Chapter 1

Day 1

1.1 Kevin Tucker - An Introduction to the work of Karen E. Smith

Handwritten.

1.2 Mel Hochster - Research Inspired by Karen Smith

Handwritten.

1.3 Eamon Quinlan-Gallego - F-jumping numbers and monodromy eigenvalues for homogeneous nondegenerate polynomials in positive characteristic

Handwritten.

1.4 Sara Faridi - The Algebra and Combinatorics of Extremal Ideals

This is work joint with T.Chau, T.Cooper, A.Duval, S.Elkhoury, T. Holleben, H. Mahmood, S. Mayestang, S. Morey, L. Sega, and S. Spiroff.

Moral idea of the talk: If I is an ideal in $R = k[x_1, \ldots, x_n]$ generated by q square-free monomials, and $\mathcal{P}$ is a property (algebraic in nature), and $r > 0$, to check $\mathcal{P}$ for I^r, it is enough to check it for $\mathcal{E}_q^r$, where $\mathcal{E}_q = q-$extremal ideal.

This was motivated by work at WICA 2019, where the coauthors asked the following question: How large can the minimal graded free resolution (MFR) of I^r be?

$$\cdots \to \bigoplus R^{\beta_{1,j}}(-j) \to \bigoplus_{j\in\mathbb{N}} R(-j)^{\beta_{0,j}} \to 0$$

Can refine this to a multigraded resolution with terms

$$\bigoplus_{m\in \mathrm{LCM}(J)} R(m)^{\beta_{i,m}}$$

Where LCM(J) denotes the poset of lcm's of generators of J. Then $\beta_{i,j} = \sum_{m\in\mathrm{LCM}(J),\deg(m)=j} \beta_{i,m}$. This is due to Hochsters Formula and the theory of Stanley-Reisner Rings. But how can we enumerate these topologically? We use a ***Taylor Resolution***: $\beta_i \le \binom{q}{i}$ for q the number of generators. For any monomial ideal $J = (m_1, \ldots, m_q)$, where each m_i is a monomial, we get

$$R \to \cdots \to R^{\binom{q}{1}} \to R^{\binom{q}{0}} \to 0$$

Viewed as a resolution of R/J. Does there exist an ideal J with q generators such that $\beta_i(R/J) = \binom{q}{i}$. However, for monomials with squares, a Taylor resolution cannot be realized as a minimal free resolution of such a J. Let's compute a Taylor Resolution for $J = (xy, yz, zw)$:

$$0 \to R \xrightarrow{\begin{bmatrix} 1 \\ w \\ x \end{bmatrix}} R^3 \to R^3 \to R \to 0$$

However, observe that J^2 never has a Taylor minimal resolution. For instance, for $J = (m_1, \ldots, m_q)$, $J^2 = \langle m_i m_j | 1 \le i \le j \le q\rangle$. However, $m_1 m_2 | \mathrm{lcm}(m_1^2, m_2^2)$. This clashes the boundary maps, and in particular, the corresponding Taylor Resolution *cannot* be minimal. This then suggests the following question: Is there a simplicial complex Δ_q^r that supports a free resolution of I^r $\forall I$ that is generated by q-square-free monomials $\forall r$ AND that $\exists I$ for which it is minimal for I^r.

When $r = 2$, there does exist such a simplicial resolution! For instance if $q = 3$, i.e. $I = (m_1, m_2, m_3)$, then I^2 always has a free resolution supported on a "tri-force" style simplex. We refer to this as $\mathbb{L}_q^2$ (in honor of the Lyubeznik Resolution). But what about the corresponding $\mathbb{L}_q^r$ for r large? Well, these are never minimal; they are too big!

A q-Extremal ideal $\mathcal{E}_q := (e_1, \ldots, e_q) \subset S_{[q]} = K[Y_A | A \subset [q]]$ (such that $e_i = \prod_{i\in A} Y_A$) has q-square free generators.

Theorem 1.4.1 (The WICA Group). $\beta_{i,j}(I^r) \le \beta_{i,j}(\mathcal{E}_q^r)\ \forall r > 0$.

But how does $\mathcal{E}_q = (e_1, \ldots, e_q)$ relate to $I = (m_1, \ldots, m_q)$? The answer relies on a ring map $\psi_I : S_{[q]} \to R$. We first build subsets $A_1, \ldots, A_n$ of $[q]$. Each $m_i = x_{i_1} \cdot x_{i_2} \cdot \cdots \cdot x_{i_t}$, as they are all square-free monomials, so we let $A_j = \{i \in [q] | x_j | m_i\}$. Then we let

$$\psi_I(Y_A) = \begin{cases} \prod_{i\in A, A=A_u} x_i & \text{if } A = A_u, \exists u \in [q] \\ 1 & \text{otherwise} \end{cases}$$

Theorem 1.4.2. *Let I be any square free monomial ideal of $R = k[x_1, \dots, x_n]$ generated by q-monomials. Then,*

1. $\psi_I : S_{[q]} \to R$ *is a ring homomorphism.*
2. $\psi_I(e_i) = m_i \ \forall i \in [q]$. *In particular,* $\psi_I(\mathcal{E}_q) \subset I$.
3. $\psi_I(\mathcal{E}_q^r) = I^r$ *for* $r \geq 1$.
4. $\psi_I(\mathrm{LCM}(\mathcal{E}_q^r)) = \mathrm{LCM}(I^r)$.
5. $\beta_{i,\psi(m)}(I^r) \leq \beta_{i,m}(\mathcal{E}_q^r)$*, for* $m \in \mathrm{LCM}(\mathcal{E}_q^r)$*. where* $r \geq 1$.
6. $\psi_I(J \cap K) = \psi_I(J) \cap \psi_I(K)$*. for any ideal* $J, K \subset S_{[q]}$.
7. $\psi_I(\mathrm{Ass}(\mathcal{E}_q^r)) = \mathrm{Ass}(I^r)$.
8. $\psi_I(\{$*primary decomposition of* $\mathcal{E}_q^r\}) =$ *Primary decomposition of* I^r.
9. $\psi_I(\overline{\mathcal{E}_q^r}) = \overline{I^r}$*, where* $\overline{*}$ *denotes the integral closure. In particular,* $q \leq 3$ *implies that* I *is normal.*
10. $\psi_I(\mathcal{E}_q^{(r)}) = I^{(r)}$.

1.5 Jack Jeffries - Singularities in Commutative Algebra and $\mathcal{D}$-Simplicity

Let $S = \mathbb{C}[x_1, \dots, x_n]$ and $\mathcal{D}_S$ denote its ring of differential operators, which is of the form

$$\mathcal{D}_S = \mathbb{C}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle \subset \mathrm{End}_{\mathbb{C}}(S)$$

Where $x_i : S \xrightarrow{\cdot x_i} S$ and $\partial_i := \frac{\partial}{\partial x_i}$. $\mathcal{D}_S$ is a noncommutative ring, and S is naturally a left-$\mathcal{D}_S$ module. This action relates to many things, such as:

- Invariant theory (where this is referred to as the Weyl Algebra)
- Symbolic powers (via Zariski-Nagata)
- Finiteness properties of Local Cohomlogy modules (via Lyubeznik)
- Bernstein Sato polynomials (relates to log canonical thresholds, multiplier ideals, etc.)

