Preliminaries

These are notes from the "Notions of Singularity in Different Characteristics" workshop at the Banff International Research Station from October 5 - 10, 2025. This program was organized by Javier Carvajal-Rojas (CIMAT), Jenny Kenkel (Grinnell College), Karl Schwede (University of Utah), Lisa Seccia (University of Neuchâtel), and Matteo Varbaro (University of Genova). You can find additional information by clicking this. These notes were (quite hastily) typed up by me during the talks, and thus are rife with typos and missing citations. All lectures given at this workshop are recorded and available here, if you'd like to watch the speakers while reading this document and catch all my mistakes. I can provide assurance that all errors here were caused by me and not by whoever was giving the lecture. As such,

$$\textbf{reading these notes is allowed} \iff \textbf{you submit any typos/errors to me at vjagat2 (at) uic (.edu).}$$

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0.1 Luis Nunez-Betancourt - The Defect of the F-Pure Threshold

Let $p > 0$ be a prime, and R a Noetherian domain of characteristic p of dimension d. Take $R^{1/p^e} = \{r^{1/p^e} \mid r \in R\}$. We assume that this is a finitely generated R-Module.

Theorem 0.1.1 (Kunz's Theorem). *R is regular if and only if R is a projective R-Module for some (equiv. all) $e > 1$.*

We can weaken this condition to introduce a mild singularity condition on R. Namely, A ring is F-pure if $\exists \varphi : R^{1/p^e} \to R$ such that $\varphi(1) = 1$. This equivalent to saying that R^{1/p^e} admits a splitting, i.e. $R^{1/p^e} \cong R \oplus M$. Examples of such rings are ubiquitious; any Stanley-Riesner ring is F-pure. By this splitting perspective, it is also clear that direct summands of F-pure rings are F-pure. Normal semigroup rings (e.g. normal toric varieties) are F-pure as well. We recall the bracket notation for ideals:

$$I^{[p^e]} = (f^{p^e} \mid f \in I)$$

Using this, we state a criterion for checking for F-purity:

Theorem 0.1.2 (Fedder). *Let $(A, \mathfrak{m})$ be a regular local ring, with $I \subset S$ an ideal and $R := A/I$. Then R is F-pure if and only if $(I^{[p^e]} : I) \not\subset \mathfrak{m}^{[p^e]}$.*

But how do we test for the failure of F-purity? Via the F-pure threshold! The following definition is due to Takagi-Watanabe:

Let $(R, \mathfrak{m})$ be a local F-pure ring and let

$$b_e : + \max\{\mathrm{ord}_{\mathfrak{m}}(f) \mid R^{1/p^e} \cong (f^{1/p^e}) \oplus M\}$$

for M some R-module. We define

$$\mathrm{fpt}(R) := \lim_{e \to \infty} \frac{b_e}{p^e}$$

$\mathrm{fpt}(R) = d = \dim(R)$ if and only if R is regular, and is a rational number much of the time. $\mathrm{fpt}(R) \leq \mathrm{depth}(R)$. But we have a localization problem! Let $R = \frac{\mathbb{F}_7[x,y,z]}{x^3+y^3+z^3}$. Localizing at the maximal deal or the 0 ideal gives $\mathrm{fpt} = 0$, but localizing at any other prime gives us $\mathrm{fpt} = 1$! To change thing, we can define $\mathrm{dfpt}(R) = \dim(R) - \mathrm{fpt}(R)$ for a local ring, the differential F-pure Threshold. Via our previous statements it is clear that

Lemma 0.1.3. $\mathrm{dfpt}(R) = 0$ *if and only if R is regular.*

We justify the "differential" adjective via utilizing differential operators. We recall that in characteristic p, differential operators are R^p linear. Thus, we can filter $\mathcal{D}_R$ as $\mathcal{D}_R = \bigcup_{e \in \mathbb{N}} \mathcal{D}_R^{(e)}$, for $\mathcal{D}_R^{(e)} := \mathrm{Hom}_{R^{p^e}}(R, R)$. We call these the "level e" differential operators. Now define

$$\mathcal{D}^{(n,e)} = \mathcal{D}_R^n \cap \mathcal{D}_R^{(e)}$$

Lemma 0.1.4. *If $(A, \mathfrak{m})$ is regular, $I \subset S$ and $R = S/I$, then*

$$\max\{n \mid (I^{[p^e]} : I) \subset \mathfrak{m}^{[p^e]} + \mathfrak{m}^n\} = \max\{n \mid \mathcal{D}_A^{(n,e)} \cdot (I^{[p^e]} : I) = A)\}$$

Further, defining the quantity above as $\theta_e(I)$,

$$\theta_e(I) = \dim(A_{\mathfrak{m}})(p^-1) - b_e(R) - 1$$

And so,

$$\lim_{e \to \infty} \frac{\theta_e(I)}{p^e} = \dim(A_{\mathfrak{m}}) - \mathrm{fpt}(R_{\mathfrak{m}})$$

This augmentation of the fpt definition rectifies the localization problem discussed earlier.

Theorem 0.1.5. $\mathrm{dfpt}(R_{Q_1}) \leq \mathrm{dfpt}(R_{Q_2})$ *when $Q_1 \subset Q_2$. Further* $\mathrm{dfpt}(-) : \mathrm{Spec}(R) \to \mathbb{R}_{\geq 0}$ *assigning $Q \mapsto \mathrm{dfpt}(R_Q)$ is stringly upper semi-continuous, implying that the set* $\{Q \in \mathrm{Spec}(R) \mid \mathrm{dfpt}(R_Q) \leq a\}$ *is open.*

Finally, as long as $\oplus \omega^{(-n)}$ is finitely generated (e.g. R is $\mathbb{Q}$-Gorenstein) then the set $\{\mathrm{dfpt}(R_Q) \mid Q \in \mathrm{Spec}(R)\}$ *is finite, and further, has a maximum.*

The second part of this theorem lets us define the differential F-pure threshold for a $\mathbb{Q}$-Gorenstein variety X:

$$\mathrm{dfpt}(X) := \max\{\mathrm{dfpt}(\mathcal{O}_{X,x}) \mid x \in X\}$$

With this definition we can also show the following:

Theorem 0.1.6. *If $X \subset \mathbb{P}_k^n$ is an irreducible variety over an algebraically closed field and $\lambda \in \mathbb{R}_{\geq 0}$ such that* $\mathrm{dfpt}(X) < \lambda$, *then* $\mathrm{dfpt}(H \cap X) < \lambda$ *for a general hyperplane section.*

0.2 Anne Fayolle - Centers of Perfectoid Purity

Let R be a Noetherian domain of (for now) positive characteristic $p > 0$. We will further assume that R is F-finite. We will define F-purity slightly differently than in (yet still equivalent to) the last talk: R is F-pure if and only if $R \to R_{\text{perf}} = \varinjlim R^{1/p^e}$ is pure.

