Preliminaries

These notes follow the Methods of Mixed Characteristic Geometry (MMCG) Fall School, held at the Johannes Gutenberg University in Mainz, Germany in October 2024. This workshop was organized by Manuel Blickle, Karl Schwede and Kevin Tucker, with talks involving a wide variety of people intersecting nontrivially but not completely with the prior list. All lectures have been TeXed by Vignesh Jagathese (me!) in chronological order, with each lecture properly attributed. I had missed one lecture Friday morning, but otherwise all notes should be available here. Following an introduction to basic positive characteristic algebraic geometry (1.1, Schwede), one can split the lecture series into 3 primary categories:

(1) An introduction to Prismatic Cohomology (sections 1.2, 2.1, 3.1, 4.3)

(2) The p-adic Riemann Hilbert Correspondence (sections 1.3, 3.1, 3.2, 4.2, 5.2)

(3) Individual talks (sections 2.2, 3.3, 4.1, 5.1)

As they were TeXed live, these notes are undoubtedly filled with errors; these are all to be attributed to me and not the speakers. On this note,

reading these notes is allowed
$$\iff$$
you submit any typos/errors to me at vjagat2 (at) uic (.edu).

In many cases, the speakers above have pre-written a set of companion notes to their talks, which will almost certainly be better than mine. You can fine these at the following URL:

`https://sites.google.com/view/mmcg2024/program`

along with suggested prerequisite reading.

Contents

Chapter 1

Day 1

1.1 Schwede, introduction to postiive characteristic geometry

This talk will mostly serve as background for the later talks.

Let X be a Noetherian Scheme over $\mathbb{F}_p$. Any scheme of positive characteristic has an equipped Frobenius $F : X \to X$, with associated map on structure sheaves $F_* \mathcal{O}_X \leftarrow \mathcal{O}_X$. We will assume here that F is a finite map. If there exists a splitting $\mathcal{O}_X \to F_* \mathcal{O}_X \xrightarrow{\varphi} \mathcal{O}_X$, we say that X is ***F-split***.

We will focus on the affine case, i.e. where $X = \text{Spec}(R)$. When R is F-split, $R \to F_*R$ assigns $r \mapsto r^p$. If there exists a splitting of this, it is clear that R is reduced. Indeed, R is reduced $\iff$ F is injective. In this case, we are allowed (and encouraged, really) to write F_*R as $R^{1/p}$. We view applying Frobenius as equipping R with its p-power roots.

Regular rings are certainly F-split (as a consequence of Kunz's Theorem) but it is rare that regular schemes are globally F-split. In general, a regular scheme X is globally F-split if and only if its cone is F-split. Consider the following examples:

- $\mathbb{P}^1$ is F-split (and smooth)
- E an elliptic curve is F-split iff it is ordinary.
- Higher genus curves (i.e. $g > 1$) are NEVER F-split.

1.1.1 Local Cohomology

Let A be a ring and $J = (f_1, \dots, f_n) \subset A$ finitely generated ideal. Let $Z = V(J)$ and $K \in \mathrm{D}(A)$ some object in the derived category of A-Modules. We define ***local cohomology*** $\mathrm{R}\Gamma_Z(K)$ to be the complex

$$(A \to \prod A_{f_i} \to \prod A_{f_i,f_j} \to \cdots \to A_{f_1,\dots,f_n}) \otimes^{\mathbb{L}} K$$

Applying this functor to $A \to F_*A \to A$ yields the diagram of complexes $R\Gamma_Z(A) \to R\Gamma_Z(F_*A) \to R\Gamma_Z(A)$, yielding a splitting on local cohomology $H^i_Z(A) \to H^i_Z(F_*A) \to H^i_Z(A)$, assigning $[\eta] \mapsto [F_*\eta^p] \mapsto [\eta]$.

Well, why is this useful? Well, suppose we wanted to show that A is CM, i.e. we want to verify that these local cohomology modules vanish. It can then be pertinent to ask, does F annihilate local cohomology modules? Well, it turns out that F-split things need not be CM, take for example any ordinary abelian variety, which has nontrivial cohomology at the origin but is F-split. We can, however, use Frobenius techniques to test for being CM. To do this, we will need a stronger notion of F-splitting.

Let $(A, \mathfrak{m})$ be a Noetherian local ring. A finitely generated A-Module M is CM if for $Z = V(\mathfrak{m})$, $H^i_Z(M) = 0$ $\forall i < \dim A$. We say that A is ***Strongly F-regular*** (or SFR) if $\forall c \in A$ nonzero, $\exists e$ such that the map $A \to F^e_*A$ assigning $a \mapsto F^e_* c a^{p^e}$ splits as an A-module morphism. Now, suppose that $\exists c_i \neq 0$ for which $c_i \cdot H^i_Z(A) = 0$ for $i < \dim(A)$.

Lemma 1.1.1. *If such c_i exist, then R being SFR implies that R is CM.*

Proof. Take the composition

$$H^i_Z(A) \to H^i_Z(F^e_*A) \xrightarrow{F^e_*c_i} H^i_Z(F^e_*A) \to H^i_Z(A)$$

Where, by construction, the middle map is the zero map. Thus the local cohomology has a 0 map splitting, and hence, is 0 identically. □

Thus we just need to show that these c_i exist. First we'll introduce some duality techniques.

1.1.2 Matlis and Local Duality

Let $(A, \mathfrak{m}, \mathfrak{K} := A/\mathfrak{m})$ be a Noetherian local ring. Further, assume that A is $\mathfrak{m}$-adically complete. Let $E = E_A(\mathfrak{K})$ be the injective hull of $\mathfrak{K}$ as an A-Module. As E is injective (by definition), the functor $(-)^\vee := \mathrm{Hom}_A(-, E)$ is exact. Applying this functor gives you the ***Matlis Dual***. It can easily be checked that $M^{\vee\vee} \cong M$. In fact, the siginificantly stronger fact is also true:

Theorem 1.1.2. $(-)^\vee$ *determines a contravariant equivalence of categories between Noetherian and Artinian A-Modules.*

We have a more general notion of duality, utilizing the dualizing complex $\omega^\bullet_A \in {D^b}_{f.g}(A)$, which implies that $\omega^\bullet_A$ has finite injective dimension and $A \cong \mathrm{RHom}_A(\omega^\bullet_A, \omega^\bullet_A)$. For $K \in D(A)$,

$$\mathrm{RHom}_A(K, \omega^\bullet_A)^\wedge \cong \mathrm{RHom}_A(R\Gamma_Z(K), E)$$

Where $(-)^\wedge$ denotes derived $\mathfrak{m}$-adic completion, to be discussed later. If $K \in {D^b}_{f.g}(A)$, then the derived completion is unnecessary. Further, if $K = A[0]$, then $H^i_Z(A)^\vee \cong \mathrm{Ext}^{-i}_A(A, \omega^\bullet_A)$, where the latter term is just $\mathrm{H}^{-i}(\omega_A^\bullet)$. This module is supported precisely where A is not CM; thus it clearly is not supported everywhere for $i < \dim(A)$. Thus, we can find a c_i that annihilates it. Matlis Duality denoting a contravariant equivalence of categories implies that finding an annihilator of $\mathrm{H}_Z^i(A)^\vee$ is equivalent to finding an annihilator of $\mathrm{H}_Z^i(A)^{\vee\vee} \cong \mathrm{H}_Z^i(A)$.

1.1.3 Derived I-adic Completion

Now assume that A is a ring, not necessarily Noetherian. Nevertheless, take $I = (f_1, \dots, f_n) \subset A$ to be some finitely generated ideal. We say that $M \in \mathrm{D}(A)$ is ***derived complete*** if one (equivalently all) of the following hold:

- $T(M, f) := \mathrm{R}\varprojlim(\dots \xrightarrow{f} M \xrightarrow{f} M)$ is zero $\forall f \in I$. Recall that the derived inverse limit is the unique object fitting in the exact triangle $T(M, f) \to \prod_{n\geq 0} M \xrightarrow{\varphi_f} \prod_{n\geq 0} \xrightarrow{+1}$, where $\varphi_f(\dots, m_i, \dots) = (\dots, m_i - f_{i+1}, \dots)$.
- $\mathrm{Ext}_A^n(A_f, M) = 0 \; \forall n \in \mathbb{Z}, f \in I$.

there are far more equivalent definitions; see the stacks project for more. Note that the "$\forall f \in I$" mentions above can be replaced with $\forall f_i \in I$, i.e. there are finitely many checks.

While the category of I-complete A-Modules is pretty bad (i.e. not Abelian), the category of derived I-complete A-Modules is Abelian. We note that being derived complete is a weaker condition than being complete, indeed any derived complete module is complete iff it is I-adically separated. Just as you can complete any module in the classical sense, once can also construct the derived completion of a (derived) R-Module $M \in \mathrm{D}(A)$. There exists a functor $(-)^{\wedge_I}$ assigning M to

$$M^{\wedge_I} = \mathrm{RHom}((A \to \prod A_{f_i} \to \prod A_{f_i, f_j} \to \dots \to A_{f_1, \dots, f_n}), M)$$

Be warned! if N is just an R-Module, $N^\wedge$ still exists in $\mathrm{D}(A)$, it need not be an R-Module! However, $\mathrm{H}^0(N^{\wedge_I})$ is derived complete, and in some sense is the module theoretic derived completion of N. $K \in \mathrm{D}(A)$ is derived complete $\iff$ $\mathrm{H}^i(K)$ are all derived complete. As expected, Local Cohomology (an I-torsion construction) plays well with derived-I-completing.

Theorem 1.1.3. *Let $I \subset A$ be finitely generated and $K \in \mathrm{D}(A)$. Let $Z = V(I)$. Then,*

$$\mathrm{R}\Gamma_Z(K^{\wedge_I}) = \mathrm{R}\Gamma_Z(K)$$

and

$$\mathrm{R}\Gamma_Z(K)^{\wedge_I} = K^{\wedge_I}$$

1.2 Tucker, Prisms and Perfectoid Rings

For now and forever, all rings are commutative with 1 and fix a prime $p > 0$.

1.2.1 δ-rings

A ***δ-ring*** is a pair (A, δ) where A is a commutative ring and $\delta : A \to A$ such that: $\delta(0) = \delta(1) = 0$ and

$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$$

$$\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p}$$

We note that the last term is formal; one can alternatively write it as $\sum_{i=0}^{p-1} \binom{p-1}{i} x^i y^{p-i}$. If (A, δ) is a δ-ring, then we get a ring homomorphism $\varphi : A \to A$ assigning $a \mapsto a^p + p\delta(a)$. Note that if A is p-torsion free, φ determines δ completely. Consider the following examples:

- $\mathbb{Z}$ or $\mathbb{Z}_p$ with $\varphi = \text{Id}$ (where, as these are p-torsion free, φ induxes the definition $\delta(x) = \frac{x - x^p}{p}$). We note that, as expected, $(\mathbb{Z}, \delta)$ for this δ is initial in the category of δ-rings.
- $\mathbb{Z}_p[x_1, \ldots, x_n]$ with $\varphi(x_i) = x_i^p + pg_i(x)$, where g_i can be any polynomial I'd like.For $g_i = 0$ for instance, $\varphi(x_i) = x_i^p$, and this induces a δ map as follows: $\delta(f(\underline{x})) = \frac{f(x_1^p, \ldots, x_n^p) - (f(\underline{x})^p}{p}$.
- (The free δ-ring) $\mathbb{Z}\{x\} := \mathbb{Z}[x_0, x_1, \ldots]$ with $\delta(x_i) = x_{i+1}$.
- $\mathbb{Z}_p[x]/(px, x^p)$ has a unique δ-ring structure (it is forced that $\delta(x) = 0$).
- $W(A)$ for any ring A. There exists a Witt Frobenius $\varphi_W : W(A) \to W(A)$. If R is perfect of characteristic p, then $\varphi(r_0, r_1, \ldots) = (r_0^p, r_1^p, \ldots)$

This gives us a well defined way to talk about perfect rings in arbitrary characteristic; indeed a δ-ring A is ***perfect*** if φ is an isomorphism.

Lemma 1.2.1. *A perfect δ-ring A is p-torsion free. (exercise)*

Lemma 1.2.2. *$A \mapsto A/(p)$ determines an equivalence of categories between the category of p-complete δ-rings and perfect rings of characteristic p. The inverse assignment is $R \mapsto W(R)$.*

Proof. For perfect rings, the cotangent complex $\mathbb{L}_{R/\mathbb{F}_p}$ vanishes, and thus via deformation theory we have an equivalence of categories between the category of perfect rings of characteristic p and p complete p-torsion free rings that are perfect modulo p. The latter category is equivalent to the one we want via the prior lemma/exercise.