S is also a simple $\mathcal{D}_S$ module. $\mathcal{D}_S$ also has many incredible ring/module-theoretic properties:

- It has finite global dimension:

- It satisfies Bernstein's inquality, implying that it has dimension[1] that is $\geq \dim(S)$.
- They are all classified by the Riemann-Hilbert correspondence.

Related, $\mathcal{D}_S$ is a simple ring, meaning that there are no nonzero proper 2-sided ideals.

We will now study differential operators' effect on singularities. Let $A \subset R$ be an inclusion of commutative rings.

- $\mathcal{D}^0_{R/A} = \mathrm{Hom}_R(R, R)$
- $\mathcal{D}^i_{R/A} = \{\delta \in \mathrm{End}_A(R) \mid \delta \cdot r - r \cdot \delta \in \mathcal{D}^{i-1}_{R/A} \ \forall r \in R\}$
- $\mathcal{D}_{R/A} = \bigcup_{i \geq 0}$ forms a noncommutative ring, and R is a left $\mathcal{D}_{R/A}$-Module.

Theorem 1.5.1. *Let $S = \mathbb{C}[x_1, \ldots, x_n]$ and G acts on S.Implicitly, S is viewed as a $\mathbb{C}$-algebra.*

- *If G is finite with no pseudoreflections, then $\mathcal{D}_{S^G} = (\mathcal{D}_S)^G$.*
- *For many infinite gorup actions, something similar is true: $(\mathcal{D}_S)^G \twoheadrightarrow \mathcal{D}_{S^G}$. This is true for Toric rings and various classes of determinental rings.*

Things fail spectacularly under quotients. For instance, consider $\frac{\mathbb{C}[x,y,z]}{(x^3+y^3+z^3)}$. Via a theorem of Bernstein-Gelfand-Gelfand, $\mathcal{D}_R$ is no longer a finitely generated $\mathbb{C}$-algebra, is neither left nor right Noetherian, and no operators 'lower degree' as in the above case. This motivates the following definition:

Let k be a field and $k \subset R$ be a commutative noetherian ring. We say R is ***$\mathcal{D}$-simple*** if R is a simple $\mathcal{D}_R$-module. The example above is not $\mathcal{D}$-simple.

Theorem 1.5.2 (Traves)**.** *Let $R = \mathbb{C}[x_1, \ldots, x_n]/J$ with J a square-free monomial ideal. Then the $\mathcal{D}_R$-submodules of R are the intersections of sums of minimal primes.*

In particular, this tells us that a broad class of rings are not $\mathcal{D}$-simple.

Theorem 1.5.3 (Smith)**.**

- *Let S be a polynomial ring over a field and $R \subset S$ a direct summand. Then R is $\mathcal{D}$-simple.*
- *Let R be F-finite and reduced. Then R is strongly-F-regular if and only if R is $\mathcal{D}$-simple and F-split.*

Theorem 1.5.4 (Mallory)**.** *$R = \frac{\mathbb{C}[x,y,z,w]}{(x^3+y^3+z^3+w^3)}$ is KLT, of strongly-F-regular type, but not $\mathcal{D}$-simple.*

Theorem 1.5.5 (Mukhopadhyay - Smith)**.** *There are 1-dimensional regular $\mathbb{C}$-algebras that are not $\mathcal{D}$-simple.*

[1]Not krull dimension, a different thing

This implies that we need some reasonable finiteness conditions to classify these. Building on $\mathcal{D}$-simplicity, we have a connection with Bernstein-Sato Polynomials.

Theorem 1.5.6 (Alvarez-Montaner, Huneke, Nunez-Betancourt)**.** *If R is a direct summand of a polynomial ring in characteristic 0, then every nonzero $f \in R$ admits a Bernstein-Sato polynomial.*

There is a characteristic p variant of this statement, too.

Theorem 1.5.7 (Quinlan-Gallego, Jeffries, Nunez-Betancourt)**.** *For direct summands of polynomial rings (or graded with FFRT), a similar statement holds.*

Further, in characteristic 0 we know that in many cases $\mathcal{D}_R$ is simple.

Theorem 1.5.8. *Let R be a $\mathbb{C}$-algebra. For invariant of finite groups, toric rings, and various determinental rings, $\mathcal{D}_R$ is a simple ring.*

This is notable because, if $\mathcal{D}_R$ is simple ,then R is $\mathcal{D}$-simple.

Theorem 1.5.9 (Smith-Van der Bergh)**.** *Let S be a polynomial ring over a perfect field of characteristic $p > 0$. Let $R \subset S$ be a direct summand. Then $\mathcal{D}_R$ is a simple ring.*

Theorem 1.5.10. *Bernstein's inquality holds for $\mathcal{D}_R$, i.e. every nonzero $\mathcal{D}_R$ module has dimension $\geq \dim(R)$, when R is strongly F-regular with FFRT in characteristic p.*

Chapter 2

Day 2

2.1 Shunsuke Takagi - Uniform positivity of F-signature under reduction modulo p

Let $(A, \mathfrak{m})$ be an F-finite local domain. For simplicity we further assume that $\mathfrak{K} = A/\mathfrak{m}$ is perfect.

$$F_*^e A \cong A^{\oplus a_e} \oplus M$$

where M admits no direct summands of A; a_e is thus the maximal number of copies of A that split off of $F_* A$. If A is regular, then $F_* A$ would be a free module and thus $a_e = p^{ed}$; in general $a_e \leq p^{ed}$. In the sense of Huneke-Lenschke, we define the notion of F-signature to because

$$s(A) = \lim_{p \to \infty} \frac{a_e}{p^{ed}}$$

This limit exists (due to Kevin Tucker) and $s(A) > 0$ if and only if A is strongly F-regular. Now let

$$R = \left(\frac{\mathbb{C}[x_1, \ldots, x_n]}{(f_1, \ldots, f_r)} \right)_{(x_1, \ldots, x_n)}$$

Where we choose $f_i \in \mathbb{Z}[x_1, \ldots, x_n]$. Taking the mod p quotient yields

$$R_p := R \bmod \mathrm{p} = \left(\frac{\mathbb{F}_p[x_1, \ldots, x_n]}{(f_1 \bmod \mathrm{p}, \ldots, f_r \bmod \mathrm{p})} \right)_{(x_1, \ldots, x_n)}$$

Well, does $\lim_{p \to \infty} s(R_p)$ exist? Due to Shideler, we have an example where it does; indeed the for $r = 1, n = 4$ and $f_1 = x^3 + y^3 + z^3 + z^3$, $\lim_{p \to \infty} s(R_p) = \frac{1}{8}$. We don't know if this exists in general, but we can define the liminf quantity:

$$s(R) := \liminf_{p \to \infty} S(R_p)$$

Associated to this is the following conjecture of Schwede-Smith, stated locally. Let $X = \mathrm{Spec}(R)$ be a normal affine variety. Then X is KLT type if and only if R_p is strongly F-regular $\forall p \gg 0$. $\Rightarrow$ is known due to Hara, Mehta-Srinivas, and Takagi and $\Leftarrow$ holds if R

is $\mathbb{Q}$-Gorenstein. due to Smith, Hara and Watanabe. Outside of this case, it the conjecture remains open in dimension 3 and greater.

