunately, $\overline{\mathcal{M}}_{0,0}(Q, 2)$ is NOT a collection of points. For now we can just consider $\mathcal{M}_{0,0}(Q, 2)$; i.e. we can restrict our study to classes of smooth curves. Q contains 2,874 lines, and for each line w we get a component of $\mathcal{M}(Q, 2)$ that is isomorphic to $\mathcal{M}_{0,0}(\mathbb{P}^1, 2)$. However, this guy has dimension 2. To see why, to define a map $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree 2 we must pick 2 sections on $\mathcal{O}_{\mathbb{P}^1}(2)$ (so, 6 conditions), then lose one condition from scaling and another 3 conditions by considering morphisms up to automorphism invariance (and $\text{Aut}(\mathbb{P}^1)$ is dimension 3). This leaves 2 dimensions, which means that there are 2,875 components of nonzero dimension and thus, there isn't a finite set of points to count up.

Our goal is to define a "natural" (and correct, in the sense that it matches the outcomes of physics theorems) class $[\overline{\mathcal{M}}_{0,0}(Q, d)]^{\text{vir}} \in A_0(\overline{\mathcal{M}}_{0,0}(Q, d))$. Where $A_0(-)$ denotes the Chow group. This virtual class computes the number of points that are SUPPOSED to comprise $\overline{\mathcal{M}}_{0,0}(Q, d)$. In this talk, we will often take the Chow ring $A_*(X)$ where X is a DM-stack to be $A_*(X)_{\mathbb{Q}}$ with a natural functoriality for pushforwards $f_* : A_*(X) \rightarrow A_*(Y)$ where $f : X \rightarrow Y$ is a proper morphism of DM stacks. Further, given a regular embedding $i : X \hookrightarrow Y$ of codimension d ; we have a map $i^! : A_*(Y) \rightarrow A_*(X)$ assigning $[V] \mapsto [V] \cdot [X]$, where the product corresponds to the transverse intersection of V with X .

If E is a vector bundle on Y , its top chern class is the self intersection of the 0 section $0 : Y \rightarrow E$, i.e.

$$0^! 0_* [Y] = c_{\text{top}}(E) \cap [Y]$$

If Y is of pure dimension, $i : V(s) = X \hookrightarrow Y$ with $s : Y \rightarrow E$ is a section, we have the following:

Theorem 0.9.1 (Fulton). $\exists [X]_E^{\text{vir}} \in A_{\dim Y - \text{rk} E}(X)$ such that:

- $i_* [X]_E^{\text{vir}} = c_{\text{top}}(E) \cap [Y]$.
- If s is regular, then $[X]_E^{\text{vir}} = [X]$.

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Y & \xrightarrow{0} & E \end{array}$$

One can generally define $[X]_E^{\text{vir}} := 0^! [Y]$.

If you assemble these two bullet points with the definition we gave of the top Chern class, you can see that $0^! 0_* [Y] = [X]$ when $X = V(\text{regular section})$. For example, let $X = \mathbb{P}^1$, $Y = X$, $E = 0$, and $s = 0$. In this case, $i_* [X]_E^{\text{vir}} = 1 \cap [X] = [X]$. For a more difficult example, take $Y = \mathbb{P}^3$, $E = \mathcal{O}(2)^{\oplus 3}$, and choose s such that $X = \mathbb{P}_{[s:t]}^1 = V(s)$ and $X \hookrightarrow Y$ is the veronese embedding $[s^3 : s^2 t : s t^2 : t^3]$. Then

$$i_* [X]_E^{\text{vir}} = c_{\text{top}}(\mathcal{O}_{\mathbb{P}^3}(2)^{\oplus 3}) \cap [\mathbb{P}^3] = 8[\text{pt}] \in A_0(\mathbb{P}^1)$$