□

1.2.2 p-typical Witt Vectors

For any ring A, $W(A) := \{(a_0, a_1, \dots) \mid a_i \in A\}$, with $+, \cdot$ defined by universal polynomials. We want the following assignment to always be a ring map $W(R) \to R$:

$$(\underline{x}) \mapsto x_0^{p^i} + px_1^{p^{i-1}} + \dots + p^i x_i$$

These extraneous terms are referred to as the ghost components; we want to define multiplication in the Witt Ring to force this fact for all i. In practice if $\underline{x} + \underline{y} = \underline{s}$, then $s_0 = x_0 + y_0, s_1 = x_1 + y_1 + \frac{x_0^p + y_0^p - (x_0+y_0)^p}{p}$ and so forth. Similarly if $\underline{x} \cdot \underline{y} = \underline{t}$, then $t_0 = x_0 y_0$ and $t_1 = x_0^p y_1 + y_0^p x_1 + px_1 y_1$. We also have an equipped surjective ring homomorphism $\varepsilon : W(A) \to A$ restricting a witt vector $\underline{a}$ down to its first component a_0. Both $W(A)$ and ε are functorial constructions in A, as is $W_n(R)$ (the n-th truncation). In fact, δ-structures on A are in bijective correspondence with maps $w : A \to W_2(A)$ where $\varepsilon \circ w = \mathrm{Id}$, via the identification $(A, \delta) \mapsto (A \to W_2(A))$ where $A \to W_2(A)$ is defined by $a \mapsto (a, \delta(a))$.

$W(R)$ has some natural equipped maps, namely $V : W(R) \to W(R)$ assigning $(x_0, x_1, \dots) \mapsto (0, x_0, x_1, \dots,)$. This is φ_W linear, and further, $\varphi_W \circ V = V \circ \varphi_W = p$. This implies that $p = (0, 1, 0, \dots)$. In fact when R is perfect, $\ker(\varepsilon) = \mathrm{im}(V) = (p)$, $W(R)$ is p-complete, and $W(R)/(p) \cong R$. We also define $[-] : R \to W(R)$ assigning $r \mapsto (r, 0, \dots)$ to be the ***Tiechmüller lift*** on R. It can be easily seen that $[r] \cdot [s] = [r \cdot s]$, but this is not true for addition. $\varepsilon([r]) = r$, and $[r] \cdot \underline{s} = (rs_0, r^p s_1, r^{p^2} s_2, \dots)$. In general we can always decompose a Witt Vector into a sum of Teichmüller lifts:

$$(r_0, r_1, \dots) = \sum V^n([r_n]) = \sum V^n \varphi_W^n([r_n^{1/p^n}]) = \sum [r_n^{1/p^n}] p^n$$

1.2.3 (Un)Tilting

For A any ring, $A^\flat := \varprojlim A/(p)$ is the tilt, where the inverse limit is taken upon iterated applications of the Frobenius on $A/(p)$. In practice, these are infinite sequences $\underline{a}$ where $a_{n+1}^p \equiv a_n$ modulo p. There exists a natural projection map onto the first component $A^\flat \to A/(p)$. When A is p-complete, $\exists$ a $\sharp$ map $A^\flat \to A$ (or, untilting map) assigning a sequence $\{a_n\}$ to $\lim_{n \to \infty} a_n^{p^n}$. This is a Cauchy sequence in the p-adic topology, and as A is p-complete, this limit will always exist. Further, $\sharp$ is multiplicative.

Further, $\sharp$ determines an isomorphism of multiplicative monoids $\varprojlim_{x \mapsto x^p} A \to A^\flat$. When R is a perfect ring, the sharp construction agrees with the Tiechmüller lift, as $R \cong R^\flat \cong W(R)^\flat \xrightarrow{\sharp} W(R)$. From what we've already constructed, we have maps

$$W(A^\flat) \xrightarrow{\varepsilon} A^\flat \to A/(p) \leftarrow A$$

Via some universal property things, when A is perfect there exists a unique map $\theta : W(A^\flat) \to A$, assigning

$$\theta(a_0, a_1, \dots) = \theta\left(\sum [a_n^{1/p^n}] p^n\right) := a_0^\sharp + (a_1^{1/p})^\sharp p + (a_2^{1/p^2})^\sharp p^2 + \dots$$

that fits into the diagramatic square with the aforementioned maps. When A is p-complete and p-torsion free where $R := A/(p)$ is perfect, it follows that $\theta : W(R) \cong A$ is a canonical isomorphism.

1.2.4 Prisms

A ***prism*** is a pair (A, I) where:

- A is a δ-ring
- $I \subset A$ is a finitely generated ideal which defines a Cartier divisor (or in other words, I is locally principal on $\mathrm{Spec}(A)$)
- A is derived (p, I)-complete.
- $p \in I + (\varphi(I))$.

A prism (A, I) is perfect if and only if it is perfect as a δ-ring. Here are some examples:

- $(\mathbb{Z}_p, p)$ is a perfect prism.
- More generally, $(W(R), p)$ is a perfect prism when R is perfect of characteristic p.
- $(\mathbb{Z}_p[[u]], (p-u))$ where $\varphi(u) = u^p$ is a prism. (One can check that $I + \varphi(I) = (p, u)$)
- If R is perfectoid, $(W(R^\flat), \ker(\theta))$ is a perfect prism.

Theorem 1.2.3. *There exists an equivalence of categories between the category of perfect prisms and perfectoid rings, defined via the assignment $A \mapsto A/I$ with inverse $R \mapsto (W(R^\flat), \ker(\theta))$.*

1.3 Witaszek, Singularities and Splinters

All rings discussed are normal, excellent, and admit a dualizing complex. When discussing rings of positive characteristic, we assume further that R is F-finite.

We say R is a ***splinter*** if any injective finite map $R \to S$ splits as an R-module map. The Direct Summand Conjecture predicts that all regular local Noetherian rings are splinters; this was shown to be true in arbitrary characteristic only recently! In characteristic 0, every normal ring in characteristic 0 is a splinter, and the positive characteristic case was proven a few decades ago. In 2018, Andre proved the mixed characterstic version using p-adic methods.

Question: Can we define an ideal which measures how far a ring is from being a splinter?

Theorem 1.3.1 (Bhatt)**.** $(\mathrm{R}, \mathfrak{m}, \mathfrak{K})$ *a Noetherian local domain of mixed characteristic. Then,* $\mathrm{H}^i_{\mathfrak{m}}(R^+) = 0$ *for* $i < \dim(R)$.

Where

$$R^+ = \bigcup_{\substack{R \subset S \\ \text{finite}}} S$$

is the absolute integral closure of R.

Theorem 1.3.2 (Bhatt-Lurie)**.** *If $X \to \mathrm{Spec}(R)$ is a proper surjective map, then $\mathrm{R}\,\Gamma(X, \mathcal{O}_X{}^+)^\wedge = \widehat{R^+}$, where we are taking the derived p-adic completion.*

In conjuction with the previous result, we see that

$$\mathrm{H}^i_{\mathfrak{m}}(\mathcal{O}_X{}^+) := \mathrm{H}^i\,\mathrm{R}\,\Gamma_{\mathfrak{m}}\,\mathrm{R}\,\Gamma(\mathcal{O}_X{}^+) = \mathrm{H}^i\,\mathrm{R}\,\Gamma_{\mathfrak{m}}(R^+) = 0$$

for $i < \dim(R)$.

Lemma 1.3.3. *All splinters are CM.*

Proof. Say $R \to S$ is finite. then $\mathrm{H}^i_{\mathfrak{m}}(R) \hookrightarrow \mathrm{H}^i_{\mathfrak{m}}(R)$, and in particular, $\mathrm{H}^i_{\mathfrak{m}}(R) \hookrightarrow \mathrm{H}^i_{\mathfrak{m}}(R^+)$. Well, $\mathrm{H}^i_{\mathfrak{m}}(R^+) = 0$ via the previous statment, so the local cohomology modules vanish. $\square$

1.3.1 Primer on Rational Singularities

We'll quickly introduce various singularity types that arise naturally through commutative algebra.To start, we say R is ***+-rational*** $\iff$ the trace map $\mathrm{Tr} : \omega_S \to \omega_R$ is surjective $\forall$ finite extensions $R \to S$. We note that when R is local and +rational, then the map $\mathrm{H}^i_{\mathfrak{m}}(R) \to \mathrm{H}^i_{\mathfrak{m}}(S)$ is injective.

Lemma 1.3.4. *If R is quasi-Gorenstein (i.e. $\omega_R \cong R$) then any +-rational ring is a splinter.*

Proof. $\omega_S \twoheadrightarrow R$, so choose $\omega \in \omega_S$ such that $\omega \mapsto 1$. Then the map $1 \mapsto \omega$ yields the desired splitting. $\square$

In positive characteristic, we say R is ***F-rational*** if R is CM and $\forall 0 \neq c \in R$, $\exists e$ such that $F^e_* \xrightarrow{F^e_* c} F^e_* \omega_R \xrightarrow{\mathrm{Tr}_{F^e}} \omega_R$ is surjective.

Theorem 1.3.5 (Smith)**.** *+-rationality and F-rationality are equivalent.*

Similarly, in characteristic 0 we say that R is ***rational*** if and only if $\mathrm{R}^i\,\pi_*\,\mathcal{O}_X = 0$ for $i > 0$, where $\pi : X \to \mathrm{Spec}(R)$ is a resolution of singularities. Beware though that +-rational is not equvalent to rational, indeed +-rationality is quite weak; every normal is +-rational in characteristic 0.

In any characteristic we can say R is ***alteration rational*** if $\forall$ alterations $g : Y \to \mathrm{Spec}(R)$, the trace map $g_*\omega_Y \to \omega_R$ is surjective.

Theorem 1.3.6. *If $(\mathrm{R}, \mathfrak{m})$ is Noetherian and local, as well as a positive or mixed characteristic ring, then R is +-rational $\iff$ is it alteration rational.*

Table 1.1: Equivalence of Rationality

	arb. char	char p	mixed	char 0
type	Alterational rational	F-rational	$+$-rational	rational
maps	Alterations	Frobenius	finite maps	birational maps

Lemma 1.3.7. *In characteristic* 0*, R is alteration rational $\iff$ it is rational.*

In positive characteristic, the study of finite maps is very much related to the study of Frobenius + rammification. In mixed characteristic, finite maps on an integral model of a characteristic 0 variety detect properties of birational maps. Using the language of Bhatt-Lurie, this can be phrased as follows in mixed characteristic:

$$\left(\varinjlim_{\substack{g:Y\to \mathrm{Spec}(R)\\ \text{alteration}}} \mathrm{R}\, g_* \mathcal{O}_Y\right)^{\wedge} = \left(\varinjlim_{\substack{f:Y\to \mathrm{Spec}(R)\\ \text{finite}}} \mathrm{R}\, f_* \mathcal{O}_Y\right)^{\wedge}$$

Now let $(R, \mathfrak{m}, \mathfrak{K})$ be a local domain of mixed characteristic. Choose $f \in \mathfrak{m}$. Then if $R/(f)$ is $+$-rational, then R is $+$-rational. Further, a characteristic 0 rational singularity on R is equivalent to an F-rationality of $\widehat{R}/(p)$ is for some p.

1.3.2 Vanishing Theorems

Theorem 1.3.8 (Kodaira Vanishing)**.** *Let X be a smooth projective scheme over k, characteristic* 0*. Let $\mathcal{L}$ be an ample line bundle; The $\mathrm{H}^i(X, \omega_X \otimes \mathcal{L}) = 0$ for all $i > 0$.*

Lemma 1.3.9. *If X is as above but of characteristic p, then being globally F-split mplies that Kodaira Vanishing holds.*

By Serre Duality, $\mathrm{H}^i(X, \omega_X \otimes \mathcal{L}) \cong \mathrm{H}^{n-i}(X, \mathcal{L}^{-1})^*$

Theorem 1.3.10 (Bhatt)**.** *Suppose X is regular projective over a mixed characteristic DVR, and $\mathcal{L}$ a big semiample line bundle. Then $\exists$ a finite surjecitve map $f : y \to X$ such that $\mathrm{H}^i(X, \mathcal{L}^{-1}) \xrightarrow{0} \mathrm{H}^i(Y, f^*\mathcal{L}^{-1})$.*

The analogous statement in positive characteristic is easy to show, given the prior tlemma. Indeed, let $f = F^e$ for $e \gg 0$. This proved quite useful; via the 7 author conglomerate one can use this to develop a mixed characteristic MMP in dimension 3.

Chapter 2

Day 2

2.1 Tucker, Perfectoid Rings

Recall that a ring R is ***Perfectoid*** if:

- R is p-complete (classically)
- $\exists \varpi$ such that $\varpi^p = pu$, for $u \in R^\times$.
- The Frobenius endomorphism $F : R/(p) \to R/(p)$ is surjective.
- $\ker(\theta)$ is principal (recall; this is Fontaine's θ-map)

Our third requirement forces θ to be surjective, so $W(R^\flat)/\ker(\theta) \cong R$, where $\ker(\theta)$ is principal. We say that d is a ***distinguished element*** provided that $\delta(d)$ is unit; this occurs when $d \in \ker(\theta)$ and generates it. Thus, for a distinguished element d we see that

$$R \cong \frac{W(R^\flat)}{(d)}$$

From an exercise from yesterday, one can see that d is distinguished $\iff d = [a_0] + p \cdot \widetilde{u}$, where $\widetilde{u} \in W(R^\flat)^\times$. It turns out that this description of R is both a necessary and sufficient condition for being perfectoid.