If $S(R) > 0$, then R_p is Strongly F-regular $\forall p \gg 0$, so if this conjecture holds, then R is KLT type. But how about the converse? This is a conjecture of Carvas-Rojas, Schwede, and Tucker (CRST). They conjecture that if R is KLT type, then $s(R) > 0$. It is sufficient to exhibit a $c > 0$ such that $s(R_p) \geq c$ for all p sufficiently large. This conjecture holds in dimension 2, as here KLT type singularities are precisely the finite quotient singularities and thus F-signature is readily computable; in dimension 3 and higher this is still unknown.

Theorem 2.1.1 (Takagi-Yamaguchi)**.** *Let* $(\mathrm{R}, \mathfrak{m}) \to (S, \mathfrak{n})$ *be a pure local* $\mathbb{C}$*-algebra homomorphism of KLT type essentially of finite type algebras over* $\mathbb{C}$*. If* $s(S) > 0$*, then* $s(R) > 0$*.*

As a corollary, we see that the conjecture of CRST holds for reductive quotient singularities. We also see that the conjecture can be reduced to the case where R is Gorenstein. We prove this below:

Proof. Take $S = \left(\bigoplus_{i \geq 0} R(-iK_R)\right)_M$ where M is the homogeneous maximal ideal of S. If R is KLT type, then S is Noetherian. In particular, S will be Gorenstein and KLT. $R \to S$ is pure, so we can cite the theorem above. □

This conjecture can also be reduced to the graded case; we will get back to justifying this later if time. The key ingredients to prove the theorem are local F-alpha invariants and ultra-F-regularity, an invariant of characteristic 0 rings. We define the F-alpha invariant below. For $(A, \mathfrak{m})$ Strongly F-regular and local, we define

$$\alpha_F(A) = \inf_{x \in \mathfrak{m}} \operatorname{fpt}(x) \overline{\operatorname{ord}}_{\mathfrak{m}}(x)$$

Where $\overline{\operatorname{ord}}_{\mathfrak{m}}(x)$ denotes the maximal $r \geq 1$ such that $x \in \mathfrak{m}^r$. We note htat $0 \leq \alpha_F(A) \leq 1$ and if A is regular, then $\alpha_F(A) = 1$. Due to work of Pande, for any R of the form above, $s(R) > 0 \iff \liminf \alpha_F(R \bmod p) =: \alpha_F(R) > 0$.

Now to define ultra-F-regularity. Let $\underline{p}$ denote the set of all prime numbers, and $\{A_p\}_{p \in \underline{p}}$ a family of rings. Then

$$A_\infty := \mathrm{ulim}_p A_p = \prod_{p \in \underline{p}} A_p / \sim$$

Where $(a_p) \sim (b_p)$ if and only if $a_p = b_p$ for almost all p.For instance, $\mathrm{ulim}_p \mathbb{F}_p \cong \mathbb{C}$! This is non-canonical, so we fix such an isomorphism. In particular, $R_\infty := \mathrm{ulim}_p R_p$ is a $\mathbb{C}$-algebra. Pick $e_p > 0$ for all p and $\mathcal{E} = \mathrm{ulim}_p e_p$. We have a morphism $F^{\mathcal{E}} : R \to R_\infty$, the ultra-Frobenius, assigning $x \mapsto \mathrm{ulim} x_p^{p^{e_p}}$. Using this ultra-Frobenius, we define ultra F-regularity.

Choose $0 \neq f_p \in R_p$ and $t_p \in \mathbb{R}_{>0}$. Define $R_\infty \ni f := \mathrm{ulim}_p f_p$ and $t = \mathrm{ulim}_p t_p$. We say that (R, f^t) is Ultra-F-regular if and only if $\forall c \in R_\infty$ nonzero, $\exists \mathcal{E}$ such that the map $R \to F_*^{\mathcal{E}} R_\infty$ assigning $x \mapsto F_*^{\mathcal{E}} \left(\mathrm{ulim} c_p f_p^{\lceil t_p p^{e_p} \rceil} F^{\mathcal{E}}(x) \right)$ is pure.

Lemma 2.1.2. *For $x \in R$ for R as above, $t_p \in \mathbb{R}_{>0}$, and $t = \mathrm{ulim}_p t_p$ such that $(R_p, x_p^{t_p})$ is Strongly F-regular for almost all p, then (R, x^t) is Ultra F-Regular.*

Lemma 2.1.3. *Let $(\mathrm{R}, \mathfrak{m}) \to (S, \mathfrak{n})$ be a pure local map of $\mathbb{C}$-algebras. Then* $\mathrm{fpt}(R \text{ mod } p, x \text{ mod } p) \geq \mathrm{fpt}(S \text{ mod } p, x \text{ mod } p)$ *for all but finitely many p.*

2.2 Kenta Sato - Extending one-forms on F-regular singularities

Let X be a normal variety over $k = \bar{k}$. we denote its regular locus by X_{reg}. and take

$$\Omega_{X/k}^{[i]} := i_* \Omega_{U/k}^{\wedge i}$$

Where : $U \hookrightarrow X$ is the standard inclusion. Assume that $f : Y \to X$ is a log resolution with E corresponding to the exceptional divisor of f. We say that X has regular extension theorem for i-forms (RET_i) if and only if $f_* \Omega_Y^i = \Omega_{X/k}^{[i]}$. When X is affine, this is equivalent to checking that following restriction map is surjective on sections:$\mathrm{H}^0(Y, \Omega_Y^i) \twoheadrightarrow \mathrm{H}^0(U, \Omega_{U/k}^i)$. Equivalently, all i-forms on X extend to i-forms on Y. Similarly, we say that X has log extension theorem for i-forms (LET_i) if and only if $f_* \Omega_{Y/k}^i (\log E) = \Omega_X^{[i]}$.

There exists a deep connection between these notions and singularities in MMP. For instance, assume that we are in characteristic 0 and X is CM. Then X has rational singularities if and only if it satisfies RET_d. X has Du Bois singularities if and only if it satisfies LET_d. But we can do better! In fact, haiving rational singularities implies RET_i for all i, as a result of Kebekus and Schnell, and having Du Bois singularities implies LET_i) for all i, due to Ben Tighe.

Well, how about in positive characteristic? The analogous question for rational singularities would be to ask whether F-rational singularities satisfy some form of RET. Unfortunately, this is not true! Via an example of Graf, consider the p-fold Veronese embedding of $k[x, y]$; this is F-regular but does not satisfy RET_1. But how about the F-injective/LFT connection? Graf has shown that, if X is a 2-dimensional F-regular variety over an algebraically close field, then it satisfies LET_1.

Theorem 2.2.1 (Sato, Kawakami)**.** *The above statement holds in all dimensions. In particular if X is an F-regular variety over $k = \bar{k}$, then it satisfies* LFT_1*.*

And we have a related result:

Theorem 2.2.2 (Sato, Kawakami)**.** *Let $(\mathrm{R}, \mathfrak{m})$ be a local, normal essentially of finite type over $k = \bar{k}$. Further assume that* $\dim(R) = 2$*. Then Strong F-regularity implies* LFT_1*.*

This theorem is the same as the result of Graf when $R/\mathfrak{m} = k$.