We say that R is F-regular if for all $0 \neq r \in R$, there is an $e > 0$ such that $\exists \varphi \in \text{Hom}_R(R_{\text{perf}}, R)$ such that $\varphi(r^{1/p^e}) = 1$. We can also characterize F-regularity via uniformly compatible ideals. Recall: an ideal $u \subset R$ is (uniformly) compatible if $\forall e > 0$, $\varphi \in \text{Hom}_R(R^{1/p^e}, R)$, $\varphi(u^{1/p^e}) \subset u$.

Lemma 0.2.1. *R is F-regular if and only if (0) and (1) are the only compatible ideals. The set of compatible ideals is closed under taking intersection, sums, and associated primes. If R is F-pure, all compatible ideals are radical, and there are finitely many of them. Further, the quotient of an F-pure ring by a compatible ideal remains F-pure.*

This motivates the following definition: If R is F-pure, a prime compatible ideal is called a **center of F-purity**. Here are some examples:

- The test ideal $\tau(R)$ is the smallest nonzero compatible test ideal.This makes sense; $\tau(R) = R$ if and only if R is F-regular, if and only if the smallest nonzero compatible ideal is (1).
- The conductor ideal $\mathfrak{c} = \{r \in R \mid rR^n \subset R\}$ is a compatible ideal.
- If R is F-pure, normal, $\mathbb{Q}$-Gorenstein, and $\mathfrak{p}$ defines a log canonical center, then $\mathfrak{p}$ is uniformly compatible. This provides another avenue to showing that F-regular rings are KLT; indeed the multiplier ideal is also a compatible ideal.

We now define a new ideal in a local ring $(\text{R}, \mathfrak{m})$: We define the **splitting prime** to be the following:

$$\beta(R) = \{r \in R \mid \varphi(r^{1/p^e} \cdot R^{1/p^e}) \subset \mathfrak{m} \text{ for all } e > 0, \varphi \in \text{Hom}_R(R^{1/p^e}, R)\}$$

Aberbach and Enescu proved that $\beta(R) \neq R$ iff R is F-pure, in which case it is the largest compatible ideal.

We will now assume that $(\text{R}, \mathfrak{m}, \mathfrak{K})$ is a Noetherian complete local ring with $\mathfrak{K}$ characteristic $p > 0$. Rings to consider are things like $\mathbb{Z}_p[x_1, \ldots, x_n]/I$ for some I. We have no Frobenius,but we do have an analogue of perfect rings, namely a perfectoid ring. We won't define a perfectoid ring here, but it is basically a "really big" ring with a lot of pth power roots.

Theorem 0.2.2 (Bhatt, Iyengar, Ma)**.** *R is regular if and only if there exists a faithfully flat perfectoid R-algebra B.*

This theorem motivates the following definition: We say that R is **perfectoid pure** if there exists a pure perfectoid R-algebra B. Now we are ready to define centers of perfectoid purity: $u \subset R$ is a (uniformly perfectoid) compatible if for all perfectoid R-algebras

B, $\varphi \in \mathrm{Hom}_R(B,R)$, $\varphi(\sqrt{uB}) \subset u$. If R is perfectoid pure and $\mathfrak{p}$ is a prime compatible ideal, we say that R is a center of perfectoid purity. These centers obey similar properties to their positive characteristic counterparts.

- If R is F-pure, then this definition agrees with the characteristic $p > 0$ notion.
- The set of compatible ideals is closed under taking sums, intersections, and associated primes.
- If R is perfectoid pure, then all compatible ideals are radical and there are finitely many of them.
- A quotient of a perfectoid pure ring by a center of perfectoid purity is still perfectoid pure.

Here are some examples:

- As usual, (0) and (1) are compatible.
- The conductor ideal $\mathfrak{c}$ as defined before.
- The splitting prime, which we will reintroduce in this mixed characteristic case.
- If R is normal Gorenstein, and perfectoid pure, ideals defining log canonical centers are centers of perfectoid purity.
- The multplier ideal.

We define the (first iteration of the) splitting prime below:

$$\beta_1(R) = \{r \in R \mid \varphi(\sqrt{rB}) \subset \mathfrak{m}\ \forall B \text{ perfectoid}, \varphi \in \mathrm{Hom}_R(B,R)\}$$

Then interatively define

$$\beta_{i+1}(R) = \{r \in R \mid \varphi(\sqrt{rB}) \subset \beta_i(R)\ \forall B \text{ perfectoid}, \varphi \in \mathrm{Hom}_R(B,R)\}$$

Using these, we define $\beta(R) := \bigcap_{i>0} \beta_i(R)$. This detects perfectoid purity: $\beta(R) \neq R$ if and only if R is perfectoid pure, in which case it is th largest compatible ideal of R. Just as in the positive characteristic case, $\beta(R)$ detect the mixed characteristic notion of F-regularity, which is BCM regularity (we won't define this today). If R is the completion of a ring of finite typ over a DVR, normal, and Gorenstein, then BCM regularity implies there is no nontrivial compatible ideal. If $\beta_1(R)$ were compatible, then the converse also holds.