Lemma 2.1.1. *R is a perfectoid ring $\iff R \cong W(B)/(d)$ where B is perfect, $d = (a_0, a_1, \dots)$ is a distinguished element (i.e. $a_1 \in B^\times$) and B is a_0 complete. In fact, if R can be written like this, it's forced that $B \cong R^\flat$.*

Here are some basic facts about perfectoids:

- If R is characterstic p, then the perfectoid condition is the same thing as perfect.
- If R is perfectoid, then we have an induced map over Frobenius $\underbrace{R/(\varpi) \xrightarrow{F} R/(\varpi^p)}_{\cong} \cong R/(p)$, which characterizes the kernel of F. This implies that $\sqrt{pR} = (\varpi^{1/p^\infty})$.

- If R is p-torsion free, then R is perfectoid $\iff$ R is p-complete, $\exists \varpi$ where $\varpi^p = pu$ for $u \in R^\times$, and $R/\varpi \cong R/(\varpi^p)$, where the latter term is of course $R/(p)$. This effectively says that the prior fact is a sufficient condition for being perfectoid in the p-torsion free case, and thus you don't have to worry about the θ map much.

- Perfectoid rings are reduced.
  - R is characteristic p: Obvious.
  - R is p-torsion free: $x^p = 0$ implies that $x \in (\varpi)$. Thus $x = y\varpi$, so $x^p = y^p\varpi^p = 0$, so $py^p = 0$. As we are p-torsion free, $y^p = 0$, and thus $y \in (\varpi)$ as well, as $x \in (\varpi^2)$. Repeating this argument again and again, we see that $x \in \bigcap(\varpi^n) = \bigcap(p^n)$. As R is p-separated, it follows that $\bigcap(p^n) = 0$, so $x = 0$.
  - General case follows from a gluing argument, though it is fairly nontrivial. One can take a canonical decomposition of a perfectoid ring into a characteristic p component and a p-torsion free component, then conclude by the prior two parts. One should check the perfectoid notes for a more thorough exposition of this, including a discussion of what the canonical decomposition is.

Now here are some examples of Perfectoid rings:

- $\mathbb{Z}_p$ is not perfect (as there is no ϖ), but we can adjoin infinitely many pth roots of p, then p-complete. This yields $\mathbb{Z}_p[p^{1/p^\infty}]^{\wedge_p}$, which is in some sense the easiest example of a perfectoid ring.

- Take a perfect field of characteristic p, so $k = k^p$. As this is perfect of characteristic p, it is immediately perfectoid. However, one can consider a power series ring over the Witt ring $W(k)$; we will need to add infinity many pth roots of both p and all the variables, then p-complete again. This yields the ring

  $$W(k)[[x_2, \dots, x_d]][p^{1/p^\infty}, x_2^{1/p^\infty}, \dots, x_d^{1/p^\infty}]^{\wedge_p}$$

  Which is perfectoid.

- Say R was perfectoid to begin with. Choose $f \in R$ such that it admits a compatible system of p-power roots. Then $(R/(f^{1/p^\infty}))^{\wedge_p}$ is perfectoid.

Recall from last lecture that there is an equivalence of categories between the category of perfect prisms and the category of perfectoid rings. A perfect prism (A, I) yields a perfectoid ring A/I, and a perfectoid ring R yields a perfect prism $(W(R^\flat), \ker(\theta))$. Note that $W(R^\flat)$ is sometimes referred to as $\mathbb{A}_{\text{inf}}(R)$. We will sketch a proof of this now.

Proof sketch. We will focus on showing that a perfectoid ring R induces a perfect prism. We first must show that $W(R^\flat)$ is a δ-ring. Indeed, let d be the distinguished element that generates $\ker(\theta)$. Then $p \in (d, \varphi(d))$. d is a nonzero divisor (see: exercise from yesterday) so (d) gives a Cartier divisor. We then need to show that $W(R^\flat)$ is derived d-complete. Again via an exercise yesterday, we decompose $d = [a_0] + p \cdot \widetilde{u}$, so it is enough toshow that $W(R^\flat)$ is derived $[a_0]$-complete. We can write $W(R^\flat) = R\varprojlim_n W(R^\flat)/(p^n)$ and $W(R^\flat)/(p^n) \cong W_n(R^\flat)$ are each derived $[a_0]$ complete. This condition pushes through the limit and we can conclude; checking the other direction of this categorical equivalence is similar. $\square$

2.1.1 The Direct Summand Theorem

Theorem 2.1.2 (Hochster(1973), Andre(2018)). *Suppose A is a Noetherian regular ring and $A \subset R$ is a module finite extension. Then $A \subset R$ splits as A-modules.*

Hochster conjectured and proved this in the equicharacteristic case, and Andre proved the mixed characteristic case much later using perfectoid techniques. The proof is related to the existence of a ***BCM algebra***, or ***Big Cohen Macaulay Algebra***, i.e. an R-algebra B such that every system of parameters of R is a regular sequence on B.

Theorem 2.1.3 (Hochster-Huneke(1992), Andre (2020)). *Any complete local Noetherian ring admits a BCM algebra.*

This can be made explicit.

Theorem 2.1.4 (Hochster-Huneke(1992), Bhatt(2020)). *Let $(\mathrm{R}, \mathfrak{m})$ be a complete local noetherian ring with $p \in \mathfrak{m}$. Then $(R^+)^{\wedge_p}$ is a perfectoid BCM R-algebra.*

We note the following chain of implications for $(\mathrm{R}, \mathfrak{m})$ Noetherian local and M a Big R-Module:

$$M \text{ is BCM} \Rightarrow \exists \text{ a SOP that is a regular sequence on } M \Rightarrow \mathrm{H}^i_{\mathfrak{m}}(M) = 0 \text{ for } i < \dim(R)$$

Where these statements are equivalent if $M = \widehat{M}$. Here are some easily checkable facts that we will leave as exercises:

- For our first theorem, we can assume that $(A, \mathfrak{m}_A, k = \overline{k}) \subset (\mathrm{R}, \mathfrak{m}, \mathfrak{K})$ where both A and R are complete local domains.
- If $(A, \mathfrak{m}_A)$ is regular local Noetherian and B is a BCM A-algebra, then B is faithfully flat over A (this is sometimes referred to as miracle flatness)
- If $(A, \mathfrak{m}_A)$ is complete and B is faithfully flat as an A-algebra, then $A \to B$ splits.

From these, it follows that our second theorem implies the first. We will thus focus on proving the second theorem, heavily utilizing Andre's flatness lemma, which will honorifically be given the title of theorem:

Theorem 2.1.5 (Andre's Flatness Lemma). *Suppose A is a perfectoid ring and $g \in A$. Then $\exists$ a perfectoid A-algebra A' such that:*

- $\{g^{1/p^e}\}_{e \in \mathbb{Z}_{\geq 0}} \subset A'$.
- *$A/(p^n) \to A'/(p^n)$ is faithfully flat $\forall n$.*

- *Can take A' to be p-torsion free if A is.*

When A is characteristic p, this is obvious. ndeed, perfectoid rings are perfect, and we can just take $A = A'$. The mixed characteristic case is where all the meat is.

Proof sketch. Without loss of generality, let $(R, \mathfrak{m}, \mathfrak{K})$ be a complete local Noetherian domain where $\mathfrak{K} = \overline{\mathfrak{K}}$ is perfect and R is characteristic 0 and of dimension d. Via Cohen Structure Theorem, we can write $R = S/Q$ where S is the regular ring $W(k)[[z_2, \dots, z_n]]$ and $Q \subset S$ is a prime ideal of height $n - d$.

Choose $f_1, \dots, f_c \in Q$ and $x_2, \dots, x_d \in S$ such that:

- Q is minimal over $(f_1, \dots, f_c)$
- $f_1, \dots, f_c, p, x_2, \dots, x_d$ are a system of parameters of the regular ring S.
- Pick $g \notin Q$ with $gQ \subset \sqrt{(f_1, \dots, f_c)}$. Q is minimal over this radical ideal, so we can decompose it assume

$$\sqrt{(f_1, \dots, f_c)} = Q \cap Q_1 \cap \cdots \cap Q_m$$

where $g \in \bigcap Q_i \backslash Q$.

Using this process, we can construct a perfectoid ring

$$S_\infty := S[p^{1/p^\infty}, z_2^{1/p^\infty}, \dots, z_n^{1/p^\infty}]^{\wedge p}$$

that is p-torsion free, perfectoid, and faithfully flat over S. Via Andre's flatness lemma, there exists $S_\infty \to S'_\infty$ where $\{f_1^{1/p^e}, \dots, f_c^{1/p^e}, g^{1/p^e}\} \subset S'_\infty$. S'_∞ is p-torsion free and $S_\infty/(p) \to S'_\infty/(p)$ is faithfully flat. Therefore, $p, f_1, \dots, f_c, x_2, \dots, x_d$ are a regular sequence on S, and hence, on S^∞. After killing p, we see that $f_1, \dots, f_c, x_2, \dots, x_d$ is a regular sequence on S'_∞, via faithful flatness of $S_\infty/(p) \to S'_\infty/(p)$. As S'_∞ is p-torsion free, it follows that $p, f_1^{1/p^e}, \dots, f_c^{1/p^e}, x_2, \dots, f_d$ is a regular sequence on S'_∞. With a bit more work, it follows that $p, x_2, \dots, x_d$ is a regular sequence on

$$T := \left(\frac{S'_\infty}{(f_1^{1/p^\infty}, \dots, f_c^{1/p^\infty})} \right)^{\wedge p}$$

Now define

$$T' := \mathrm{Hom}_T((g^{1/p^\infty}), T)$$

Using the fact that the ideal (g^{1/p^∞}) is self square, we can deduce that T' is a commutative T-algebra, and moreover, it kills Q and is thus a $R = S/Q$ algebra. T and T' are the same up to almost mathematics; i.e. as (g^{1/p^∞}) kills the kernel and cokernel of $T \to T'$, T and T' are (g^{1/p^∞})-almost isomorphic. Thus, $p, x_2, \dots, x_d$ is a (g^{1/p^∞})-almost regular sequence on T'. To conclude, we use Gabber's Trick, or a method to make something that is almost true into something that is true on the nose. Take the mapping

$$T \to \prod_{\mathbb{Z}_{\geq 0}} T' \to \left(\prod_{\mathbb{Z}_{\geq 0}} T'\right)\left[\frac{1}{w}\right] =: B'$$

Where $w = (g, g^{1/p}, g^{1/p^2}, \dots)$. The following checks are all that's left, and will be left as an exercise:

- $p, x_2, \dots, x_d$ is a regular sequence on B'.
- $B'/(p, x_2, \dots, x_d)B' \neq 0$.
- $B := (B')^{\wedge \mathfrak{m}}$ is the desired BCM-algebra.

□

2.2 Bernasconi, Failure of F-stable version of GR Vanishing

2.2.1 A review of Kodaira Vanishing

Let k be a field and X a smooth projective variety over k. Let ω_X be the canonical sheaf. In characteristic 0, we have the Kodaira Vanishing theorem:

Theorem 2.2.1 (Kodaira Vanishing). *For $i > 0$, $\mathrm{H}^i(X, \omega_X \otimes A) = 0$ for any ample line bundle A.*

This has been vital in characteristic 0 to the development of MMP and the study of pluricanonical maps. It is fundamentally a topological theorem over $\mathbb{C}$.

Proof Sketch. Pick $H \in |A|$ smooth. By Serre Duality it is enough to show that $\mathrm{H}^i(X, \mathcal{O}_X(-H)) = 0$ for $i < \dim X$. We check this at the level of the short exact sequence

$$0 \to \mathcal{O}_X(-H) \to \mathcal{O}_X \to \mathcal{O}_H \to 0$$

after taking a long exact sequence in cohomology, vanishing is governed by verifying the isomorphism

$$\mathrm{H}^i(X, \mathcal{O}_X) \cong \mathrm{H}^i(H, \mathcal{O}_H)$$

for $i < \dim X$. Taking the Hodge decomposition of both sides respectively yields

$$\bigoplus_{p+q=i} \mathrm{H}^q(X, \Omega_X^p), \qquad \bigoplus_{p+q=i} \mathrm{H}^q(H, \Omega_H^p)$$

Taking the excision sequence

$$\cdots \to \mathrm{H}^i(X \backslash H, \mathbb{C}) \to \mathrm{H}^i(X, \mathbb{C}) \to \mathrm{H}^i(H, \mathbb{C}) \to \mathrm{H}^{i+1}(X \backslash H, \mathbb{C}) \to \dots$$

It is then enough to show that $\mathrm{H}^i(X\backslash H, \mathbb{C}) = 0$ for $i < \dim X$. Via Poincare duality (utilizing smoothness of X!) this cohomology group is precisely $\mathrm{H}^{2\dim X - i}(X\backslash H, \mathbb{C})$. $X\backslash H$ is affine as H is ample; this cohomology then vanishes by Amolreatti-Frankel-Artin. $\square$

This result famously does not immediately port over the characteristic p in dimension > 1.