2.3 Shiji Lyu - Approximating Complete Local Rings

Computer died.

2.4 Eleanor Faber - Noncommutative Resolutions of Normal Toric Varieties and Beyond

This is joint work with Greg Muller and Karen Smith.

Let $X = \operatorname{Spec}(R)$ where R is commutative/Noetherian overa field k. We can consider the ring of differential operators $\mathcal{D}_k(R)$, which is notably *not* commutative. When $k = \mathbb{C}$ this is just the Weyl Algebra, and in positive characteristic $\mathcal{D}_k(R)$ is non-Noetherian (but still simple for $R = k[x]$, for instance!). It is worth studying how singular $\mathcal{D}_k(R)$ is, notably, what is its global dimension? For rings R, it is known due to Auslander-Buchsbaum and Serre that R is regular if and only if $\operatorname{glDim}(R) < \infty$. Over $\mathcal{D}_k(R)$, it is known in char 0 and char p that if R is regular, $\operatorname{glDim}(\mathcal{D}_k(R)) = \dim(R) < \infty$. The characteristic 0 variant is due to Roos/Chase, and the characteristic p case is due to P.Smith. But is the reverse case true? Are there other rings R for which $\mathcal{D}_k(R)$ has finite global dimension?

The answer is yes, so the converse does not hold. In characteristic p,

$$\mathcal{D}_k(R) = \varinjlim_e \operatorname{End}_{R^{p^e}}(R) = \varinjlim_e \operatorname{End}_R(R^{1/p^e})$$

$\operatorname{End}_R(M)$ for M any R-Module can be studied via NCRs (Noncommutative Resolutions of Singularities) and NCCRs (Noncommutative Crepant Resolutions).

Theorem 2.4.1 (Faber-Muller-Smith, 2019)**.** *Let $R = k[x]$ be the coordinate ring of an affine toric variety X. Then,*

1. *$\exists$ a reflexive MCM finitely generated R-Module A such that* $\operatorname{glDim}(\operatorname{End}_R(A)) = \dim(R)$.
2. *In characteristic p,* $\operatorname{glDim}(\mathcal{D}_k(R)) \leq \dim(R) + 1$. *Notably, global dimension is finite.*
3. *The ring* $\operatorname{End}_R(A)$ *is an NCCR $\iff$ X is simplicial.*

What is this A? It has an explicit description: $\operatorname{End}_R(A)$ is Morita equivalent to $\operatorname{End}_R(R^{1/p^e})$.

2.5 Michel Van den Bergh -

Computer died.

Chapter 3

Day 3

3.1 Lawrence Ein - Syzygies and singularities of secant varieties of curves

Let C be a smooth complex projective curve of genus g, and L a very ample line bundle on C. Then $C \hookrightarrow \mathbb{P}\,\mathrm{H}^0(L)) = \mathbb{P}^r$ where $h^0(L) = r+1$. Such objects have been studied for a long time, and there are many classical results:

Theorem 3.1.1 (Castelnuovo)**.** *If* $\deg(L) \geq 2g+1$*, then* L *is very ample and* C_H *is projectively normal.*

We can define $R(C, L) = \bigoplus_{m \geq 0} \mathrm{H}^0(L^{\otimes m})$ and $S = \mathrm{Sym}\,\mathrm{H}^0(L) \cong \mathbb{C}[x_0, \ldots, x_r]$. We have that $S \twoheadrightarrow R(C.L)$ if and only if $\mathrm{H}^0(L) \otimes \mathrm{H}^0(L^{\otimes m}) \twoheadrightarrow \mathrm{H}^0(L^{\otimes m+1})$ for any $m \geq 0$.

Theorem 3.1.2 (Fujita, 1980)**.** *If* $\deg(L) \geq 2g+2$*, then* $I_{C/\mathbb{P}^r}$ *is generated by the degree* 2 *elements.*

Consider the minimal graded free resolution

$$\oplus S(-2) \xrightarrow{\text{Fujita's Theorem}} S \to R(C, L) = S(C, L) \to 0$$

There is a notable conjecture of Green-Lazarsfeld: Assuming $\deg(L) \gg 0$, can one determine the gonality of C from the minimal resolution? This was answered in the affirmative by Ein-Lazersfeld; The best way to interpret the minimal resolution is to dualize it (as the curve is CM)

3.2 Linquan Ma - Lech's inequality and stability of local rings

This is joint work with Ilya Smirnov.

Theorem 3.2.1 (Lech, 1960s)**.** $(R, \mathfrak{m})$ *is a local ring and I an $\mathfrak{m}$-primary ideal. Then*

$$e(I) \leq d! e(R) \ell(R/I) \tag{3.1}$$

Here, $e(I) := \lim_{n\to\infty} \frac{\ell(R/I^n)}{n^d} d! \in \mathbb{Z}$ is the Hilbert-Samuel Multiplicity. We define $e(R) := e(\mathfrak{m})$. About this, we remark the following:

- For $I = J^N$, $e(J^N) = N^d e(J)$, so $\ell(R/J^N) \sim N^d e(J) \cdot \frac{1}{d!}$.
- Along these lines, $e(J^N) \sim d! \ell(R/J^N)$.
- 3.1 is asymptotically sharp with $e(R) = 1$.
- For $R = k[[x_1, \ldots, x_d]]$ and I an $\mathfrak{m}$-primary monomial ideal, then we can relate the Newton Polygon:

  $$\mathrm{NP}(I) := \text{ convex hull of } \{\underline{a} \mid \underline{x}^{\underline{a}} \in I\} \subset \mathbb{R}^d_{\geq 0}$$

  to $e(I)$. Namely,

  $$e(I) = d! \mathrm{vol}(\mathbb{R}^d_{\geq 0} \setminus \mathrm{NP}(I))$$

  and $\ell(R/\overline{I})$ counts the number of integer points in the complement of $\mathrm{NP}(I)$.

Proof of Lech's Inequality. In characteristic p,

$$e(I) \leq d! e_{HK}(I) \leq d! e_{HK}(R) \ell(R/I) \leq d! e(R) \ell(R/I)$$

□

We now define the following invariant for a local ring $(R, \mathfrak{m})$ of dimension d:

$$C_{LM}(R) := \sup_{\sqrt{I} = \mathfrak{m}} \left\{ \frac{e(I)}{d! \ell(R/I)} \right\}$$

Called the Lech-Mumford constant of R. We remark the following:

- $C_{LM}(R) = C_{LM}(\widehat{R})$.
- It is sufficient to consider only the integrally closed ideals I when evaluating over the sup.
- Via Lech's inequality, we see that $1 \leq C_{LM}(R) \leq e(R)$.

Theorem 3.2.2 (Huneke, Ma, Quy, Smirnov)**.** $C_{LM}(R) = e(R) \iff$ *one of the following conditions hold:*

- $\dim(R) \leq 1$.
- $\dim(R) \geq 2$ *and* $e(\widehat{R}_{\mathrm{red}}) = 1$.