0.3 Alessandro De Stefani - Depth Properties of $\mathrm{gr}_{\mathfrak{m}}(R)$

Let (R, $\mathfrak{m}$) be equicharacteristic and complete. As R is complete, we can fix a Cohen representation $T := k[[x_1, \ldots, x_n]] \twoheadrightarrow R$. We can assume that $I \subset \mathfrak{m}_T^2$. We may just refer to $\mathfrak{m}_T$ as $\mathfrak{m}$ going forward. To R we can associate $G := \mathrm{gr}_{\mathfrak{m}}(R) = \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$. Ignoring the degree 0 component, we can also define $G_+ = \bigoplus_{i>0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$. We let $\mathrm{in}(I) := \ker(P := k[x_1, \ldots, x_n] \twoheadrightarrow G)$ be the ***initial ideal*** associated to G. For $0 \neq f \in T, f = \sum_{i \geq v(f)} f_i$ for $v(f)$ the $\mathfrak{m}$-adic order of f. In this sense $\mathrm{in}(f) = f_{v(f)}$ and $\mathrm{in}(I) = \{\mathrm{in}(f) \mid 0 \neq f \in I\}$. For an example, consider the parametric curve

$$R = k[[t^7, t^7, t^{15}]] \cong \frac{k[[x, y, z]]}{(z^2 - x^5, xz - y^3)}$$

In this setting, $G \cong \frac{k[[x,y,z]]}{(z^2, xz, y^3z, y^6)}$. This can be read off the Betti Resolution

$$0 \to P \to P^4 \to P^4 \to P \to G \to 0$$

the depth of G is 0, which is notably less than the depth of R which is 1. For another example where things can get bad, consider

$$R = \frac{k[[x, y, w, t]]}{(x^3 - y^7, x^2y - xt^3 - w^6)}$$

This is a 2 dimensional complete intersection with $\mathrm{in}(I) = (x^3, x^2y, x^2t^3, xt^6, x^2w^6, xy^9 - xw^6t^3, xy^8t^3, y^7z^9)$. There are 8 generators and 12 relations between them. Continuing this process yields the resolution

$$0 \to P \to P^6 \to P^{12} \to P^8 \to G \to 0$$

G is again depth 0; strictly less than R which is depth 2. In general we have that $\mathrm{depth}(G) \leq \mathrm{depth}(R)$, implying that $\mathrm{H}^i_{G_+}(G) = 0 \Rightarrow \mathrm{H}^i_{\mathfrak{m}}(R) = 0$.

Let's work towards something more general. Let $J \subseteq P$ be a homogeneous ideal, with $<$ tracking the monomial order. Further assume that the initial ideal of J with respect to this order, denoted $\mathrm{in}_<(J)$, be monomial.

Theorem 0.3.1 (Conca-Varbaro)**.** *If* $\mathrm{in}_<(J)$ *is square free, then* $\mathrm{depth}(P/J) = \mathrm{depth}(P/\mathrm{in}_<(J))$. *Further,*

$$\dim_k \left(\mathrm{H}^i_{(\underline{x})}(P/J)_j\right) = \dim_k \left(\mathrm{H}^i_{(\underline{x})}(P/\mathrm{in}_<(J))_j\right)$$

If you take $P = k[x_1, \ldots, x_5, y_1, \ldots, y_5]$ and cut out the graph G of a 5 cycle via the binomial edge ideal $J_G = (x_iy_j - x_jy_i \mid \{i, j\} \in E(G))$. Taking the lexiographic order where $x > y$, you get a square free initial ideal $\mathrm{in}_<(J_G)$ with 10 generators. This motivates the following (naive) question: If (R, $\mathfrak{m}$) is local and the associated graded G is reduced, does this imply that $\mathrm{depth}(G) = \mathrm{depth}(R)$? The answer is probably no, but we don't know.

Recall from Luis's talk that, in positive characteristic, square free monomial ideals correspond to F-split monomial ideals.

We now slightly switch gears. We say that a local ring $(A, \mathfrak{m})$ is cohomologically full if $\forall (B, \mathfrak{m}) \twoheadrightarrow (A, \mathfrak{m}) = B/Q$ with B the same characteristic as A, the map on local cohomology $\mathrm{H}^i_{\mathfrak{m}}(B) \twoheadrightarrow \mathrm{H}^i_{\mathfrak{m}}(A)$ is surjective for all i. In this setting one should consider B a "thickening" of A, in the sense that Q as above is a nilpotent ideal. Now consider the case $A = T/I$ where T is n-dimensional and regular. By local duality, being cohomologically full is saying that $\mathrm{Ext}^i(A, T) \hookrightarrow \mathrm{Ext}^i_T(B, T)\ \forall i$. It is sufficient to ask that $\mathrm{Ext}^i(A, T) \hookrightarrow \mathrm{H}^i_I(T)$ for all i. Here are some examples:

- Cohen Macaulay rings are cohomologically full.
- In characteristic $p > 0$, F-split/F-pure rings are cohomologically full. (but F injective rings need not be!)
- In characteristic 0, Du Bois singularities are cohomologically full.

Cohomologically full rings have a key feature. If $(A, \mathfrak{m})$ is local and t is a nonzero divisor, then if $A/(t)$ is cohomologically full, then $\mathrm{H}^i_{\mathfrak{m}}(A) \xrightarrow{\cdot t} \mathrm{H}^i_{\mathfrak{m}}(A)$ is surjective for all i. If $A = T/I$ i complete, local duality says that t acts on $\mathrm{Ext}^i_T(A, T)$ injectivetly, i.e. $\mathrm{Ext}^i(A, T)$ is a flat $k[[t]]$-Module.

Theorem 0.3.2. *If* $(\mathrm{R}, \mathfrak{m})$ *is equicharacteristic and the associated graded G is cohomologically fully, then* $\mathrm{H}^i_{G^+}(G) = 0 \iff \mathrm{H}^i_{\mathfrak{m}}(R) = 0$. *In particular,* $\mathrm{depth}(R) = \mathrm{depth}(G)$.