Theorem 2.2.2 (Deligne-Illusie)**.** *X is a smooth projective variety over k, where k is perfect of characteristic p. If X admits a lifting to $W_2(k)$ and $p \geq \dim(X)$, then Kodaira Vanishing holds on X.*

Further, Kodaira vanishing is true up to Frobenius, in some sense; for an ample line bundle A, $\exists e \gg 0$ such that $\mathrm{H}^i(X, A^{-1}) \xrightarrow{F^e} \mathrm{H}^i(X, F^e_* A^{-1}) = \mathrm{H}^i(X, A^{-p^e}) = 0$, via Serre duality/vanishing. It follows then that Kodaira Vanishing holds when X is a smooth, projective, globally F-split variety. For many applications, however, Kodaira Vanishing is not enough.

Theorem 2.2.3 (Kawamata-Viehweg Vanishing)**.** *Let X be a smooth projective variety over $\mathbb{C}$. If L is a big semi-ample line bundle on X, then $\mathrm{H}^i(X, \omega_X \otimes L) = 0$ for $i > 0$.*

Theorem 2.2.4 (Hochster-Huneke, Bhatt)**.** *For X a smooth projective variety over a characteristic p field, For L big and semi-ample, $\exists \pi : Y \to X$ a finite morphism such that*

$$\pi^* : \mathrm{H}^i(X, L^{-1}) \xrightarrow{0} \mathrm{H}^i(Y, \pi^* L^{-1})$$

for $i < \dim(X)$.

This is effectively saying that Kawamata-Viehweg vanishing holds up to finite cover. Do be aware that π is not necessarily a power of F.

2.2.2 Graurt-Riemenschneider Theorem

In characteristic 0, Kawamata-Viehweg implies the following:

Theorem 2.2.5 (Graurt-Riemenschneider)**.** *Say X is a normal variety with a proper resolution of singularities $f : Y \to X$. Then $\mathrm{R}^i f_* \omega_Y = 0$ for $i > 0$.*

This result turns out to have a beautiful applications. As a result of Elkik, rational singularities deform and KLT singularities are CM and rational. Further, Kovacs/Bhatt have shown that rational singularities are equivalent to derived splinters. However, in characteristic p Graurt-Riemenschneider is true in dimension ≤ 2, but false otherwise.

This motivates the following question: Can we find a weakening (up to the action of Frobenius, or equivalently, up to universal homomorphism) of Graurt-Riemenschneider that is still true in characteristic p?

2.2.3 F-Crystals, Cartier Crystals, and Riemann-Hilbert

An F-***Module*** is a pair (M, τ) where $M \in \mathrm{QCoh}(X)$ and $\tau : M \to F_* M$ is an $\mathcal{O}_X$-Module homomorphism. We say (M, τ) is nilpotent if $\exists m > 0$ such that $\tau^m = 0$. It is locally nilpotent if it is the union of nilpotent F-modules. For instance, $(\mathcal{O}_X, F)$ is an F-module. We will focus on one key example; consider $f : Y \to X$ a proper birational morphism and $\mathrm{R}^i f_* \mathcal{O}_Y \xrightarrow{F_* \mathrm{R}^i f_*} F_* \mathrm{R}^i f_* \mathcal{O}_Y$ the Frobenius on higher direct images. The pair $(\mathrm{R}^i f_* \mathcal{O}_Y, F_* \mathrm{R}^i f_*)$ is an F-Module We then define the category of F-crystals, denoted Crys^F, of the category of F-Modules modulo the locally nilpotent ones (as a Serre quotient category). This is an abelian category.

Theorem 2.2.6 (Riemann-Hilbert Correspondence, Blöckle-Pink, Bhatt-Lurie)**.** *If X is a separated finite type scheme over $\mathbb{F}_p$, then $\exists$ an equivalence of abelian categories between Crys_X^F and the category of constructible sheaves on X_{et} over $\mathbb{F}_p$.*

We will denote this equivalence of categories $\mathrm{Crys}_X^F \cong \mathrm{Sh}^c(X_{et}, \mathbb{F}_p)$, where we assign $[(M, \tau)] \mapsto \ker(\mathrm{Id} - \tau)$. This is referred to as the ***Solution Functor***, or $\mathrm{Sol}(-)$. Applying the Riemann-Hilbert Correspondence to our key exmaple above, we see that

$$\mathrm{Sol}(\mathrm{R}^i f_* \mathcal{O}_Y, F_* F^i f_*) = \mathrm{R}^i f_* \underline{\mathbb{F}_p}, y$$

This suggests that we can define a weaking of rational singularities. Indeed, we say that X is $\mathbb{F}_p$***-rational*** if for any $Y \to X$ a resolution of singularities, $R^i f_* \mathcal{O}_Y = 0$ as an F-crystal for $i > 0$. This condition is equivalent via the Riemann-Hilbert Correspondence to checking that $\mathrm{R}^i f_* \underline{\mathbb{F}_p}, y = 0$. To discuss Graurt-Riemenschneider up to Frobenius, we need a dual version.

(N, φ) is a ***Cartier Module*** if $N \in \mathrm{QCoh}_X$ and $\varphi : F_* N \to N$ is an $\mathcal{O}_X$-Module homomorphism. As before, we can define nilpotent and locally nilpotent Cartier Modules, and similarly define the category of ***Cartier Crystals*** as the category of all Coherent Cartier Modules moduli the locally nilpotent ones. For an example, consider the Cartier operator on ω_X; applying Grothendieck duality to $F : \mathcal{O}_X \to F_* \mathcal{O}_X$ yields $C : F_* \omega_X \to \omega_X$, so (ω_X, C) is a Cartier Module. When X is a cone singularity, then $R_*^f \omega_Y = 0$ as a Cartier Crystal.

Theorem 2.2.7 (BBK)**.** *3-fold KLT singularities are $\mathbb{F}_p$-rational.*

Lemma 2.2.8. *If X has $\mathbb{F}_p$-rational singularities and $f : Y \to X$ is a resolution, then $\mathrm{R}^i f_* \omega_Y = 0$ as a Cartier Crystal $\iff$ $\mathrm{H}^i_{\mathfrak{m}_x}(X, \mathcal{O}_X) = 0$ as an F-crystal for $i < \dim(\mathcal{O}_{X,x})$.*

Thus to find a counterexample to Graurt-Riemenschneider in the Frobenius stable setting, it is sufficient to find a KLT singularity for which the action of Frobenius on local cohomology is NOT nilpotent.

Theorem 2.2.9 (BBK)**.** *For $p \in \{2, 3, 5\}$, $\exists$ a KLT singularity for which Frobenius Stable Graurt-Riemenschneider fails.*

For $p = 2$, there is an explicit example due to Totaro. Consider the action $\mathbb{Z}/2\mathbb{Z} \hookleftarrow \mathbb{G}_m^3$ defined by $(x, y, z) \mapsto \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$, and take $X = \mathbb{G}_m^3/(\mathbb{Z}/2\mathbb{Z})$. $\mathrm{H}^2(X, \mathcal{O}_X)$ has a non-nilpotent F-action. As a corollary of this, Bhatt vaishing really needs the map to be separable. In other words, for $p \in \{2, 3, 5\}$ $\exists$ a smooth projective 3-fold Y and L big and semi ample such that $\forall$ purely inseparable maps $Z \to Y$,

$$\mathrm{H}^2(Y, L^{-1}) \to \mathrm{H}^2(Z, \pi^* L^{-1})$$

is nonzero.

2.3 Cai, Introduction to Prismatic Cohomology

We will start off with some baseline assumptions. In particular, All prisms (A, I) are bounded, i.e. A/I has bounded p^∞ torsion. Further, X will be a p-adic smooth formal scheme/$\mathbb{Z}_p$ or A/I.

First consider the affine case $X = \mathrm{Spec}(R)$, where R is smooth over A/I. The ***Prismatic Site***, denoted $(R/A)_{\Delta}$, is a category such that:

- The objects are diagrams $A/I \to R \to B/IB$, descended from morphisms $A \to B$, Where (B, IB) is a prism over (A, I) (note: a map of prisms $(A, I) \to (B, J)$ is a map of δ-rings such that $\delta(I) \subset J$. It turns out that $J \sim IB$, so the general case reduces to the given case).
- Morphisms are maps between the diagrams that pairwise commute.
- This is indeed a site. The map $(B \to B/IB \leftarrow R) \to (C \to C/IC \leftarrow R)$ induces a map of prisms $(B, IB) \to (C, IC)$, which is flat as $B \to C$ is (p, I)-complete and flat. This induces the corresponding Grothendieck topology.
- One can define presheaves $\mathcal{O}_{\Delta}, \mathcal{O}_{\overline{\Delta}}$ where $\mathcal{O}_{\Delta}(R \to B/IB \leftarrow B) := B$ and $\mathcal{O}_{\overline{\Delta}}(R \to B/IB \leftarrow B) = B/IB$.

Lemma 2.3.1. *(Bhatt, Scholze) $\mathcal{O}_{\Delta}, \mathcal{O}_{\overline{\Delta}}$ are sheaves.*

We will define $\Delta_{R/A} := R\Gamma^{\mathrm{Site}}((R/A)_{\Delta}, \mathcal{O}_{\Delta}) \in D(A)$ to be the prismatic complex (which is (p, I) derived complete) and $\overline{\Delta}_{R/A} := R\Gamma^{\mathrm{site}}((R/A)_{\Delta}, \mathcal{O}_{\overline{\Delta}}) \in D(A/I)$ (which is p-complete) to be the Hodge-Tate Complex. These are both Commutative algebra objects by general nonsense (i.e. they are equipped with a derived tensor product satisfying desirable properties of an algebra multiplication operator).

Theorem 2.3.2 (Bhatt, Scholze). $\mathrm{H}^i(\overline{\Delta}_{R/A}) \cong \Omega^i_{R/(A/I)}$ *as Modules.*

In characterstic p, this is (up to Frobenius twist) a Cartier isomorphism. In some sense, Hodge-Tate Cohomology is the ”same thing” as De Rham cohomology of Frobenius twists.

2.3.1 Stack Theoretic Approach

We define the ***Generalized Cartier-Witt Divisor*** on Sas a pair (I, α) where I is an invertible $W(S)$-module for S a p-nilpotent ring, and $\alpha : I \to W(S)$ a $W(S)$-Module map, such that:

- $\text{im}(I \to W(S) \twoheadrightarrow S)$ is nilpotent in S
- the ideal generated b $\text{im}(I \to W(S) \xrightarrow{\delta} W(S))$ is the unit ideal in $W(S)$.

We note that α could be anything (i.e., even 0), motivating the "general" moniker. For example, $W(S) \xrightarrow{V} W(S)$ is a generalized Cartier-Witt Divisor. Consider the functor

$$\mathbb{Z}_p^{\mathbb{\Delta}} : \{p\text{-nilpotent rings}\} \to \text{Groupoids}$$

Assigning S to the groupoid of generalized Cartier-Witt divisors on S. This functor parameterizes all prisms in a sense; more precisely, for every prism (A, I), there exists a map $\rho_A : \text{Spf}(A) \to \mathbb{Z}_p^{\mathbb{\Delta}}$, where Spf denotes the formal spectrum, or the Spectrum under the (p, I)-complete topology. This is referred to as the ***Absolute Primsmaticization***. To show this map exists, it is sufficient to define $\rho_A(S) : \text{Spf}(A)(S) \to \mathbb{Z}_p^{\mathbb{\Delta}}(S)$, where $f : A \to S \in \text{Spf}(A)(S)$ must be assigned. Well, A is a δ-ring, so there exists a unique lift of $A \to S$ to $A \to W(S)$, where the latter map is a map of δ-rings. The associated generalised Witt Divisor is $I \otimes W(S) \xrightarrow{\alpha} W(S)$ on S; it may no longer injective, so we do in fact need to consider generalized prisms.

Similarly to the prior functor, define $\mathbb{Z}_p^{\text{HT}}$: p-nilpotent rings $\to$ Groupoids[1] assigning a ring S to the set of Cartier Witt Divisors on S such that $I \to W(S) \to S$ is the zero map. $\mathbb{Z}_p^{\text{HT}}$ parameterizes all quotients of prisms; for every prism (A, I) $\exists \text{Spf}(A/I) \to \mathbb{Z}_p^{\text{HT}}$, where the corresponding square determined by maps $\text{Spf}(A/J) \to \mathbb{Z}_p^{\text{HT}}$ and $\text{Spf}(A) \to \mathbb{Z}_p^{\mathbb{\Delta}}$.