In particular, for complete local domains of dimension at least 2*, we have this equality precisely when R is regular.*

Thus, $c_{LM}(R)$ is rarely maximal. this motivates the following question: when is it *minimal*? i.e. Whe does $C_{LM}(R) = 1$?

Theorem 3.2.3 (Mumford)**.** *If X is a projective variety over k and L is an ample line bundle, suppose (X, L) was asymptotically Chow semi-stable. Then for every closed $x \in X$, $C_{LM}(\mathcal{O}_{X,x}[[t]]) = 1$.*

This implies that $\forall n \gg 0$, $X \xrightarrow{|L^n|} \mathbb{P}^N$ is injective, and the Chow form of this embedding is semi-stable under the SL_{N+1} action. This also implies the following lemma:

Lemma 3.2.4. *For* $(\mathrm{R}, \mathfrak{m})$ *a local ring,*

$$e(R) \geq C_{LM}(R) \geq C_{LM}(R[[t]]) \geq C_{LM}(R[[t_1, t_2]]) \geq \cdots \geq 1)$$

Via this lemma, we have the ollowing definition. We say that a local ring $(\mathrm{R}, \mathfrak{m})$ is semi-stable if $C_{LM}(R[[t]]) = 1$ and stable if it is semistable and the sup in the definition of $C_{LM}(R[[t]])$ is *not* attained. Further, $(\mathrm{R}, \mathfrak{m})$ is Lech stable if $C_{LM}(R) = 1$ and lim stable if $\lim_{n\to\infty} C_{LM}(R[[t_1, \ldots, t_n]]) = 1$. We naturally have that Lech stable $\Rightarrow$ semi-stable $\Rightarrow$ lim-stable. In fact, we even have that Lech stability implies stability, which of course implies semi-stability, as long as $\dim(R) \geq 1$.

Let's consider the case of Artinian rings, e.g. $\dim(R) = 0$. In this setting, $C_{LM}(R[[t_1, \ldots, t_n]]) = \ell(R)\ \forall n$. in particular, Lech-stable, semi-stable, and lim-stable are all equivalent, and hold precisely when R is a field. But what about stability? Well, $\dim(R[[t]]) = 1$, so sup is always acheived at the maximal ideal. Thus, an Artinian ring will never be stable.

Let's compute an example. Take $R = k[[x, y]]/(xy)$. This is dimension 1, so $C_{LM}(R) = 2$. Mumford has shown that R is semi-stable, i.e. $C_{LM}(k[[x, y, t]]/(xy)) = 1$. However, R is not stablee as (x, y, z) hits hits the maximal.

Theorem 3.2.5 (Ma, Smirnov)**.** *Let* $\dim(R) = 1$ *and R be CM. Then,*

- *Lech-Stable* $\iff$ *Stable* $\iff$ *R is a DVR.*
- *Semi-stable* $\iff$ *lim-stable* $\iff$ *R is a node* $\iff$ *R is Grenstein and semi-normal.*

For a higher dimensional example, consider a simple normal crossing. In this case $R = \frac{k[[x_1, \ldots, x_n]]}{x_1 x_2 \ldots x_n}$. For $n = 1$ R is a field, so it is Lech-stable. For $n = 2$, this recovers the previous example and is semi-stable but not stable. For $n \geq 3$, R is stable but not Lech-stable.

Theorem 3.2.6. $(\mathrm{R}, \mathfrak{m})$ *is a complete normal local ring of dimension* 2*.*

- *(Goto-Iai-Watanabe) R is Lech-stable* $\iff$ *R is an RDP. (i.e. Gorenstein and pseudo-rational)*

- *(Ma-Smirnov) If R is lim-stable, then R is $\mathbb{Q}$-Gorenstein and log canonical.*

Theorem 3.2.7 (Ma-Smirnov)**.** *Suppose $(R, \mathfrak{m})$ is essentially of finite type over k, with k characteristic 0. Further suppose that R is normal and $\mathbb{Q}$-Gorenstein. Then*

- *If R is lim-stable then R is log-canonical.*
- *If R is Lech-stable and R is an isolated singularity then R is canonical.*

We remark the following:

- It is not true that lim/Lech-stable implies $\mathbb{Q}$-Gorenstein.
- You cannot drop the isolated singularity requirement in the second point in the theorem above.
- The converse of either statement is false.

3.3 Tatsuki Yamaguchi - F-pure singularities in equal characteristic zero

Computer died.

Chapter 4

Day 4

4.1 Sándor Kovács - The injectivity theorem for m-Du Bois singularities

For now, let X be smooth over $\mathbb{C}$. We can resolve $\mathbb{C}$ by differential forms:

$$\mathbb{C} \to \mathcal{O}_X \to \Omega_X \to \Omega_X^2 \to \dots$$

In particular, $\mathbb{C} \cong \Omega_X^\bullet$ in the derived catagory. This gives rise to the Hodge-de-Rham Spectral Sequence $H^q(X, \Omega_X^p) \Rightarrow \mathrm{H}^{p+q}(X, \mathbb{C})$. When X is singular, however, such a decomposition does not exist. Due to work of Deligne and DuBois, one can construct a related complex $\underline{\Omega}_X^\bullet$, the filtered Deligne-DuBois complex, where

$$\underline{\Omega}_X^\bullet := \mathrm{Gr}^p \underline{\Omega}_X^\bullet[p]$$

there is a natural filtered morphism $\Omega_X^\bullet \to \underline{\Omega}_X^\bullet$, yielding morphisms on graded components. Notably, even outside of the smooth case we get the degeneration $\mathbb{H}^q(X, \underline{\Omega}_X^p) \Rightarrow \mathrm{H}^{p+q}(X, \mathbb{C})$. We say that X has Du-Bois Singularities (DB Singularities) if these complexes agree on the 0th part, i.e. $\Omega_X^0 = \mathcal{O}_X \to \underline{\Omega}_X^0$ is a quasi-isomorphism. Phrased differently, $\mathrm{H}^0(\mathcal{O}_X) \cong \mathrm{H}^0(\underline{\Omega}_X^0)$. But how about checking whether isomorphism holds on higher parts of the Deligne-DuBois Complex? This definition came about nearly 40 years later! We say that X has m-DB singularities if $\Omega_X^p \cong \underline{\Omega}_X^p$ for $p \leq m$.

This definition works very well for LCIs, but outside of varieties with LCI singularities, this condition is rarely acheived. Thus, recent work has adjusted the definition. We now say that X is m-DB if it is semi-normal, $\mathrm{H}^i(\underline{\Omega}_X^p) = 0$ for all $i > 0$ and $p \leq m$, some nebulous codimension condition, and $\widetilde{\Omega}_X^p := \mathrm{H}^0(\underline{\Omega}_X^p)$ is reflexive.