Let us sketch the proof. Pick $0 \neq f \in T = k[[x_1, \ldots, x_n]]$ and write $f = \sum_{i \geq v(f)} f_i$. Taking $S = T[[t]]$, we can homogenize f by setting $\mathrm{hom}(f) := \sum_{i \geq v(f)} f_i t^{i - v(f)}$. For an ideal $I \subset T$, define $\mathrm{hom}(I) := (\mathrm{hom}(f) \mid f \in I)$. One can check that $\mathrm{hom}(R) := S/\mathrm{hom}(I)$ is a flat $k[[t]]$-Module. We have two reductions from this: inverting t gives $R \widehat{\otimes}_k k[[t]]$ and killing t yields $\mathrm{hom}(R)/(t) = \widehat{G}$. If $\widehat{G}$ is cohomologically full, then $M_i := \mathrm{Ext}^i(\mathrm{hom}(R), S)$ are flat $k[[t]]$-Modules for all i. We can use M_i to represent both of the following:

$$\mathrm{Ext}^i_T(\widehat{G}, T) \cong M_i/(t)M_i$$

$$\mathrm{Ext}^i_T(R, T) \cong \widehat{(M_i)_t}$$

One can then directly show that, via flatness, these modules have the same krull dimension, and further, vanishing of one determines the vanishing of the other. In particular, as a corollary of the proof, we see that if $(\mathrm{R}, \mathfrak{m})$ is equicharacteristic and R, G are eqidimensional and G is cohomologically full, R satisfies (S_r) if and only if G does, for (S_r) Serre's condition as in the criterion for normality.

It is naturally to ask: when is G cohomologically full? At least when G is not Cohen-Macaulay? Well, In characteristic $p > 0$, for $R = T/I$, if $\exists f \in (I^{[p]} : I)$ such that $\mathrm{in}(f) \notin \mathfrak{m}^{[p]}$, then G is F-pure and thus cohomologically full.

0.4 Janet Page - The Geometry of the F-Pure Threshold

The F-Pure Threshold is a measure of singularities, defineable for any ideal in a ring:

$$\text{fpt}(I, R) = \sup\{\lambda \in \mathbb{R}_{\geq 0} \mid (R, I^{\lambda}) \text{ is } F\text{-Pure}\}$$

We caution that, in his earlier talk, Luis studied $\text{fpt}(R) := \text{fpt}(\mathfrak{m}, R)$, whereas we will study the case where $R = S/f := k[x_1, \ldots, x_n]/f$. We will study $\text{fpt}(f, S)$, which is fundamentally an invariant of the polynomial f. In this setting we define

$$\text{fpt}(f) = \sup\left\{\frac{N}{p^e} \,||f^N \notin \mathfrak{m}^{[p^e]}\right\}$$

$\text{fpt}(f)$ measures the failure of F-purity of $V(f)$. Indeed, $0 < \text{fpt}(f) \leq 1$ and $V(f)$ is F-pure if and only if $\text{fpt}(f) = 1$. As $\text{fpt}(f)$ lowers, the singularities become worse. We note that the F-pure Threshold is intimately related to the Log canonical threshold: Indeed due to work of Mustata, Takagi, and Watanabe, The LCT of f, a polynomial in characteristic 0, can be realized as the limit of the fpts of the mod p reductions of f, as $p \to \infty$.

$\text{fpt}(f)$ depends on the degree of f. In fact, for f homogeneous, $\text{fpt}(f) \geq \frac{1}{d}$; this can easily be checked by a degree argument. This bound is attained as $\text{fpt}(x^d) = \frac{1}{d}$,but it turns out that this is (basically) the only way to hit that bound.

Theorem 0.4.1. *Let $f \in k[x_1, \ldots, x_n]$ be homogeneous, reduced, and of degree d. Then* $\text{fpt}(f) \geq \frac{1}{d-1}$*, with equality if and only if $d = p^e + 1$ for some e and $f \in \mathfrak{m}^{[p^e]}$.*

As f is of degree $p^e + 1$, it follows that f must be of the form $\sum x_i^{p^e} L_i$ for L_i a linear form. Further, assuming we are over an algebraically closed field, the only smooth projective hypersurface that is extremal is precisely the Fermat hypersurface $\sum x_i^{p^e+1}$. But what makes this unique? Let's restrict to surfaces: Why is $V(x^{p^e+1} + y^{p^e+1} + z^{p^e+1} + w^{p^e+1})$ so special?

To do this, we will do a diversion into the study of smooth projective surfaces: the classification of the singularities of which depends on the degree. Let $X \subset \mathbb{P}^3$ be a smooth projective surface of degree d. In $d = 1, 2$ you either have a plane or a ruled quadric; these have infinitely many lines and infinite automorphism group. For $d = 3$, it is clasically known that a cubic surface has 27 lines, and the automorphism group is finite. For $d \geq 4$, the surface *generally* contains no lines and has finite automorphism group. When $d \geq 3$, work has been done to bound the number of lines N on the surface X:

- In characteristic 0: $N \leq 11d^2 - 24d$ and $N \leq 11d^2 - 28d + 12$.
- In characteristic 0 or $p > d$, $N \leq 11d^2 - 30d + 18$.
- In any characteristic, a surface has $\leq d^4$ lines due to a Bezout's theorem computation.

In recent work, this has been generalized:

Theorem 0.4.2. *Let $X \subset \mathbb{P}^3_k$ be a smooth surface of degree d. Then $N \leq d^4 - 3d^3 + 3d^2$ with equality if and only if X is of positive characteristic $p > 0$ and is a Fermat hypersurface of degree $p^e + 1$.*

0.5 Shiji Lyu - Approximation of Schemes over Local Rings

Let $A = \varinjlim_{\lambda \in \Lambda} A_\lambda$ Where A_λ are simple objects, like essentially of finite type algebras over a base field. The goal is to understand A via the A_λ. We consider a simple example first. Let $A = k[[T_1, \dots, T_n]]$ be a power series ring. This can be realized as the completion of $k[T_1, \dots, T_n]$ at the maximal ideal at the origin; by Popescu's theorem, this geometrically regular completion can be approximated by a colimit of smooth k-algebras.