Similarly, for X a smooth p-adic formal scheme, we an take the ***relative Primsmaticization*** $X^{\mathbb{\Delta}}$ assigning a p-complete ring $S \mapsto$ the pair $(I, \alpha) \in \mathbb{Z}_p^{\mathbb{\Delta}}$ and morphism $\text{Spf}((W(S)/I)^{\mathbb{L}}) \to X$. From this we define X^{HT} to be the pullback of the diagram $X^{\mathbb{\Delta}} \to \mathbb{Z}_p^{\mathbb{\Delta}} \leftarrow \mathbb{Z}_p^{\text{HT}}$. When X is over A/I for some prism A/I, both $(X/A)^{\mathbb{\Delta}}$ and $(X/A)^{\text{HT}}$ remember A and A/I structures respectively.

Theorem 2.3.3 (Drinfeld, Bhatt-Lurie)**.**

$$R\Gamma^{Site}((X/A)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}) \cong R\Gamma((X/A)^{\mathbb{\Delta}}, \mathcal{O})$$

$$R\Gamma^{Site}((X/A)_{\mathbb{\Delta}}, \overline{\mathcal{O}}_{\mathbb{\Delta}}) \cong R\Gamma((X/A)^{HT}, \mathcal{O})$$

Where $\mathcal{O}$ denotes taking the Stack Cohomology.

[1]The HT superscript stands for Hodge-Tate, as in the Hodge-Tate Comparison to be stated later.

Theorem 2.3.4 (Drinfeld, Bhatt-Lurie). *there exists a canonical map $\pi_{HT} : (X/A)^{HT} \to X$ such that $(X/A)^{HT}$ is identified as a $BT_{X/(A/I)}\{1\}^\sharp$ torsor, and this storsor splits if X is affine.*

These results then imply the ***Hodge Tate Comparison***. First,

$$\overline{\mathbb{\Delta}}_{R/A} := \mathrm{R}\,\Gamma^{\mathrm{site}}((X/A)_{\mathbb{\Delta}}, \overline{\mathcal{O}}_{\mathbb{\Delta}}) \cong \mathrm{R}\,\Gamma^{\mathrm{site}}((X/A)^{\mathrm{HT}}, \mathcal{O})$$

via the first theorem, and

$$\mathrm{R}\,\Gamma^{\mathrm{site}}((X/A)^{\mathrm{HT}}, \mathcal{O}) \cong \mathrm{R}\,\Gamma^{\mathrm{site}}\left(\mathrm{BT}_{X/(A/I)}\{1\}^\sharp, \mathcal{O}\right) \otimes^{\mathbb{L}} I^i/I^{i+1}$$

via the second theorem. Using some linear algebra, this reduces to

$$\mathrm{R}\,\Gamma^{\mathrm{site}}\left(BT_{X/(A/I)}\{1\}^\sharp, \mathcal{O}\right) \otimes^{\mathbb{L}} \left\{I^i/I^{i+1}\right\}_\bullet \cong \bigoplus \bigwedge^i \Omega^1_{R/(A/I)}\{i\}[-i]$$

Yielding the Hodge Tate Comparison

$$\overline{\mathbb{\Delta}}_{R/A} \cong \bigoplus \bigwedge^1 \Omega^1_{R/(A/I)}\{i\}[-i]$$

Chapter 3

Day 3

3.1 Witaszek, (singular) Riemann-Hilbert over $\mathbb{C}$

Finsihing up last lecture, let R be an excellent domain which admits a dualizing complex. We would like to define an "alteration rational" variant of the parameter test ideal. To that end, define

$$\tau_{\text{alt}}(\omega_R) := \bigcap_{\substack{g:Y\to \text{Spec}(R)\\ \text{alteration}}} \text{im}\,(\text{Tr} : \omega_Y \to \omega_R)$$

One can easily see that $\tau_{\text{alt}}(\omega_R) = \omega_R \iff R$ is alteration rational. One goal is to prove that alteration test ideals localize:

Lemma 3.1.1. *$\tau_{alt}(\omega_R)\left[\frac{1}{f}\right] = \tau_{alt}(\omega_{R[1/f]})$ for any $0 \leq f \in R$.*

Or more generally, $\tau_{\text{alt}}(\omega_R) = \text{Tr}(\omega_Y)$ for some distinguished alteration $g : Y \to \text{Spec}(R)$. Not everything is known about these test ideals, for instance an open conjecture of Datta and Tucker asks when R is essentially of finite type over an excellent local ring, whether or not the alterational or splinter locus is open.

3.1.1 Characteristic 0 Riemann-Hilbert

Suppose that X is a smooth projective variety over $\mathbb{C}$. X then of course has a Hodge decomposition:

$$\text{H}^k(X^{an}, \mathbb{C}) = \bigoplus_{i+j=k} \text{H}^i(X, \Omega_X^d)$$

A natural question: Can we replace $\mathbb{C}$ be a local system $\mathbb{L}$ (i.e. a locally constant $\mathbb{C}$-sheaf in the analytic topology)? It turns out we can, provided that $\mathbb{L}$ underlies a variation of Hodge structure.

Theorem 3.1.2 (Zucker)**.** *Let C be a Riemann surface. $\text{H}^i(C, j_*\mathbb{V})$ admits a Hodge Structure when $j : C\backslash\{x_0, \ldots, x_n\} \hookrightarrow C$ and $\mathbb{V}$ is a local system underlying a variation of Hodge structure.*

Note that $j_*\mathbb{V}$ underlies a Hodge module.

Theorem 3.1.3 (Riemann-Hilbert)**.** *The set of local systems in the analytic topology correspond to vector bundles with integrable connection.*

If I have a local system $\mathbb{L}$, then $(L \otimes \mathcal{O}_X, \nabla_{\text{can}})$ where ∇_{can} is the canonical integrable connection on $L \otimes \mathcal{O}_X$, namely $1 \otimes d$. Similarly, if I get an integral connection $(\mathcal{E}, \nabla)$, I can take $\mathcal{E}^{\nabla=0}$ to be my local system.

3.1.2 D-Modules

A *D-**Module*** is effectively just an $\mathcal{O}_X$-sheaf with an integrable connection, or in other words, a moxule over hte sheaf of differential operators D_X. We will skip an exact definition for time, but it is not that pertinent. For a notable example, for $X = \mathbb{A}^1$, then $D_X = \mathbb{C}\langle x, \partial_x \rangle$. $\mathcal{E} = \mathbb{C}[x]$ is naturally a D-module, as is $\mathbb{C}[x^{\pm 1}]$. Note that the latter D-module is not a coherent $\mathcal{O}_X$-Module, but it is a coherent D-Module; this is because $\partial_x x^{-1} = -1/x^2$, so we can generate everything from x^{-1}.

This motivates the construction of ***perverse sheaves***, i.e. local systems with singularities. Let the $\text{Cons}(X^{an}, \mathbb{C}) \subset D(X^{an})$ be the collection of ***constructible sheaves***, i.e. those who's closed stratification have local systems on each locally closed subset.They are built from local systems $\mathbb{L}_Z$ where $Z \subset X$ is locally closed under extensions. These are not very well behaved in general (for instance, they do not satisfy Poincare duality) so we can further restrict to the category of perverse sheaves, which is the category of things that look like $\mathbb{L}_Z[\dim Z]$. Funnily enough, perverse sheaves are neither perverse nor sheaves. We let $\text{Perv}_{\text{cons}}(X^{an}, \mathbb{C})$ be the category of these perverse sheaves.

For instance, if $d = \dim X$ for X smooth, then $\mathbb{L}[d]$ is a perverse sheaf. If $i : Z \to X$ is a smooth closed map, then $i_* \mathbb{L}_Z[\dim Z]$ is a perverse sheaf on X. For a more interesting example, consider $\text{R} j_* \mathbb{C}[1]$, where $j : \mathbb{A}^1 \backslash \{0\} \hookrightarrow \mathbb{A}^1$. This is a perverse sheaf. To see why, notice that $\text{H}^1(\text{disk minus origin}, \mathbb{C}) = \mathbb{C}$. This gives us an exact triangle

$$\mathbb{C} \to \text{R} j_* \mathbb{C} \to i_* \mathbb{C}[-1] \xrightarrow{+1}$$

And shifting everything by [1], we see that $\text{R} j_* \mathbb{C}[1]$ is sorrounded by perverse sheaves, and is thus itself a perverse sheaf.

Theorem 3.1.4 (Singular Riemann-Hilbert)**.** *$\mathbb{C}$-perverse sheaves on X^{an} are in direct bijection with regular holonomic D-modules.*

Consider $X = \mathbb{A}^1$ and $j : \mathbb{A}^1 \backslash \{0\} \hookrightarrow \mathbb{A}^1$ the inclusion of a point. $\mathbb{C}$-perverse sheaf $\mathbb{C}[1]$ would be bijectively assigned to $\mathbb{C}[x]$ (this functor in general is thought of as the RH functor, with the inverse considered to be the DR, or DeRham, functor). Similarly if we have an honest complex on the left side, say $\text{R} j_* \mathbb{C}[1]$, this corresponds to the D-module $\mathbb{C}[x^{\pm 1}]$.

3.1.3 Hodge Modules

We will throw another definition in; a ***Hodge Module*** is a tuple $(\mathcal{K}, M, F_\bullet M)$ for $\mathcal{K}$ a perverse $\mathbb{Q}$-sheaf, $M = \mathrm{RH}(\mathcal{K} \otimes \mathbb{C})$ a D-Module, and $F_\bullet M$ a sufficiently good filtration of M, along with various properties that make these good objects to study. In practice, we think of a Hodge Module as a $\mathbb{Q}$-perverse sheaf such that the corresponding D-Module admits a good filtration. For example, $\mathbb{Q}^H := (\mathbb{Q}[d], \mathcal{O}_X, F_\bullet \mathcal{O}_X)$, for $F_i \mathcal{O}_X = \mathcal{O}_X$ for $i \geq 0$ and $F_i \mathcal{O}_X = 0$ otherwise, is in some sense the simplest Hodge module you can think of. Taking j as the inclusion of the point in $\mathbb{A}^1$ again, $(\mathrm{R}\, j_* \mathbb{C}[1], \mathbb{C}[x^{\pm 1}], F_\bullet \mathbb{C}[x^{\pm 1}])$, for $F_i \mathbb{C}[x^{\pm 1}] = x^{-i-1}$ for $i \geq 0$ and $\mathbb{C}[x]$ otherwise, is also a Hodge Module.

We will now define what Jakub defines as his favorite functor. Let $\mathrm{Gr}_\bullet \mathrm{DR}$ be a functor that maps filtered regular holonomic D modules to objects in $D^b \operatorname{Coh}(X)$, where $\mathrm{Gr}_\bullet \mathrm{DR}(M) := \bigoplus_i \mathrm{Gr}_i \mathrm{DR}(M)$. A key fact about this is that it commutes with proper pushforwards, and the t-lift is exact for an appropriate choice of perverse t-structure. For example, again for X smooth, $\mathrm{DR}(\mathcal{O}_X) = \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^d_X = 0$, and

$$F_k \mathrm{DR}(\mathcal{O}_X) = F_k \mathcal{O}_X \to F_{k+1} \mathcal{O}_X \otimes \Omega^1_X \to \cdots \to F_{k+d} \mathcal{O}_X \otimes \Omega^d_X = 0$$

Thus $F_{-d} \mathrm{DR}(\mathcal{O}_X) = \Omega^d_X[0]$ and $F_{-d+1} \mathrm{DR}(\mathcal{O}_X) = \Omega^{d-1}_X \to \Omega^d_X = 0$. Thus,

$$\mathrm{GR}_\bullet \mathrm{DR}(\mathcal{O}_X) = \mathcal{O}_X[d] \oplus \Omega^1_X[d-1] \oplus \cdots \oplus \Omega^d_X[0]$$

Theorem 3.1.5 (Saito's Vanishing). *Let X be a smooth projective variety over $\mathbb{C}$. Let $(M, F_\bullet M)$ be a filtered D-module underlying a Hodge module and $\mathcal{L}$ an ample line bundle. Then*

$$\mathbb{H}^i(X, Gr_\bullet DR(M) \otimes \mathcal{L}) = 0 \qquad i > 0$$

Where $\mathbb{H}$ denote hypercohomology. This recovers basically all vanishing theorems. For $M = \mathcal{O}_X$, this says that

$$\mathbb{H}^i(X, \lfloor [d] \oplus \Omega^1_X \otimes \mathcal{L}[d-1] \oplus \cdots \oplus \Omega^d_X \otimes \mathcal{L}[0]) = 0$$

Which in particular implies that $\mathrm{H}^i(X, \Omega^d_X \otimes \mathcal{L}) = 0$ for $i + j > d$.

3.2 Witaszek, t-Structures and Constructible Sheaves

3.2.1 t-Structures

The notion of a t-structure generalize $\mathrm{D}(X)$, or the derived category of sheaves of abelian groups on X. We will define them now.