Lemma 4.1.1. *If X is a 1-DB LCI, then $\mathrm{H}^0(\underline{\Omega}_X^p) \cong \Omega_X^p$ for all p.*

In other words, the definitions coincide in the LCI case. Thus the old definition is referred to as strict m-DB, and the new definition without the codimension condition is sometimes referred to as weakly-m-DB. If only the $h^i(\underline{\Omega}^p_X)$ condition is satisfied, we say that X is pre-m-DB. We now introduce a DuBois notion of the dualizing complex:

$$\underline{\omega}^\bullet_X := R\,\mathcal{H}om_X(\underline{\Omega}^0_X, \omega^\bullet_X)$$

The notation comes from the fact that, if you replace $\underline{\Omega}^0_X$ with $\Omega^0_X = \mathcal{O}_X$ you recover the definition of $\omega^\bullet_X$ for trivial reasons.

Theorem 4.1.2 (Kovács-Schwede)**.** $\mathrm{H}^i(\underline{\omega}^\bullet_X) \hookrightarrow \mathrm{H}^i(\omega^\bullet_X)$ *for all* i*.*

But does this hold for higher order variants? We define

$$\underline{\omega}^p_X := R\,\mathcal{H}om_X(\underline{\Omega}^p_X, \omega^\bullet_X)$$

$$\widetilde{\omega}^p_X := R\,\mathcal{H}om_X(\widetilde{\Omega}^p_X, \omega^\bullet_X)$$

Thus one can ask, if X has $(m-1)$-DB singularties, is $\mathrm{H}^i(\underline{\omega}^m_X) \to \mathrm{H}^i(\widetilde{\omega}^m_X)$ injective for all i?

$$\underline{\omega}^\bullet_X := R\,\mathcal{H}om_X(\underline{\Omega}^0_X, \omega^\bullet_X)$$

Theorem 4.1.3 (Popa-Shen-Vo)**.** *This is true for X with isolated singularities.*

Theorem 4.1.4 (Kovács)**.** *This is true in general!*

Chapter 5

Day 5

5.1 Adela Vraciu - Possible values of F-pure thresholds of Homogeneous Polynomials

We set $f \in k[x_1, \ldots, x_n]$ for k an algebraically closed field of characteristic $p > 0$. We further assume that f is homogeneous. The F-pure Threshold is a measure of singularity of the hypersurface corresponding to f. $0 < \text{fpt} \le 1$, where $\text{fpt}(f) = 1$ if and only if R/f is F-pure, and lower F-pure thresholds correspond to having worse singularties.

Theorem 5.1.1 (Hara-Yoshida, Takagi-Watanabe). $\text{lct}(f) = \sup_p\{\text{fpt}(f_p)\}$ *where f_p corresponds to the mod p reduction of f.*

It is famously conjectured that, for fixed f, that there exists infinitely many p for which $\text{lct}(f) = \text{fpt}(f_p)$. For example, $\text{lct}(x^2 - y^3) = 5/6$, but

$$\text{fpt}(f_p) = \begin{cases} \frac{5}{6} & p \equiv_6 1 \\ \frac{5}{6} - \frac{1}{6p} & p \equiv_6 5 \end{cases}$$

If $\deg(f) = d$, then $\text{fpt}(f) \ge \frac{1}{d}$. This bound is acheived precisly when $f = x^d$. Further, when f is homogeneous of degree d in n variables, then $\text{fpt}(f) \le \frac{n}{d}$.

Theorem 5.1.2 (HNWZ). *If f is an isolated singularity, then $\text{fpt}(f) = \frac{n}{d}$ or a truncation of $\langle n/d \rangle_L$, with the distinction subject to necessary conditions to be discussed later.*

Lemma 5.1.3. *Let $X_\lambda = \{f \mid \text{fpt}(f) \le \lambda\}$ is a closed set, and there are finitely many possible values of λ. Further, there is a largest possible value, and hence set of polynomials in which this is acheived is Zariski open.*

Theorem 5.1.4 (Smith,Vraciu). *The generic value of the* fpt *is* 1 *if $n \ge d$ if $n < d$, it is $\langle n/d \rangle_e$, where e is the smallest number such that d does not divide np^e and the remainder np^e mod $d < n$. If no such e exists, then the generic value is $\frac{n}{d}$.*

As a consequence of this theorem, if $p \equiv_d 1$, then we land in the last case. Thus the generic $\text{fpt} = n/d$, which is the lct so the conjecture above holds in this case.

Theorem 5.1.5 (Smith, Vraciu). *Assume that $d \geq 4$ and p does not divide d. Then there exists reduced polynomials $f \in k[x, y]$ of degree d that satisfy* $\mathrm{fpt}(f) < 2/d$.

Theorem 5.1.6 (Smith, Vraciu). *For $d \leq 8$ and L satisfying the conditions of the HNWZ theorem, $\exists$ a polynomial $f \in k[x, y]$ with* $\mathrm{fpt}(f) = \langle 2/d \rangle_L$.

5.2 Bhargav Bhatt - The Deligne-DuBois Complex

We first construct the algebraic DeRham complex, the smooth variant of what we will soon discuss. Let k be a fixed field, of any characteristic for now. To a smooth variety X/k, Grothendieck associated to it the algebraic DeRham complxe $\Omega^\bullet_{X/k} = \left(\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/k} \xrightarrow{d} \dots\right)$. As X is smooth, these are all vector bundles. d is not $\mathcal{O}_X$-linear, but that is OK.

Theorem 5.2.1 (Grothendieck). *If $k = \mathbb{C}$, then*

$$\mathrm{H}^*_{dR}(X/k) := \mathrm{H}^*(X, \Omega^\bullet_{X/k})$$

is isomorphic to $H^*(X^{an}, \mathbb{C})$.

This is quite remarkable, as these cohomology theories do not really seem related. This also forms the starting point of Hodge theory, as we can use algebraic methods to understand topological properties of complex manifolds. Well, what happens when X is not smooth? If X is singular, $\Omega^i_{X/k}$ are no longer vector bundles, and Grothendieck's theorem fails. This is why we need the Deligne-DuBois (DDB) complex.

Now we let k be characteristic 0.

Theorem 5.2.2 (D-DB). *FIx $j \geq 0$. Then $\exists$ a unique assignment $X \in \mathrm{Sch}^{ft}_k \mapsto \underline{\Omega}_{X/k} \in \mathrm{D}^b_{Coh}(X)$ plus a map*

$$\alpha : \Omega^j_{X/k} \to \underline{\Omega}^j_{X/k}$$

such that α is an isomorphism when X is smooth. This also satisfies 'cdh'-descent, i.e. given a cartesian square

$$\begin{array}{ccc} E = \pi^* Z & \longrightarrow & X \\ \downarrow & & \downarrow \pi \\ Z & \overset{i}{\hookrightarrow} & Y \end{array}$$

such that π is proper and an isomorphism outside Z and i is a closed immersion, the natural maps give an exact triangle

$$\underline{\Omega}^j_{Y/k} \to R\pi_* \underline{\Omega}^j_{X/k} \oplus Ri_* \Omega^j_{Z/k} \to Rg_* \underline{\Omega}^j_{E/k} \xrightarrow{+1}$$

For instance, take $X_{red} \hookrightarrow X$. This implies that $\underline{\Omega}^j_{X/k} \cong Ri_*\underline{\Omega}^j_{X_{red}/k}$. Thus the DDB complex does not care about nilpotents, yet is still defined for them. Now consider $X = V(y^2 - x^3) \subset \mathbb{A}^2$ and take $\pi : Y \to X$ to be the normalization. The theorem gives us a way to pullback the DDB complex and relate it to the normalization.