Say $X \to \mathrm{Spec}(A)$ is of finite type. Via EGA, $\exists X_\lambda \to \mathrm{Spec}(A_\lambda)$ with morphism $X \to X_\lambda$ that forms a Cartesian square. Further, for a morphism $f : Y \to X$ there exists a morphism $f_\lambda : Y_\lambda \to X_\lambda$ that makes the following diagram commute:

$$\begin{array}{ccccc} Y & \xrightarrow{f} & X & \longrightarrow & \mathrm{Spec}(A) \\ \downarrow & & \downarrow & & \downarrow \\ Y_\lambda & \xrightarrow{f_\lambda} & X_\lambda & \xrightarrow{\text{finite type}} & \mathrm{Spec}(A_\lambda) \end{array}$$

One can verify properties of X by verifying them on X_λ.

Theorem 0.5.1. *If X is reduced (or normal, regular, R_k, S_k, CM, Gorenstein, LCI, etc.) Then X_λ is also reduced (or ...) for large λ. If X is characteristic 0 and X has (KLT,DuBoi, Rational...) singularities,s than so does X_λ for large enough λ.*

This relies on a key technical result; for λ chosen large enough, X_λ is Tor independent with A and $A_{\lambda'}$ (with $\lambda' \geq \lambda$). We also note that, for f proper and $\mathscr{G}$ a coherent sheaf on Y, then for all large λ, $R^i f_* \mathscr{G} = (R^i f_{\lambda *} \mathscr{G}_\lambda) \otimes_{A_\lambda} A$. These techniques can be used to recover Murayama's results on Kodaira vanishing for complete local rings, by approximating them by essentially of finite type algebras over a field. One can also find new results. For instance, every $K_X + \Delta$-negative extremal ray is spanned by a rational curve: this result is known for varieties but can be extended to excellent schemes. One can also show that, for a quasiexcellent $\mathbb{Q}$-scheme X, if X has log canonical singularities, then X has DuBois singularities, again using the theorem to reduce to the case of varieties where these results are already known.

In characteristic 0, for all large λ, $\mathcal{J}(X_\lambda)\mathcal{O}_X = \mathcal{J}(X)$, for $\mathcal{J}$ the multiplier ideal sheaf. Related to this, Yamaguchi showed that, for (A, Δ) a log pair essentially of finite type over k, A admits a BCM algebra B such that $\tau_B(A, \delta) = \mathcal{J}(A, \Delta)$: this theorem extends this to A a complete local ring.

However, not everything can be approximated; take for instance local cohomology. A theorem of Dao and Takagi says that, if R is a regular local ring that is essentially of finite typ over k of char 0, then if $I \subset R$ and $\mathrm{depth}(R/I) \leq 3$, then $\mathrm{H}^c_I(R) = 0 \; \forall c \geq \dim(R) - 2$. One cannot replace R with a complete local regular ring.

0.6 Marta Benozzo - Anti-Iitaka Inequality in characteristic $p > 0$

In birational geometry, one classifies varieties using the positivity of the canonical divisor. For instance, for curves we have the following examples:

- $\mathbb{P}^1$ has $K_{\mathbb{P}^1} < 0$.
- Elliptic curves have $K_E = 0$
- Curves of genus > 2 have $K_C > 0$, via a Riemann-Roch computation.

But what about in higher dimension. Here X will be a smooth projective variety over an algebraically closed field (of arbitrary characteristic, for now). K_X is the divisor associated to the top wedge of the sheaf of 1 forms on X, which because X is smooth, is a line bundle.

Recall that a line bundle on X, along with defining a divisor, also determines an embedding into $\mathbb{P}^n$. $\mathrm{H}^0(X, L^{\otimes m}) = \langle s_0, \dots, s_N \rangle$, in particular, determines an embedding $x \mapsto [s_0(x) : \cdots : s_N(X)]$. We define the Iitaka dimension as follows:

$$\kappa(X; L) = \begin{cases} -\infty & \mathrm{H}^0(X, L^{\otimes m}) = 0 \ \forall m \\ \max_{m>0}\{\dim \varphi_{|mL|}(x)\} & \text{else} \end{cases}$$

For $L = K_X$, this is just the Kodaira dimension. We will focus on the anti-cononical Iitaka dimension (e.g. $\kappa(X, -K_X)$). Recall that $f : X \to Y$ is a fibration if $f_* \mathcal{O}_X = \mathcal{O}_Y$. How does $\kappa(X, -K_X)$ behave under fibrations?

Let $f : X \to Y$ be a fibration such that $X_y := f^{-1}(y)$ is normal for a general $y \in Y$. This is true in characteristic 0 but not always true in characteristic p. There is an easy additivity theorem: $\kappa(X, L) \leq \kappa(X_y, L_{X|_y}) + \dim(Y)$ that holds in any characteristic. There is a notable conjecture that claims the following:

$$\kappa(X, K_X) \geq \kappa(X_y, K_{X_y}) + \kappa(Y, K_Y)$$

There are counter examples to this in positive characteristic, but it is still open (outside of a few known cases) in characteristic 0.

Theorem 0.6.1 (Chang, 2022). *Let $f : X \to Y$ be a fibration over $\mathbb{C}$ such that $\mathbb{B}(-K_X)$ does not dominate Y. Then $\kappa(X, -K_X) \leq \kappa(X_y, -K_{X_y}) + \kappa(Y, -K_Y)$. Here we are defining $Bs(-mK_X) = \{x \in X \mid s(x) = 0 \ \forall s \in \mathrm{H}^0(-mK_X)\}$ and $\mathbb{B}(-K_X) = \bigcap_{m>0} Bs(-mK_X)$.*

The assumption on $\mathbb{B}(-K_X)$ is a necessary assumption, though it is enough to assume that $\mathcal{J}(X_y, 0; \|K_X\||_{X_y}) = \mathcal{O}_{X_y}$ for general y, where this denotes the multiplier ideal sheaf. But what about positive characteristic? The proof of the theorem above utilized Bertini Theorems and the canonical bundle formula, which are not known to work (in this generality) in positive characteristic. $\varphi_{|-mK_Y|}$ may also have non-normal fibers.