Let $\mathcal{D}$ be some triangulated category. A ***t-structure*** is a collection of subcategories $(D^{\leq 0}, D^{\geq 0})$ such that $\mathcal{D}^{\leq i} := D^{\leq 0}[-i]$, where

- $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ for $X \in \mathcal{D}^{\leq 0}, Y \in \mathcal{D}^{\geq 1}$.

- $\mathcal{D}^{\leq 0} \leq \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 0} \geq \mathcal{D}^{\geq 1}$.
- $\forall A \in \mathcal{D}$, $\exists X \to A \to Y \xrightarrow{+1}$ where $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$.

The ***heart*** of the t-structure is denoted $\mathcal{D}^{\heartsuit} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$. It is nontrivial to show, but $\mathcal{D}^{\heartsuit}$ is an abelian subcateegory. Further, $0 \to A \to B \to C \to 0$ is exact in $\mathcal{D}^{\heartsuit}$ if and only if $\exists$ a triangle $A \to B \to C \xrightarrow{+1}$.

Suppose $\mathscr{F} : \mathcal{D}_1 \to \mathcal{D}_2$ is a functor between triangulated categories with t-structures. Then $\mathscr{F}$ is left (right) t-exact if $\mathscr{F}(\mathcal{D}_1^{\geq 0}) \subset \mathcal{D}_2^{\geq 0}$ ($\mathscr{F}(\mathcal{D}_1^{\leq 0}) \subset \mathcal{D}_2^{\leq 0}$). It is t-exact is it is both t-left and t-right exact. We can also define ${}^t\mathscr{F} : \mathcal{D}_1^{\heartsuit} \to \mathcal{D}_2^{\heartsuit}$ to be the associated functor on hearts. Notice that ${}^t\mathscr{F}(M)^{\heartsuit} = \mathrm{H}^0(\mathscr{F}(M))$.

3.2.2 Perverse Coherent Sheaves

Let $K \in \mathrm{D}^{\mathrm{b}}\,\mathrm{Coh}(X)$ for X a variety that is finite type over k, where notably k can be any field. Then

- $K \in^p D^{\leq 0}(X) \iff K_x \in D^{\leq -\dim \overline{\{x\}}}\ \forall x \in X$.
- $K \in^p D^{\geq 0}(X) \iff R\Gamma_x(K) \in D^{\geq -\dim \overline{\{x\}}}\ \forall x \in X$.

This is not symmetric! (Not a typo.) It turns out, there exists a symmetric condition that, a-priori, only must be checked on closed points.

Lemma 3.2.1. *Take K, X as before.*

- $K \in^p D^{\leq 0}(X) \iff R\Gamma_x(K) \in D^{\leq -\dim \overline{\{x\}}}\ \forall x \in X$ *that are closed.*
- $K \in^p D^{\geq 0}(X) \iff R\Gamma_x(K) \in D^{\geq -\dim \overline{\{x\}}}\ \forall x \in X$ *that are closed.*

From here, we will denote ${}^p\mathcal{D}^{\heartsuit}(X)$ as $\mathrm{Perv}\,\mathrm{Coh}(X)$ (suggestive; we will get to this later). $K \in \mathrm{Perv}\,\mathrm{Coh}(X) \iff R\Gamma_x(K) \in \mathrm{Art}(\mathcal{O}_{X,x})\ \forall x \in X$ closed. As a consequence, for any $K \in \mathrm{Coh}(X)$, $K[d]$ is perverse coherent $\iff$ K is CM. In this way, can think of $\mathrm{Perv}\,\mathrm{Coh}(X)$ to be the category of "Cohen Macaulay complexes".

We now define the duality functor $\mathbb{D} : \mathrm{D}^{\mathrm{b}}\,\mathrm{Coh}(X) \to \mathrm{D}^{\mathrm{b}}\,\mathrm{Coh}(X)$ such that $\mathbb{D}(M) := \mathrm{RHom}(M, \omega_X^{\bullet})$. The key here is that $\mathbb{D}$ interchanges the category $\mathrm{Perv}\,\mathrm{Coh}(X)$ with $\mathrm{Coh}(X)$; thus K is a perverse coherent sheaf $\iff$ $\mathbb{D}(K)$ is coherent. Further, $\omega_X^{\bullet}$ is a coherent perverse sheaf, as it is the image of $\mathcal{O}_X$ under $\mathbb{D}$. As a consequence of Matlis Duality, we also have that $K \in pD^{\leq 0}(X) \iff \mathbb{D}(K) \in D^{\geq 0}$, and similarly if we switch the signs. Combining all these with the prior lemma, we see that $\mathrm{D}^{\mathrm{b}}\,\mathrm{Coh}(X) \xrightarrow{R\Gamma_x} \mathrm{D}^{\mathrm{b}}\,\mathrm{Art}(\mathcal{O}_{X,x})$ is t-exact, and the subcategory $\mathrm{Perv}\,\mathrm{Coh}(X) \subset \mathrm{D}^{\mathrm{b}}\,\mathrm{Coh}(X)$ maps to the category of Artinian $\mathcal{O}_{X,x}$-Modules under this assignment. Viewing "perverse cohomology" as the functor ${}^p\mathrm{H}^i(-) := ({}^t\tau^{\geq 0}) \circ ({}^t\tau^{\leq 0})(-[i])$, we see from this that $\mathrm{R}\Gamma_x({}^p\mathrm{H}^i(F)) =$

$H^i_X(F)$. After we complete, as a consequence of Matlis duality we get the equivalence of categories

$$D^b \operatorname{Coh}(\mathcal{O}^\wedge_{X,x}) \cong D^b \operatorname{Art}(\mathcal{O}^\wedge_{X,x})$$

With an equivalance of subcategories

$$\operatorname{Perv}\operatorname{Coh}(\mathcal{O}^\wedge_{X,x}) \cong \operatorname{Art}(\mathcal{O}^\wedge_{X,x})$$

We will now apply all this machinery to the case we are interested in. Consider the finite map $R \to R^+$. R is $+$-rational $\iff$ $H^d_{\mathfrak{m}}(R) \hookrightarrow H^d_{\mathfrak{m}}(R^+)$ for all maximal ideals $\mathfrak{m} \subset R$. This is equivalent to checking the corresponding map on perverse cohomology is injective, i.e. ${}^p H^d(R) \hookrightarrow^p H^d(R^+)$ (WARNING: to avoid coherence issues, we define ${}^p H^d(R^+) := \varinjlim^p_{R \subset S \text{finite}} H^d(S)$; this makes sense as direct limits commute with cohomology in the traditional sense).

3.2.3 Perverse Constructible Sheaves

Suppose X is a topological space, and $M \xrightarrow{j} X$ and $Z \xrightarrow{i} X$ are both inclusions with j open and $U = X \backslash Z$. We have our six functor formalism:

$$j_*, j_! : \operatorname{Ab}(M) \to \operatorname{Ab}(X)$$
$$j^* = j^! L\operatorname{Ab}(X) \to \operatorname{Ab}(M)$$
$$i_* = i_! : \operatorname{Ab}(Z) \to \operatorname{Ab}(X)$$
$$i^*, i^! : \operatorname{Ab}(X) \to \operatorname{Ab}(Z)$$

These are all exact except for $j_*, i^!$, and we have short exact sequences

$$0 \to j_! j^! \mathscr{F} \to sF \to i_* i^* \mathscr{F} \to 0$$
$$0 \to i_* i^! \mathscr{F} \to \mathscr{F} \to j_* j^* \mathscr{F}$$

We will now move to the setting where X is a variety over k of characteristic 0, where $\dim X = d$. We consider the bounded derived category of constructible sheaves over X, viewed in the étale site, over $\mathbb{Z}/n$ for n prime. We label this category ${D^b}_{\text{cons}}(X_{et}, \mathbb{Z}/n)$. We note that all the results that follow will apply in a more classical setting, i.e. ${D^b}_{\text{cons}}(X^{an}, \mathbb{C})$. For a morphism $f : X \to Y$, our 6 functor formalism will give us adjoint pairs $(f_!, f^!)$, (f^*, f_*), and $(\operatorname{RHom}(-,-), - \otimes^{\mathbb{L}} -)$. We also have a duality functor $\mathbb{D} : {D^b}_{\text{cons}}(X, \mathbb{Z}/n) \to {D^b}_{\text{cons}}(X, \mathbb{Z}/n)$, where $\mathbb{D}(M) = \operatorname{RHom}(M, \omega_{X,et})$, where $\omega_{X,et} := \pi^! \mathbb{Z}/n$ for $\pi : X \to \operatorname{Spec}(k)$. When X is smooth, $\omega_{X,et} = \mathbb{Z}/n[2d]$. A key fact here is that $\mathbb{D}(\mathbb{Z}/n[d]) = \mathbb{Z}/n[d]$ if X is smooth.

Let $K \in {D^b}_{\text{cons}}(X, \mathbb{Z}/n)$. We say that $K \in^p D^{\leq 0}(X) \iff K_x \in D^{\leq -\dim \overline{\{x\}}}$ $\forall$ geometric points $x \in X$. Similarly, $K \in^p D^{\geq 0}(X) \iff R\Gamma_x(K) \in D^{\geq -\dim \overline{\{x\}}}$ $\forall$ geometric points $x \in X$. Using our new language, the first condition is also equivalent to $i^* K \in D^{\leq -\dim Z}$ for all closed $Z \subset X$ with embedding i, and the second condition is also equivalent to $i^! K \in D^{\geq \dim Z}$ $\forall$ $Z \subset X$ closed with embedding i.

Lemma 3.2.2. *$\mathbb{D}({}^p D^{\leq 0}(X, \mathbb{Z}/n)) =^p D^{\geq 0}(X, \mathbb{Z}/n)$ and $\mathbb{D}({}^p D^{\geq 0}(X, \mathbb{Z}/n)) =^p D^{\leq 0}(X, \mathbb{Z}, n)$. Further, the image of $Perv_{Cons}(X, \mathbb{Z}, n)$ under $\mathbb{D}$ lands inside itself.*

3.3 Niziol - Hidden Structures on De Rham Cohomology of p-adic Varieties

p is a fixed prime and K a DVR of mixed characteristic $(0, p)$. Let $K \supset \mathcal{O}_K \twoheadrightarrow k$ for k perfect. Let $C = \widehat{\overline{K}}$ and $\mathcal{G}_K := \mathrm{Gal}(\overline{K}, K)$ be the absolute Galois group of K.

3.3.1 Algebraic Varieties

Let X/K be a smooth proper variety. $\mathrm{H}^n_{et}(X_{\mathbb{C}}, \mathbb{Q}_p)$ is the étale cohomology with associated $\mathcal{G}_K$ action, and $\mathrm{H}^n_{dR}(X)$ the is De Rham cohmology. We let $F^\bullet$ be the filtration defined by $F^i\,\mathrm{H}^n_{dR}(X) := \mathrm{im}(\mathrm{H}^n(X, \Omega^{\geq i}))$. Recall that when we view $X/\mathbb{R}$, via the comparison theorem the etale cohomology can be computed via singular cochains, i.e. $\mathrm{H}^n_{et}(X_{\mathbb{C}}, \mathbb{Q}_p) \cong \mathrm{H}^n_B(X(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{Q}_p$. We have an equipped period isomorphism

$$\mathrm{H}^n_{dR}(X) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{H}^n_B(X(\mathbb{C}), \mathbb{C})$$

where the isomorphism is further a Galois equivalence. This has arithmetic analogues:

- Tate('67) formulated the Hodge-Tate conjecture, which conjectures that the comparison map below is an isomorphism.

  $$C_{HT} : \mathbb{C} \otimes_{\mathbb{Q}_p} \mathrm{H}^n_{et}(X_{\mathbb{C}}, \mathbb{Q}_p) \cong \bigoplus_{i=0}^{n} \mathbb{C} \otimes \mathrm{H}^{n-i}(X, \Omega^i)(-i)$$

  For i the cyclotomic character coming from the action $\chi : \mathcal{G}_K \to \mathbb{Z}_p^*$ on the roots of unity. Tate proved this for abelian varieties A in the $n = 1$ case, showing this holds:

  $$\mathrm{H}^1_{et}(A_{\mathbb{C}}, \mathbb{Q}_p) \otimes \mathbb{C} \cong \mathrm{H}^1(A, \mathcal{O}) \otimes_K \mathbb{C} \oplus \mathrm{H}^0(A, \Omega^1) \otimes_K \mathbb{C}(-1)$$

  Only in 2011, due to work of Scholze, was this generalized to rigid Abelian varieties. This approach to p-adic Hodge theory is referred to as 'almost étale Hodge Theory'.

- Fontaine-Messings ('85) For X projective and $p > \dim(X)$ determines a one to one correspondence between Syntomic methods/syntomic cohomologies and p-adic nearby cycles.