This is a bit unmotivated; the theorem tells us how to construct an object from soemthing else. The point of this talk is to intrisincally construct the DDB complex, and think of this result as a theorem about the construction. But why should we care: We see a natural motivation from Hodge theory: Let X/k be any variety, we have that

$$\mathrm{H}^*(X, \underline{\Omega}^\bullet_{X/k}) \cong \mathrm{H}^*_{sing}(X^{an}, \mathbb{C})$$

with a naive filtration of the DDB complex giving the Hodge filtrarion when X is proper. We can also use the DDB complex to study singularities of X. A variety X has DB singularities if $\alpha : \mathcal{O}_X \to \underline{\Omega}^0_X$ is an isomorphism. Nodal curves are DB, but cuspidal curves are not. As a theorem of Kollar and Kovács, Log canonical singularities are DB. This complex also has a relation to char p. Karl Schwede conjcetures that X has DB singularities if and only if X mod p is F-injective for infinitely many p. $\Leftarrow$ is known (due to Karl), but $\Rightarrow$ is only known given the weak ordinarity conjecture. This is an analogous relationship to the one between KLT and SFR singularities.

To construct the DDB complex, we will pass to non-archimedian geometry. Let k be a field of characteristic 0 and $K = k((t))$ a non-archimedian field.

Theorem 5.2.3 (Tate '62'). *There exists a category of rigid (analytic) spaces over K, denoted RS_K, plus an 'analytification' functor assigning Varieties over K to an analogous object in RS_K.*

This functor is quite nice, but it, and the construction of rigid spaces, are quite complicated. We will just mention some properties about these things:

- The 'building blocks' of rigid analytic spaces are called affinoids, as affine varieties are to varieties. For example, take $\mathbb{D} = \mathbb{D}[0,1]$ the closed unit disc of radius 1 at 0. the K points $\mathbb{D}(K) = \{z \in K \mid |z| \leq 1\} = \mathcal{O}_K = K[[t]]$. In fact, $\mathbb{D} = \mathrm{Spa}(R)$, where $R = \mathcal{O}_K[z]_{(t)}{}^\wedge[1/t]$. R is a Dedekind domain with reasonably nice properties. For another example, consider $(\mathbb{A}^1)^{an}$. This is not a disk on the nose, but it is a union of disks $(\mathbb{A}^1)^{an} = \bigcup_r \mathbb{D}[0,r]$.

- Any rigid analytic space has a coherent structure sheaf $\mathcal{O}_X$ with reasonable properties, similar to what happens in algebraic geometry. For instance, if $X = Y^{an}$ for some Y proper, then there exists a GAGA type comparison: $\mathrm{D}^b_{Coh}(X) \cong \mathrm{D}^b_{Coh}(Y)$. For $X = \mathrm{Spa}(R)$ an affinoid, $\mathrm{D}^b_{Coh}(X) \cong \mathrm{D}^b_{Coh}(R)$.

- Any rigid analytic space has a reasonable theory of differential forms $\Omega^j_{X/k} \in \mathrm{Coh}(X)$, yielding a corresponding DDB complex $\underline{\Omega}^j_{X/k} \in D^b_{Coh}(X)$.

- For any rigid space X, in addition to the structure sheaf, there is another sheaf floating around. In particular, we have a natural integral subsheaf $\mathcal{O}_X{}^+ \subset \mathcal{O}_X$ defined by
$$\mathcal{O}_X{}^+(U) := \{f \in \mathcal{O}_X(U) \mid |f(x)| \le 1 \; \forall x \in U\}$$
This is a sheaf of $\mathcal{O}_K$-algebras, but not of K-algebras. As an example, $\mathcal{O}_X{}^+(\mathbb{D}) = \mathcal{O}_K[z]_{(t)}{}^\wedge$. In particular, this is just like R but where we didn't invert t. However, this is not always well behaved. From our other example, $\mathcal{O}_X{}^+(\mathbb{A}^1) = k$, as there are no entire bounded non-constant functons. Also we can consider $X = V(y^2 - x^3) \subset \mathbb{D}^2$. As one would expect,
$$\mathcal{O}_X{}^+(X) = \left(\frac{\mathcal{O}_K[y,x]}{(y^2 - x^3)}\right)_{(t)}^{\wedge}$$
The normalizaiton of X is $\mathbb{A}^1$; one way to see that X is not normal is that $y/x \in \mathcal{O}_X{}^+(X)$. However, we can evaluate $\mathcal{O}_X{}^+/t^n(X)$ and see that it contains y/x for any $n \ge 1$; glue y/x on $X - \{0\}$ to 0 on a very small (dependent on n) neighborhood of 0. In particular, the reduction is not recognized via the $+$ variant of sheafification. In particular,
$$\mathrm{H}^1(X, \mathcal{O}_X{}^+)[t^n] \ne 0$$

Define $\widehat{\mathcal{O}_X{}^+} = \mathrm{D}\varinjlim_n \mathcal{O}_X{}^+/t^n \in \mathrm{D}(X, \mathcal{O}_X{}^+)$. and $\widehat{\mathcal{O}_X} = \widehat{\mathcal{O}_X{}^+}[1/t] \in \mathrm{D}(X, \mathcal{O}_X)$, and similar variants for $\widehat{\Omega}^k_{X/k}\; \forall j$. This notation comes from thinking about sheaves on the pro-etale site, where these "completed sheaves" are sheaves on the nose, and don't have to be sitting in the derived category.

Theorem 5.2.4 (Bhatt). *$K = k((t))$ with k char 0, and X/k a rigid analytic space. Then $\widehat{\Omega}^j_{X/k} \cong \underline{\Omega}^j_{X/k}$ for all j.*

We have a couple of remarks about the $j = 0$ case.

1. The $j = 0$ case (at least) also works over p-adic fields K.
2. The $j = 0$ case also implies that, for X is affinoid, $\mathrm{H}^i(X, \mathcal{O}_X{}^+)$ has unbounded t-torsion if and only if $\mathrm{H}^{i-1}(\underline{\Omega}^0_X/\,\mathcal{O}_X) \ne 0$. This alligns (in the negation) with the cusp case, as the cusp does not have DB singularities.

5.3 Peter McDonald - Ideal closure operations via resolution of singularities in characteristic zero

Theorem 5.3.1 (Smith, Hara, Mehta-Srinivas). *Let R be finite type over a field of characteristic 0. Then R has rational singularities if and only if R mod p is F-rational $\forall p \gg 0$.*

F-rationality has been studied commutative algebraically for quite some time, and there are a number of properties we would like to port back to char 0, namely the perspective of tight closure. Indeed, a ring is F-rational if and only all parameter ideals are tightly closed. But what are closure operations in equal char 0?

- Brenner: parasolid closure (as well as similar ones like axis closure and continuous closure)
- Schouner,Aschenbrenner, Yamaguchi: Closure operations via ultraproducts of ultra-Frobenius.
- Hochster and Huneke's char 0 version of tight closure.