Theorem 0.6.2. *$f : X \to Y$ a fibration over an algebraically closed field of characteristic $p > 0$. Let $y \in Y$ be a general point and assume that X_y is normal. Further assume that $Q^0_y(X, \mathcal{O}_X) = k$ (we will define this object shortly, but it is analogous to the base locus condition of th above theorem). Further assume that $\exists m$ such that $(m, p) = 1$ and $Bs(-mK_X)$ does not dominate Y. Then,*

$$\kappa(X, -K_X) \le \kappa(X_y, -K_{X_y}) + \kappa(Y, -K_Y)$$

We will now work towards defining $Q^0_y(X, \mathcal{O}_X)$. Recall that a log pair (X, Δ) (with Δ a $\mathbb{Z}_{(p)}$-Divisor) is globally F-split if $\exists e > 0$ such that $\mathcal{O}_X \to F^e_* \mathcal{O}_X((p^e - 1)\Delta)$ splits. It's worth noting that not all smooth varieties are globally F-split; for instance, an elliptic curve is globally F-split if and only if it is ordinary, whereas a curve of genus $g \ge 2$ is never globally F-split.

For a pair example, consider $(\mathbb{P}^1, (1/2)(P_1 + P_2 + P_3 + P_4))$. Associated to this pair is the unique elliptic curve rammified at $P_1, \dots, P_4$. Thus the pair is globally F-split if and only if the corresponding curve is ordinary.

Theorem 0.6.3 (Schwede, Smith)**.** *If X is Globally F-split, then $\exists \Lambda \sim_{\mathbb{Q}} -K_X$ such that (X, Λ) is Globally F-split.*

Theorem 0.6.4 (Das,Schwede)**.** *Let $f : X \to Y$ with $\Lambda \ge 0$ such that (X_y, Λ_y) is globally F-split and $K_X + \Lambda \sim_{\mathbb{Z}_{(p)}} 0$. Then $\exists B \ge 0$ such that $K_X + \Lambda \sim_{\mathbb{Z}_{(p)}} f^*(K_Y + B)$.*

This motivates the following definition. For $f : X \to Y$ and (X, Δ) a pair such that X_y is normal for general y and $\exists > 0$ such that $\mathrm{H}^0(X, (1 - p^e)(K_X + \Delta)) \ne 0$.This determines the trace map

$$T^e_{\Delta,y} : \mathrm{H}^0(X, (1 - p^e)(K_X + \Delta)|_{X_y}) \subset \mathrm{H}^0(X_y, (1 - p^e)(K_{X_y} + \Delta|_{X_y})) \xrightarrow{T^e_{\Delta,y}} \mathrm{H}^0(X_y, \mathcal{O}_{X_y})$$

We define $S^0_y(X, \Delta) := \bigcap_{e>0} \mathrm{im}(T^e_{\Delta,y})$ and

$$Q^0_y(X, \Delta) := \bigcap_{m>0, D \in |-m(K_X+\Delta)|, e_0>0} \sum_{e \ge e_0} \mathrm{im}(T^e_{\Delta + \frac{1}{p^e-1}D})$$

The condition $S^0_y(X, \mathcal{O}_X) = k$ says that thre exists a section $s \in \mathrm{H}^0((1 - p^e)K_X)$ such that $\Delta = \frac{\mathrm{div}(f)}{p^e-1}$ and $K_X + \Delta \sim_{\mathbb{Q}} 0$ and (X, Λ) is Globally F-split. Thus setting $Q^0_y(X, \mathcal{O}_X) = k$ gives a perterubed version of positivity descent.

0.7 Kenta Sato - Classification of Quasi-F-Splitting Surface Singularities

F-singularities are tied to the singularities that arise in the MMP. In particular, KLT and LC singularities, determined by a log resolution, correspond to strongly F-regular nad F-pure singularities, which arise from Frobenius, respectively. In dimension 2 over $\mathbb{C}$, KLT singularities are equivalent to having quotient singularities, so this is a mild singularity condition.

This correspondence can be made explicit. Indeed, via mod p-reduction techniques, KLT singularities have infinitely many mod p reductions for which the reduction is SFR, and vice versa. If a variety is F-pure for infinitely many mod p- reduction, then it is log canonical, with the converse still a major open question.In a fixed characteristic $p > 0$, say if X is a normal $\mathbb{Q}$-Gorenstein variety over an algebraically closed field, then SFR implies KLT and F-pure, and KLT and F-pure both imply LC.

In 1998, Hara classified surface singularities. In particular, he showd that if $p > 5$, then SFR is equivalent to KLT. However, the LC vs F-pure comparison is more subtle, even in dimension 2.For example, consider the Fermat hypersurface $x^3 + y^3 + z^3$ in $\mathbb{P}^3_{\mathbb{F}_p}$ for $p \neq 3$. This is always log canonical, but is F-pure if and only if $p \equiv 1 \bmod 3$. Our goal is to construct a class of singularities $\mathscr{C}$ such that F-Pure $\Rightarrow \mathscr{C} \Rightarrow$ LC and if dim $= 2, p > 5$, then $\mathscr{C} \equiv$ LC. It turns out one can consider $\mathscr{C} =$ "being quasi-F-split".

To introduce this condition we first must introduce witt rings $W_n(R)$. We will omit the precise definition, but essentially, $W_n(R)$ is a p^n torsion ring that is a p-nilpotent thickening of R. Indeed, they have the same Spec. Witt rings come equipped with a Witt Frobenius $F : W_n(R) \to W_n(R)$ that is a lift of the Frobenius on R. $X = \mathrm{Spec}(R)$ is n-quasi-F-split if there exists a $W_n(R)$-module homomorphism that makes the following diagram commute:

$$\begin{array}{ccc} W_n(R) & \xrightarrow{F} & F_*W_n(R) \\ \downarrow & \swarrow_{\exists} & \\ R & & \end{array}$$

We say that X is quasi-F-split if it is n-quasi-F-split for some n. As $W_1(R) = R$, it is clear that 1-quasi-F-splitting is equivalent to being F-split, and the condition weakens as n grows.One can also define the iterated version: X being quasi-F^e-split is an iterated condition that is harder to satisfy. If X is quasi-F^e-split for any e, then we say X is quasi-F^∞-split.