- Faltings ('87) developed relative almost étale theory, generalizing the almost étale theory of tate. In this context it is known that C_{HT} is proper, almost proper, and works on local systems. One can use the results of Tate to show that the Hodge-De Rham spectral sequence degenerates in mixed characteristic: $E_1^{p,q} = \mathrm{H}^q(X, \Omega^p) \Rightarrow \mathrm{H}^{p+1}_{dR}(X)$. This was proven using p-adic techniques of Deligne and Illusie.

3.3.2 Fine Structures on DeRham Cohomology

We now will consider what the geometry of these varieties look like after reducing mod p. For instance, let X/K be a semistable reduction (where $X = \mathcal{X}_K$ has a smooth proper map to Spec(K), we take the ring of integers Spec($\mathcal{O}_K$) $\subset$ Spec(K), which determines a semistable map $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$ where $X \hookrightarrow \mathcal{X}$. In this setting $\mathcal{D}_{dR}^{(i)} := \mathrm{H}^i_{dR}(X), F^\bullet$ and $D_{st}^{(i)} := \mathrm{H}^i_{HK}(\mathcal{X}) := \mathrm{H}^i_{crys}(\mathcal{X}_0, W(k))$. With some clever work, one can check that $a_{HK} : D_{st} \otimes_F K \cong \mathrm{H}^i_{dR}(X)$.

In later work, Fontaine and Jannsen introduced abstract ***(Filtered)*** (φ, N)***-Modules***. In this setting let $\mathcal{D}$ be the tuple (D_{st}, D_{dR}, i). such that:

- D_{dR} is a finite rank K vector space with a decreasing filtration $F^i, i \in \mathbb{N}$.
- D_{st} a finite rank $F = \mathrm{Frac}(W(k))$ vector space with bijective semilinear filtration such that $N\varphi = p\varphi N$.
- i an isomorphism $D_{st} \otimes_F K \cong D_{dR}$

Such a D can have the following properties:

- D is ***easily admissible*** if for $t_N(D) = v_p(\det \varphi)$, $\ell_H(D) = \sum_{j \geq 0} j \dim_K(D_{dR}) \subset D_{dR}$, $t_N(D) = \ell_H(D)$ and $t_N(D') \geq \ell_H(D')\ \forall D' \subset D$.
- for $V_{st}(D) := (D_{st} \otimes B_{st})^{\varphi=1, n=0} \cap F^0(D_{dR \otimes B_{dR}})$ a $\mathcal{G}_K$ representation, D is ***admissible*** if $\dim_{\mathbb{Q}_p} V_{st}(D) = \dim_K D_{dR}$.

Theorem 3.3.1 (Tsuji('98))**.** *De Rham Cohomology of Abelian Varieties with semistable reduction is admissible.*

What about any variety X/K? This induces (D_{dR}, D_{st}, ρ) with $\mathcal{G}_K$ action. Due to Rapoport, Dat, Orlik and Datthere exists p-adic period domains, built using only D.

3.3.3 Rigid Analytic Varieties

Suppose X/K is a proper smooth variety; rigic analytic varieties are basically such X where the p-adic hodge theory is similar to the algebraic setup.

Theorem 3.3.2 (Scholze, 2011)**.** *If your $i, n \geq 0$ and if $i > 0$, $\mathrm{H}^i_{et}(X_{\mathbb{C}}, \mathbb{Z}_p) = 0$ for $i > \dim(X)$, and regardless it is finitely generated $\mathbb{Z}_p$-Module. Further Hodge-Tate decomposition exists, and the Hodge de Rham structure degenerates.*

Chapter 4

Day 4

4.1 Gubler, On the non-Archimedian Monge-Ampére Equation

Let X be a complex manifold of dimension n, and $L \to X$ a holomorphic line bundle with smooth metric $\|\cdot\|$, i.e. on every fiber $\pi^{-1}(x)$ the metric varies smoothly. We define the ***first Chern class*** $c_1(L, \|\cdot\|)$ as follows. Pick a local nowhere vanishing $s \in \Gamma(U, L)$. Then $c_1(L, \|\cdot\|) := -\frac{i}{2\pi}\partial\overline{\partial}\log\|s\|^2$, where $\partial = \sum \frac{\partial}{\partial z_j}dz_j$, and the complex conjugate $\overline{\partial}$ defined similarly. We note that locally,

$$c_1(L, \|\cdot\|) = \frac{i}{2}\sum_{j,k=1}^{n} h_{jk}dz_j d\overline{z}_k$$

We say that $\|\cdot\|$ is ***semipositive*** $\iff$ h_{jk} are all positive semidefinite. It remains to show that this is well defined independent of choice of s, but it is; indeed we have a globally defined $(1,1)$ form called the ***first Chern form***.

Theorem 4.1.1. *For X a complex manifold with ample line bundle L with μ a smooth positive volume form on X^{an} (i.e. it is locally a Lebesgue measure) such that $\mu(L) = \deg_L(X)$. Then $\exists$ a smooth semipositive metric $\|\cdot\|$ on L such that $c_1(L, \|\cdot\|)^n = \mu$.*

In this context, we say that the ***Monge-Ampëre Equation holds***. This was conjectured by Calabi in 1954, and he proved uniqueness up to scaling. Existence was shown by Yau in 1978, for which he won a Fields Medal.

4.1.1 Berkovich Spaces

Let R be a complete DVR with field of fractions K and residue field k. If $X = \operatorname{Spec}(A)$ is an afine variety over K, we define the ***analytification*** of X, denoted X^{an}, as the set of multiplicative seminorms $A \to \mathbb{R}_{>0}$ such that the restriction of the norm to K is simply absolute value. We endow this with the coarsest topology such that taking norms are continuous for all elements of A. It can be seen that

$$X(K) \hookrightarrow X^{an}, \qquad x \mapsto |\cdot\|_x \text{ given } |f|_x := |f(x)|$$

Thus the analytificiation can be thought of as a compactification of the K points of X. If X is any variety, then $X^{an} := \bigcup_{U \subset X} U^{an}$ for all $U \subset X$ affine subsets of X. It can be seen that X^{an} is a locally compact and locally pathwise connected space endowed with a canonical sheaf of analytic functions. We call spaces like these ***Berkovich Spaces***, and indeed they can be thought of a refinement of a rigid analytic space.

4.1.2 Semipositive Metrics

Let X be a projective variety over K, $n = \dim(X)$ and L an amply line bundle. We define $(\mathcal{X}, \mathcal{L})$ to be an R-***model*** of $(X, L) \iff \mathcal{X}$ is a projective variety over R, $\mathcal{X} \otimes_R K = X$, and $\mathcal{L}$ is an amply line bundle on X with $\mathcal{L}|_X = L$. One tool useful in studying these will be the ***reduction map*** $\text{red} : \mathbb{P}^N(K) \to \mathbb{P}^N(k)$ assigning $[x_0 : \cdots : x_N] \mapsto [\overline{x_0} : \cdots : \overline{x_N}]$, where we are taking the mod $\mathfrak{m}$ reduction. Of course this can descend down to $X(K) \subset \mathbb{P}^N(K)$ that maps to $\mathcal{X}_s(k) := \mathcal{X} \otimes_R k \subset \mathbb{P}^N(k)$. We can then take the analytification, yielding a setwise map $X^{an} \to \mathcal{X}_s$.

This induces a model metric $\| \cdot \|_{\mathcal{L}}$ on L^{an}, given at $x \in X^{an}$, as follows. Pick a neighborhood $\mathcal{U}$ of $\text{red}(x)$ in $\mathcal{X}$ such that $\mathcal{L}(\mathcal{U}) \cong \mathcal{O}(\mathcal{U})$, i.e. for $s \in \Gamma(\mathcal{U}, \mathcal{L}) \cong \mathcal{O}(\mathcal{U}) \ni \gamma$, $\|s\|_{\mathcal{L}}(x) := |\gamma(x)|$. $\| \cdot \|_{\mathcal{L}}$ is ***a semipositive model*** $\iff |cL$ is nef, i.e. $\deg_{\mathcal{L}}(C) \geq 0$ for all closed curves C in X_s.

We say a continuous metric $\| \cdot \|$ is semipositive $\iff \| \cdot \|$ is a uniform limit of metrics $\| \cdot \|_j$ such that $\| \cdot \|_j^{n_j}$ is a semipositive model metric for some $n_j > 0$.

Theorem 4.1.2 (Berkovich)**.** *For $\| \cdot \|_{\mathcal{L}}$ a semipositive model metric of L and $\mathcal{X}_s$ reduced, Fpr a generic point ρ_v of an irreducible compoent of X_s, $\exists! \, \zeta_Y \in X^{an}$ with $red(\zeta_Y) = \rho_Y$.*

In this setting

$$c_1(L, \| \cdot \|_{\mathcal{L}})^n := \sum_{Y \text{ an irreducible component of } \mathcal{X}} \deg_{\mathcal{L}}(Y) \zeta_Y$$

If $\| \cdot \|$ is a semipositive continuous metric, then $c_1(L, \| \cdot \|)^n$ is the weak limit of $c_1(L \| \cdot \|_j)^n$, i.e.

$$\int_{X^{an}} f c_1(L, \| \cdot \|)^n = \lim_{j \to \infty} \int_{X^{an}} f c_1(L, \| \cdot \|_j)^n$$

4.1.3 non-Archimedian Monge-Ampére Problem

Using this framework, one can naturally ask whether or not Calabi's conjecture holds over K¿ More precisely, assume X is a smooth projective variety over K and L is an ample line bundle. Say μ is a positive measure (supported in a skeleton of X^{an}) with $\mu(X) = \deg_L(X)$. Then does there exist a continuous semipositive metric $\| \cdot \|$ of L such that $c_1(L, \| \cdot \|)^n = \mu$ that is unique up to scaling?

Theorem 4.1.3 (Yuan-Zhang 2017)**.** *Uniqueness holds.*

Theorem 4.1.4 (Boucksom, Farre, Jonsson 2015)**.** *The answer is yes, if K has equicharacteristic 0.*

Theorem 4.1.5 (Gubler, Jell, Künnermann, Martin 2020)**.** *The answer is yes, if K has equicharacteristic p and if a resolution of singularities exists for X.*

Theorem 4.1.6 (Fang, Gubler, Künnermann)**.** *The answer is yes, if K has mixed characteristic $(0, p)$ and if a resolution of singularities exists for X.*

The proof of Boucksom, Farre, and Jonsson uses a so called variational method. Let $\|\cdot\|$ be any continuous metric of L. An ***envelope*** is

$$P(\|\cdot\|) := \inf\{\|\cdot\|' \text{semipositive metrics of L such that } \|\cdot\| \leq \|\cdot\|'\}$$

The three authors showed that if $P(\|\cdot\|)$ is continuous $\forall$ continuous metrics $\|\cdot\|$, then Calabi's conjecture is true (in particular, $P(\|\cdot\|)$ is a semipositive continuous metric). If X is a generic fiber of a regular projective scheme $\mathcal{X}$ over R and $\mathcal{A}, \mathcal{L}$ are models of $\mathcal{L}$ on $\mathcal{X}$ with $\mathcal{A}$ ample, and $\mathfrak{a}_\mathfrak{m}$ a base ideal of $\mathcal{L}^m$ that is verical for $m \gg 0$ as L is ample, then we define $|\mathfrak{a}_\mathfrak{m}|(x)$ to be the maximal value of $|\gamma(x)|$ for any $\gamma \in \mathfrak{a}_\mathfrak{m}$ and $x \in X^{an}$.

It can be shown that $|\mathfrak{a}_\mathfrak{m}|$ is continuous on X^{an}. Further, for $m \gg 0$, $\mathcal{L}^m \otimes \mathfrak{a}_\mathfrak{m}$ is globally generated, and this can be used to show that $|\mathfrak{a}_\mathfrak{m}|^{-1/m} \cdot \|\cdot\|_\mathcal{L}$ is a semipositive model metric. To show continuity of the envelope (as above, a sufficient condition for the Calabi conjecture to hold) it is enough to show that $\|\cdot\|_\mathfrak{m} = |\mathfrak{a}_\mathfrak{m}|^{-1/m}\|\cdot\|_\mathcal{L}$ converges uniformly to $P(\|\cdot\|_\mathcal{L})$. The three authors above showed that this follows if we have coherent vertical ideals $\mathfrak{b}_\mathfrak{m}$ of $\mathcal{O}_\mathcal{X}$ such that:

- $\mathfrak{a}_\mathfrak{m} \subset \mathfrak{b}_\mathfrak{m}$. *(standard)*
- $\mathfrak{b}_{\mathfrak{m}\ell} \subset \mathfrak{b}_\mathfrak{m}^\ell$ $\forall \mathfrak{m}, \ell > 0$. *(subadditivity)*
- $\exists \mathfrak{m}_0 \geq 0$ such that $\mathfrak{b}_\mathfrak{m} \otimes \mathcal{A}^{\mathfrak{m}_0} \otimes \mathcal{L}^\mathfrak{m}$ is globally generated $\forall \mathfrak{m} > 0$. *(global generation)*

This criterion was used in the three papers covering equicharacteristic 0, p, and mixed characteristic:

- Char 0: $\mathfrak{b}_\mathfrak{m} = \mathcal{J}(\mathfrak{a}_\mathfrak{m})_\mathcal{X}$, for the multiplier ideal $\mathcal{J}(-)$.
- Char p: $\mathfrak{b}_\mathfrak{m} = \tau(\mathfrak{a}_\mathfrak{m})$, for the test ideal $\tau(-)$.
- Mixed char: $\mathfrak{b}_\mathfrak{m} = \tau(\mathcal{O}_X, \mathfrak{a}_\mathfrak{m})$, for the mixed characteristic test ideal in the sense of [BMPSTWW].