Let $\mathbb{Q} \subset R$ b a reduced, excellent, with dualizing complex ω_R. For any R with these properties we say R has (*). Consider a resolution $\pi : Y \to \mathrm{Spec}(R)$, and consider the assignment $R \to R\Gamma(Y, \mathcal{O}_Y) \in \mathrm{D}^b(R)$. R has rational singularities if this map is an isomorphism. If R is also locally equidimensional, $R\Gamma(Y, \mathcal{O}_Y)$ is locally CM (as a complex). Recall that for $X \in \mathrm{D}^b(R)$ when $(\mathrm{R}, \mathfrak{m}, \mathfrak{K})$ isa local ring, we can define X to b BCM if:

- $\mathrm{H}^0(X) \to \mathrm{H}^0(k \otimes_R^{\mathbb{L}} X)$ is nonzero.
- $\mathrm{H}^i_\eta(X) = 0$ except for where $i = \dim(R)$.

Thus being 'locally CM' is just being BCM every localization at P for each $P \in \mathrm{Spec}(R)$. For R with (*) we define the Koszul-Hironaka Closure, or KH Closure of an ideal $J = (f_1, \ldots, f_n)$ to be

$$J^{KH} := \ker\left(R \to \mathrm{H}^0\left(\mathrm{Kos}\left(\underline{f}\right) \otimes_R^{\mathbb{L}} R\Gamma(Y, \mathcal{O}_Y)\right)\right)$$

This can equivalently be formulated as

$$\mathrm{Ann}_R\left(\mathrm{Kos}\,(f) \otimes_R^{\mathbb{L}} R\Gamma(Y, \mathcal{O}_Y)\right)$$

Since one can formulate $R\Gamma(Y, \mathcal{O}_Y)$ as a dg R-algebra.

Theorem 5.3.2 (Epstein-R.G.-MacDonald-Schwede)**.** *For R with (*), KH-closure is independent of choice of resolution and generating set for your ideals and commutes with localization/completion. Further, It is a 'persistent' operation, meaning that if you extend it along a ring map, the extension of the closure is contained in the closure of the extension.*

Here are some other properties:

- If $(\mathrm{R}, \mathfrak{m})$ is equidimensional with $x_1, \ldots, x_d$ a system of parameters, then $\forall 1 \leq k \leq d-1$,
$$(x_1, \ldots, x_k) : x_{k+1} \subset (x_1, \ldots, x_k)^{KH}$$
- For $J = (f_1, \ldots, f_n)$, then if R is a domain, $\overline{J^n} \subset J^{KH}$, a weak Brianson-Skoda style result. It is worth noting that stronger Brianson-Skoda fails, i.e. $\overline{J^{n+k-1}} \not\subset (J^k)^{KH}$.
- $(\mathrm{R}, \mathfrak{m})$ equidimensional has rational singularities if and only if all ideals are KH-closed. It is sufficient to check this at a single ideal generated by a maximal system of parameters.

Lemma 5.3.3. $(J^{KH})^{KH} = J^{KH}$.

5.4 Irena Swanson- Primary decompositions and powers of ideals

At the start we will start with finite/linearly bounded things, but we will branch into a primary decomposition rule before introducing the more general case.

R will always be Noetherian and commutativ, with $I \subset R$ an ideal. In the 20s, Noether proved that primary decompositions exist. In 1956, REes proved that $\forall I$, there is a finite set of valuations that determine $\overline{I^n}$ $\forall n$. In 1961, Rees further proved that $(\mathrm{R}, \mathfrak{m})$ has reduced completion $\iff$ there exist an (or equivalently, $\forall$) $\mathfrak{m}$-primary ideal I for which $\exists c$ such that $\forall n \geq c$, $\overline{I^n} \subset I^{n-c}$.

Briancon-Skoda (1974) Lipman-Sathaye (1984) and Lipman-Teissier (1984) showed that if R is regular and I is generated by d elements, $\overline{I^{n+d}} \subset I^n$ $\forall n$. First proved for R by Rathiff and later for all R-modules by Brodmann in 1976, it is also true that the chain $\bigcup_n \operatorname{Ass}(M/I^nM)$ stabilizes for large n, and indeed, the associated primes in the union can be computed at a sufficently high power. Katz and McAdam (1989) further showed that $\forall I \exists c$ such that $\forall n$ and $\forall$ ideals J, $I^n : J^{nc} = I^n : J^{nc+1}$.

In the late 1980s, ideal theory really began to focus on tight closure, to wit a notable question popped up: Does tight closure commute with localization? This was answered in the negative only in 2010 by Brenner and Monsky, but it was not known at the time. This leads to a related question: $\forall I$ in a ring R of characteristic p, dos there exist a constant c such that $\forall e > 0$, there exists a primary decomposition $I^{[p^e]} = q_{e_1} \cap \ldots q_{e_{n_e}}$ such that $\forall e, i$, $\left(\sqrt{q_{e_i}}\right)^{[p^e]c} \subset q_{e_i}$? If an ideal I of R satisfies this property, we say it has LPD for Frobenius.

If The answer is true, then tight closure of I commutes with localization at principal multiplicatively closed sets. But what about LPD for ordinary powers (i.e. remove the brackets)? In 1992, Swanson proved LPD for ordinary powers of I for when $\dim R/I \leq 1$. Katzman in 1996 constructde an example for which $\bigcup_e \operatorname{Ass}(R/I^{[p^e]})$ is not finite. This was an obstacle, since non-finiteness makes the question much harder.

Swanson and Smith proved in 1997 that LPD for Frobenius holds for the I in the example above. In addition, for R a quotient of a polynomial ring by a monomial ideal, LPD holds for ordinary and Frobenius powers with effective c. Swanson extended this the same year to show LPD for ordinary pwoers, alongside some uniform Artin-Rees style results in the same vain as prior work of Huneke in 1992. Chandler in 1997proved that, for R a polynomial ring with I a homogeneous ideal, $\exists c$ for which $\operatorname{reg}(R/I^n) \leq cn$ for all n. Henzer extended this to restriction rings shortly after.

This sets the stage for a celebrated paper of Ein-Lazarsfeld-Smith in 2001, where R is an affine regular domain over $\mathbb{C}$, with $p \in \operatorname{Spec}(R)$; they showed that $P^{(n \cdot \operatorname{ht} P)} \subset P^n$ $\forall n$.

Hochster and Huneke extended these results to characteristic p via tight closure in 2001, and Ma-Schwede extended this to mixed characteristic using perfectoid algebras in 2018.

But what about the non-linear world? Given positive integers m, d, n, $\exists B(m, d, n)$ such that $\forall$ ideals $I = (f_1, \ldots, f_m) \subset k[x_1, \ldots, x_n]$ with $\deg f_i \leq d \; \forall i$, the following are bounded above by said $B(m, d, n)$:

- Coefficients in ideal contained in I, i.e. for $g = \sum a_i f_i$, $\deg a_i \leq B(m, d, n) + \deg g$.
- The number of associated primes.
- The number of generators of said associated primes.
- The degree of said generators.
- Number of primary components, as well as the number of their generators and the degree of the generators.