Theorem 0.7.1. *If X is a normal $\mathbb{Q}$-Gorenstein variety over $k = \overline{k}$, for k a field of characteristic $p > 0$, then if X is quasi-F^∞-split, it has log canonical singularities.*

There are analogous definitions for pairs. For (X, Δ) a normal affine variety over an algebraically closed field of characteristic $p > 0$, and $\Delta \geq 0$ such that $K_X + \Delta$ is $\mathbb{Q}$-Gorenstein, then there also exists a notion of quasi-F-regularity that fits into the diagram below:

$$\begin{array}{ccccc} \text{Strongly-}F\text{-Regular} & \Longrightarrow & \text{quasi-}F\text{-Regular} & \Longrightarrow & \text{KLT} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{(Sharply)-}F\text{-Pure} & \Longrightarrow & \text{quasi-}F^\infty\text{-split} & \Longrightarrow & LC \end{array}$$

The benefit of this classification is that it is independent of positive characteristic. For instance, Hara showed that for surfaces if $p > 5$, then KLT is equivalent to SFR, but if $p \leq 5$ then there exists KLT singularities that are not strongly F-regular. KTTWYY have shown that KLT is equivalent, for any characteristic $p > 0$ surface, to quasi-F-regularity. The author shows that as long as p does not divide the index of $K_X + \Delta$, Log canonicity is equivalent to quasi-F^∞-splitting. Outside of the $\mathbb{Q}$-Gorenstein case, as long as Δ has standard coefficients, quasi-F^∞-splitting is equivalent to quasi-F-splitting. Thus, as long as p does not divide the index, being log canonical is equivalent to being quasi-F-split. If p does divide the index, then we have something stronger; that quasi-F-splitting is equivalent to KLT singularities. thus, in this case, LC singularities that are not KLT are NOT quasi-F-split.

As a corollary, a normal surface singularity X is quasi-F-split if and only if X is KLT or LC and p does not divide the index. Thus one can classify all quasi-F-splitting singularities via the corresponding dual graph.

0.8 Jefferson Baudin - A Grauert-Riemanschneider Vanishing Theorem for Witt Canonical Sheaves

We begin with the statement of Grauert-Riemanschneider Vanishing, or GR-vanishing:

Theorem 0.8.1 (GR). *Let $f : X \to Y$ be a projective birational morphism of complex varieties, with X smooth. Then, $R^i f_* \omega_X = 0$ for all $i > 0$.*

This theorem has many profound consequences. For instance, through a theore of Elkik one can show that KLT singularities are rational, and it can be used to prove the deformation of rational singularities. But what about characteristic $p > 0$? Then GR-Vanishing does not hold; indeed you can consider the action of $\mathbb{F}_p$ on $\mathbb{A}^1$ assigning $x \mapsto x + 1$. This lifts to a corresponding action on $\mathbb{P}^1$, and even on $(\mathbb{P}^1)^3$, where the action acts on each component of the vector. If one takes the groups action quotient then resolve the corresponding singularity to X, the corresponding canonical sheaf does NOT satisfy GR vanishing as stated as above; indeed $R^1 f_* \omega_X \neq 0$. When $p = 2$, this is evan an F-pure singularity, so GR-vanishing fails even in nice settings.

This fails for fairly straightforward, yet somewhat deep reasons. De Rham cohomology, unlike in characteristic 0, tends to be poorly behaved; in characteristic p we are better served looking at a thickening of De Rham Cohomology in mixed characteristic, called crystalline cohomology. Let $K = W(k)[1/p]$ for k a perfect field of characteristic p (for $k = \mathbb{F}_p$, $K = \mathbb{Q}_p$). For each $n \geq 1$, we can define the De Rham Witt complex $W_m\Omega_X^\bullet$ of $W_n \mathcal{O}_X$-Modules, then using this, define $W\Omega_X^\bullet := \varprojlim W_m \Omega_X^\bullet$.

$$R\Gamma(X/W\Omega_X^\bullet) := \text{Crystalline Cohomology} = R\Gamma_{\text{crys}}(X/k)$$

$$R\Gamma(X/W\Omega_X^\bullet)[1/p] := \text{Rigid Cohomology} = R\Gamma_{\text{rig}}(X/k)$$

Now we define $W_m\omega_X$ and $W\omega_X$ be the last piece of $W_m\Omega_X^\bullet$ and $W\Omega_X^\bullet$ respectively. When studying these, now a version of GR-vanishing holds:

Theorem 0.8.2. *Let $f : X \to Y$ be a projective birational morphism over k a perfect field of characteristic 0, with X smooth. Then $\exists e > 0$ such that*

$$p^e R^i f_* W\omega_X = 0$$

Furthermore if $\dim(X) \leq 3$, then $\exists e > 0$ such that, $\forall m \geq 1$,

$$p^e R^i f_* W_m\omega_X = 0$$

One can construct similar results relating a variant of rational singularities. Explicitly, a variety Y is $W\mathcal{O}$-rational (resp. $\mathbb{Q}_p$-rational) singularities if $\forall X \to Y$ a (quasi-)resolution, $\exists e > 0$ such that $p^e R^i f W\mathcal{O}_X = 0$ (resp. $R^i f_* \mathbb{Q}_p = 0$). $\mathbb{Q}_p$ rationality is weaker than $W\mathcal{O}$-rationality. As a corollary of this theorem we have that:

- F-rational implies $\mathbb{Q}_p$-rational.
- F-rational 3-fold singularities are $W\mathcal{O}$-rational and rational up to nilpotents (i.e. F acts on $R^i f_* \mathcal{O}_Y$ nilpotently)
- KLT and CM implies $\mathbb{Q}_p$-rational.

To prove these results, one must utilze the following result:

Theorem 0.8.3. *Let X be a smooth projective variety with $D \geq 0$ a big and semiample divisor. Then $\exists e > 0$ such that, $\forall i < \dim(X)$, $p^e \operatorname{H}^i(X, WI_D) = 0$.*

Proving these theorems relies heavily on Logarithmic Witt-Hodge theory and Artin Vanishing.