Now let's do a more relevant (stacky) example. Take $X = \overline{\mathcal{M}}_{0,0}(Q, d)$ and $Y = \overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, d)$. Then $X \rightarrow Y$ by $(f : C \rightarrow Q) \mapsto (i \circ f : C \rightarrow \mathbb{P}^4)$ is a closed embedding. In fact, $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, d)$ is smooth of dimension $5d + 1$, the pushforward of the bundle cutting out the quintic $\pi_* f^* \mathcal{O}_{\mathbb{P}^4}(5)$ is a vector bundle of rank $5d + 1$ with section with zeroes $= X$. As a consequence, we get $[\overline{\mathcal{M}}_{0,0}(Q, d)]^{\text{vir}} \in A_0(\mathcal{M}_{0,0}(Q, d))$, which is exactly what we wanted!

But not everything is perfect; indeed this embedding need not arise so easily in more general cases. Let's now consider the case where X is a smooth projective variety, $g, n \in \mathbb{Z}_{\geq 0}$, and $\beta \in H_2(X)$. Via Gromov-Witten theory, a formal mathematical package, we can put this information into this package like a black box to yield $\langle \quad \rangle_{g,n}^{X,\beta} : H^*(X)^{\otimes n} \rightarrow \mathbb{Q}$. In general, we don't know how to find a global embedding of $\mathcal{M} := \overline{\mathcal{M}}_{g,n}(X, B) \subset Y = Z(s)$ as the zero section of vector bundle. We need a more robust definition of an obstruction theory and of virtual classes. Fortunately, you can always do this embedding locally on \mathcal{M} . But how can we glue those embeddings together?

This motivates the Behrend-Fantechi Perfect Obstruction Theory. The Perfect Obstruction Theory (POT) on X/B to be a morphism $\varphi : \mathbb{E}_{X/B} \rightarrow \mathbb{L}_{X/B}$ in the derived category of quasi-coherent sheaves such that

- $\mathbb{E}_{X/B}$ is quasi-isomorphic to a complex of locally free sheaves in degree $[-1, 0]$.
- φ induces an isomorphism of cohomology at degree 0 and a surjection of cohomology in degree -1 .

The key observation is that this φ is encoding equations that locally cut out X from something smooth. But how does that work? For a smooth Y that X embeds, we always have an etale-local embedding:

$$\begin{array}{ccc} U & \hookrightarrow & Y \\ \downarrow \text{etale} & & \downarrow \text{smooth} \\ X & \longrightarrow & B \end{array}$$

For example, if X is locally finite type over $B = \text{Spec}(k)$, $Y = \mathbb{A}_k^n$, and U is an affine open subset of X , we can localize completely until the top two objects are free \mathcal{O}_U modules and the bottom two objects reduce to the map $I/I^2 \rightarrow \Omega_{Y/B}$. Plugging these into the diagram, we see through Nakayama's lemma that we can construct the desired morphism $\mathcal{O}^{\oplus r} \rightarrow I$; this determines a surjection (not just on cohomology!) in degree -1 and an isomorphism (not just on cohomology) in degree 0:

$$\begin{array}{ccccc} & & \mathcal{O}_U^{\oplus r_1} & \longrightarrow & \mathcal{O}_U^{\oplus r_2} \\ & \swarrow \text{Nak.} & \downarrow & & \downarrow \cong \\ I & \xrightarrow{\quad} & I/I^2 & \longrightarrow & \Omega_{Y/B} \end{array}$$

So φ is just encoding these local equations for X into Y . Now we want to turn $\varphi : \mathbb{E} \rightarrow \mathbb{L}$ into a way to construct/define a suitable $[X]^{\text{vir}}$. Recall: $[X]_E^{\text{vir}} = 0^! [Y] = 0_E^! [C_{X/Y}]$. Behrend and Fantechi construct an "intrinsic" normal cone $\mathcal{C}_{X/B}$ which embeds via $\varphi : \mathcal{C}_{X/B} \hookrightarrow \mathbb{E}_{X/B}$, with $[X]_E^{\text{vir}} = 0_{\mathbb{E}_{X/B}}^! [\mathcal{C}_{X/Y}]$.