0.9 Alessio Caminata - Hilbert-Kunz and F-signature functions of hypersurfaces

Let K be a field of positive characteristic $p > 0$ and $A = k[[x_1, \ldots, x_n]]$. Let $I \subset A$ and take $R = A/I$; while Hilbert Kunz multiplicity and F-signature can be defined more generally, we restrict ourselves to this setting. We define the Hilbert Kunz function as follows:

$$\mathrm{HK}_R(e) = \dim_K(R/\mathfrak{m}^{[p^e]})$$

for $e \in \mathbb{N}$. This was first investigated by Kunz, in pursuit of a characteristic p analogue of the Hilbert-Samuel multiplicity in characteristic 0. Monsky shows that

$$\mathrm{HK}_R(e) = e_{\mathrm{HK}}(R) p^{ed} + \mathcal{O}(p^{(d-1)e})$$

Where this leading coefficient, independent of e, defines the Hilbert-Kunz multiplicity. Related to this is the F-signature: we define the F-signature function to be $\mathrm{FS}_R(e) = \mathrm{freerank}_R(F_*^e R)$, and similarly to the above, Tucker proved that

$$\mathrm{FS}_R(e) = s(R)p^{ed} + \mathcal{O}(p^{(d-1)e})$$

Where this leading coefficient is the F-signature. These have become very useful invariants that are not yet fully understood. For instance:

- $e_{\mathrm{HK}}(R) \iff s(R) = 1 \iff R$ is regular. Higher HK multiplicity, and lower F-signature, correspond to worse singularities.
- $s(R) > 0$ if and only if R is strongly F-regular.
- There is no good general algorithm to compute these, and computations are only known for specific cases.
- If R is regular in codimension 1, then $\exists \beta, \beta' \in \mathbb{R}$ such that $\mathrm{HK}_R(e) = e_{\mathrm{HK}}(R)p^{ed} + \beta p^{e(d-1)} + \mathcal{O}(p^{(d-2)e})$ and, conjecturally, $\mathrm{FS}_R(e) = s(R)p^{ed} + \beta' p^{e(d-1)} + \mathcal{O}(p^{(d-2)e})$. The F-signature statement is known for $\mathbb{Q}$-Gorenstein rings, but not in general. The assumption on regularity is necessary, due to an example of Han and Monsky (in particular, the coordinate ring of the Fermat quartic surface over $\mathbb{F}_5$).

But what about as we take p large? In particular, does there exist limit notions of HK multiplicity $\lim_{p\to\infty} e_{\mathrm{HK}}(R)$ or F-signature $\lim_{p\to\infty} s(R)$? To understand these, we introduce so-called φ-functions of hypersurfaces. To do this, we restrict ourselves to the case of hypersurfaces $R = A/f$.

$$\varphi_{f,p}\left(\frac{a}{p^e}\right) := \frac{1}{p^{ne}} \dim_K \left(\frac{A}{(\mathfrak{m}^{[p^e]}, f^a)}\right)$$

for $a \in \mathbb{N}$, p not dividing a and $0 \le a < p^e$, and n the number of variables in A. These can recover the Hilbert-Kunz and F-signature functions:

$$\mathrm{HF}_{A/f}(e) = p^{ne}\varphi_{f,p}(1/p^e), \qquad \mathrm{FS}_{A/f}(e) = p^{ne}\left(1 - \varphi_{f,p}(1-1/p^e)\right)$$

Further, if $a/p^e > \mathrm{fpt}(f)$, then $\varphi_{f,p}(a/p^e) = 1$. $\varphi_{f,p}$ is a function of a dense subset $[0,1]$, and in fact, can be extended to a function on $[0,1]$ that is continuous, lipshitz, concave, and increasing, due to work of Blickle, Schwede and Tucker. They also show that

$$\varphi_{f,p}(t) = 1 - s(R, f^t)$$

where $s(R, f^t)$ is the F-signature function for pairs, defined as

$$s(R, f^t) := \lim_{e\to\infty} \frac{1}{p^{ne}} \ell_A \left(\frac{A}{(\mathfrak{m}^{[p^e]}, f^{\lceil tp^e \rceil})}\right)$$

Also, taking the right derivative of $\varphi_{f,p}$ and evaluating at 0 recovers $e_{\mathrm{HK}}(R)$, and taking the left derivative and evaluating at 1 yields $s(R)$. Monsky and Teixera use $\varphi_{f,p}$ to deterimine rationality of F-signature and $e_{\mathrm{HK}}(R)$; in particular they are rational if $\varphi_{f,p}$ is so called "p-fractal": any power series in two variables is p-fractal and the sum of p-fractal functions is p-fractal.

The goal of the author's paper is to compute the limit φ function ($\varphi_f := \lim_{p\to\infty} \varphi_{f,p}$) and compute $\varphi_{f,p}$ for f a diagonal hypersurface, and use this to compute the F-signature.

Theorem 0.9.1. *φ_f exists, is a piecewise polynomial function, and is a uniform limit. Morever, the limits of the left and right derivatives of $\varphi_{f,p}$ exist and both converse to $\varphi'(f)$ (except at finitely many points).*

The authors have also computed φ_f in explicit cases:

$$\varphi_{x^2+y^3}(t) = \begin{cases} 2t & 0 \leq t \leq 1/6 \\ -\frac{3}{2}t^3 + \frac{5}{2}t - \frac{1}{2^4} & 1/6 \leq t \leq 5/6 \\ 1 & t \geq 5/6 \end{cases}$$

let $f = x_1^r + \cdots + x_n^r$ be a Fermat hypersurface with $n > r$. For $n = r+1$, it was known that $s(A/f) \leq \frac{1}{2^{r-1}(r-1)!}$ due to work of Watanabe and Yoshida. It turns out, equality holds for $r = 2$, but not $r = 3$:

Theorem 0.9.2. *For $r = 3$,*

$$s(A/f) = \frac{3(p(p-1)(p-2))}{8(3p^3 - p \pm 2)} < 8$$

but, the limit F-signature hits the Watanabe Yoshida bound.