4.2 Witaszek, Towards p-adic Riemann–Hilbert

When we last spoke, we introduced and provided examples of perverse sheaves. Using this, we will introduce the characteristic p variant of Riemann-Hilbert. Let R be a a Noetherian F-finite domain of positive characteristic for $p \gg 0$. Further assume that

R admits a dualizing complex. We define $\mathrm{Mod}(R[F])$ to be the category of ***Frobenius Modules***, consisting of pairs (M,φ) where $\varphi : M \to F_*M$ is an R-Module morphism. Any (M,φ) induces an assignment $M^{\varphi-\mathrm{Id}} := \{m \in M \mid \varphi(m) = m\}$. For instance for $M = R = k[x_1,\ldots,x_n]$, $M^{F-\mathrm{Id}} = \mathbb{F}_p$.

There exists an exact functor (the "Solution functor") Sol: $\mathrm{Mod}(R[F]) \to \mathrm{Shv}_{et}(\mathrm{Spec}(R),\mathbb{F}_p)$ where

$$\mathrm{Sol}(M,\varphi)(S) :=_{R\to S \text{ étale}} \ker(M\otimes_R S \xrightarrow{\varphi-\mathrm{Id}} M\otimes_R S)$$

Further for any Frobenius Module M we define $M_{\mathrm{perf}} := \varinjlim M \xrightarrow{F} F_*M \xrightarrow{F} \ldots$. We say that (M,φ) is perfect if φ is an isomorphism, and holonomic if M is a perfection. there exists an equivalence of finitely generated modules as follows:

Theorem 4.2.1 (Positive Characteristic Riemann-Hilbert, Bhatt-Lurie)**.**

$$\mathrm{D^b}_{Cons}(\mathrm{Spec}(R)_{et},\mathbb{F}_p) \cong \underbrace{\mathrm{D^b}_{hol}(R[F])}_{t\text{ structure induced from } \mathrm{D^b}_{cons}(R)}$$

are equivalent categories with $\xrightarrow{RH}$ *and* $\xleftarrow{Sol}$ *being the maps between them. Further, RH is t-exact and commutes with proper pushforwards.*

The key question is, What is RH($\mathbb{F}_p$)? It turns out this is just R_{perf}! Indeed, The Reimann Hilbert functor in restricted settings simply recovers the weight 0 part of the $\mathrm{Gr}_\bullet\mathrm{DR}$ functor. We will now construct a strengthening of this functor so we can apply it to non-Noetherian rings (e.g. R^+):

$$\mathrm{D^b}(\mathrm{Spec}(R)_{et},\mathbb{F}_p) \cong D_{alg}(R[F])$$

We will use this correspondence to show that R^+ is CM for $p \gg 0$, recovering the results of Hochster and Huneke.

Theorem 4.2.2. *Y is an integral scheme with $K(Y) = \overline{K(Y)}$. Then,*

- $R\Gamma(Y_{et},\mathbb{F}_p) = \mathbb{F}_p$
- *$\rho : M \to Y$ open implies that $Rj_*\mathbb{F}_p = \mathbb{F}_p$.*

Proof. If $g : V \to Y$ is étale, then $K(V) \supset K(Y)$ is finite, implying that they are equal, so g is an open immersion. Thus, $R\Gamma$ over the étale site agrees with its interpretation over the Zariski site, which via a result of Grothendieck is just $\mathbb{F}_p$. The second statement follows from this one. □

As a corollary, for $(\mathrm{R},\mathfrak{m})$ local and $\pi : \mathrm{Spec}(R^+) \to \mathrm{Spec}(R)$, if $\mathbb{F}_{p,R^+} = \pi_*\mathbb{F}_p$ then $\mathbb{F}_{p,R^+}[d]$ is perverse.

Proof Sketch. One needs to check that

- $\mathbb{F}_{p,R^+}[d] \in^p D^{\leq 0}$

- $\mathbb{F}_{p,R^+}[d] \in^p D^{\geq 0}$

It is sufficient to check that $i^*\mathbb{F}_{p,R^+}[d]$ and $i^!\mathbb{F}_{p,R^+}[d]$ are contained in $D^{\leq -\dim \overline{\{x\}}}$ and $D^{\geq -\dim \overline{\{x\}}}$ respectively. Well, $i^*\mathbb{F}_{p,R^+}[d] = \mathbb{F}_p[d] \in D^{\leq -d}$, and we have the exact triangle

$$i^!\mathbb{F}_{p,R^+}[d] \to \mathbb{F}_{p,R^+}[d] \cong j_*\mathbb{F}_{p,R^+}[d] \xrightarrow{+1}$$

where the isomorphism follows from the theorem. It follows that $i^!\mathbb{F}_{p,R^+}[d] = 0$. □

Theorem 4.2.3 (Goal)**.** *If $(R, \mathfrak{m})$ is local, then R^+ is CM.*

Proof. Applying RH on R and utilizing the fact that RH commutes with proper pushforwards and colimits, we see that

$$\mathrm{RH}(\mathbb{F}_{p,R^+}[d]) = (R^+)^{\mathrm{perfd}}[d] = R^+[d]$$

Well, we commute if and only if we are MCM, so R^+ is CM. □

Do note: R^+ is NOT CM in characteristic 0! This is truly a positive characteristic result.

Theorem 4.2.4. $(R, \mathfrak{m})$ *local and* $X \to \mathrm{Spec}(R)$ *is proper. Then* $\mathrm{R}\Gamma(X^+, \mathcal{O}_X{}^+) = R^+$

4.2.1 Mixed Characteristic Case

Let Y be a variety over k, a field of characteristic 0. We consider the subcategory $\mathrm{D^b}_{\mathrm{cons}}(Y, \mathbb{Z}_p) \subset \mathrm{D}_{p-\mathrm{complete}}(Y, \mathbb{Z})$ of all p-complete K such that $K/p \in \mathrm{D^b}_{\mathrm{cons}}(Y, \mathbb{F}_p)$. We now would like to define the ***perverse t-structure*** on $\mathrm{D^b}_{\mathrm{cons}}(Y, \mathbb{Z}_p)$, satisfying the following properties:

- $K \in^p D^{\geq 0} \iff K/p \in^p D_{\geq 0}(Y, \mathbb{F}_p)$
- $K \in^p D^{\leq 0} \iff K/p \in^p D_{\leq 1}(Y, \mathbb{F}_p)$ and ${}^p\,\mathrm{H}^i(K/p^n)$ is killed by some c that can be chosen independent of n.

The key fact here is that if $K/p \in \mathrm{Perv}_{\mathrm{Cons}}(Y, \mathbb{Z}/p)$, then $K/p \in \mathrm{Perv}(Y, \mathbb{Z}_p)$. For instance when Y is smooth, $\mathbb{Z}_p := R\varinjlim \mathbb{Z}/p^n$ so $\mathbb{Z}_p[d]$ is perverse.

Now we move into mixed characteristic in earnest. Let R be finitely presented and flat over $\mathcal{O}_C$, a DVR of mixed characteristic. We define a ***perverse t-structure*** on $\mathrm{D^b}\,\mathrm{Coh}(\widehat{R})$ via

- $K \in^p D^{\leq 0} \iff R\Gamma_x(K) \in D^{\leq -\dim \overline{\{x\}+1}}\ \forall x \in \mathrm{Spec}(R/p)$.
- $K \in^p D^{\geq 0} \iff R\Gamma_x(K) \in D^{\geq -\dim \overline{\{x\}+1}}\ \forall x \in \mathrm{Spec}(R/p)$.

Now let $\mathcal{O}_C$ be a ring of integers for a perfectoid ring C containing $\mathbb{Q}_p$ (for isntance, $\mathcal{O}_C = \mathbb{Z}_p[p^{1/p^\infty}]^\wedge$)

Theorem 4.2.5 (Bhatt-Lurie)**.** *$\exists$ a RH functor* $\mathrm{D^b}_{cons}(R[1/p], \mathbb{Z}_p) \to \mathrm{D^b}\,\mathrm{Coh}(\widehat{R})$.

This functor relates the constructible picture (existing in characteristic 0, but with $\mathbb{Z}_p$ coefficients) with the coherent picture. It's quite remarkable! We will spend much more time on this in the final lecture on Friday.

4.3 Ma, Direct Summand Theorem via Prismatic Cohomology

The goal for today is to explain Andre's flatness lemma via the prismatic cohomology perspective. Let (A, d) be a bounded prism and $\mathrm{D}(A)$ the derived category of A-modules. Similarly define $\mathrm{D}(A/d)$. We will let $\mathrm{D}_{\mathrm{comp}}(A)$ be the full subcategory of $\mathrm{D}(A)$ spanned by derived (p, d)-complete objects (and similarly define $\mathrm{D}_{\mathrm{comp}}(A/d)$).

Recall that we have a mapping from p-complete smooth A/d algebras to commutative algebraic objects in $\mathrm{D}_{\mathrm{comp}}(A)$ with φ, assigning an object R to $\Delta_{R/A}$. Utilizing the left Kan extension yields a left derived functor between the category of derived p-complete simplicial A/d-algebras to the same commutative algebraic objects as above, assigning $R \mapsto \mathrm{L}\,\Delta_{R/A}$. We often will just denote this as $\Delta_{R/A}$.

Recall that we have the Hodge Tate comparison: for (A, d) a bounded prism (without loss of generality we assume d is a nonzero divisor) and R is a p-complete smooth A/d-algebra, then the comparison states that there is an isomorphism

$$(\Omega^*_{R/(A/d)})^{\wedge p} \cong \mathrm{H}^*(\overline{\overline{\Delta}_{R/A}})\{*\}$$

is an isomorphism of DGAs. In particular, $(\Omega^i_{R/(A/d)})^{\wedge p} \cong \mathrm{H}^i(\overline{\Delta}_{R/A})$.

Lemma 4.3.1. *Suppose that (A, d) is a bounded prism and R is a derived p-complete A/d-algebra. Then $\overline{\Delta}_{R/A} := \Delta_{R/A} \otimes^{\mathbb{L}}_A A/d$ admits an increasing multiplicative exhausted filtration Fil^{HT}_* in $D_{comp}(R)$. so that*

$$gr^{HT}_i(\overline{\Delta}_{R/A}) \cong \left(\mathbb{L}_{R/(A/d)}\right)[-i]^{\wedge}$$

Proof Sketch. If R is a p completion of a polynomial A/d-algebra, then

$$\mathrm{Fil}^{HT}_i := \tau^{\leq i}\overline{\Delta}_{R/A}$$

From this, it follows that by Hodge Tate,

$$\mathrm{gr}^{HT}_i = \mathrm{H}^i(\overline{\Delta}_{R/A})[-i] = \Omega^i_{R/(A/d)}[-i]^{\wedge}$$

Let $\mathscr{C}$ denote the category of diagrams $\{F_0 \to F_1 \to \dots\}$ in $\mathrm{D}_{\mathrm{comp}}(R)$. This yields the functor

$$\{p - \text{complete poly } A/d\text{-alg}\} \to \mathscr{C} \xrightarrow{\text{colim}} \mathrm{D}_{\mathrm{comp}}(R)$$

Left deriving this functor yields the desired result. □

For instance, when (A, d) is a perfect prism (equivlant to A/d being perfectoid) and R is a perfectoid A/d-algebra, then $\overline{\Delta}_{R/A} \cong R$ and $\overline{\Delta}_{R/A} \cong \mathbb{A}_{\mathrm{inf}}(R)$. We can see this via the proposition, coupled with the fact that the derived completion of the cotangent complex is trivial (exercise; $\widehat{\mathbb{L}}_{R/(A/d)} = 0$).

We are now ready to define the ***perfectoidization functor***. Let (A, d) be a perfect prism and let R be a derived p-complete A/d-algebra. Then:

$$\Delta_{R/A,\mathrm{perf}} := \left(\varinjlim(\Delta_{R/A} \xrightarrow{\varphi} \varphi_* \Delta_{R/A} \xrightarrow{\varphi} \dots)\right)^{\wedge}$$

And the ***perfectoidization*** of R is

$$R_{\mathrm{perfd}} := \Delta_{R/A} \otimes^{\mathbb{L}}_A A/d$$

There exists a site-theoretic definition of the perfectoidization. Recall that when we used the site to define the prismati complex (i.e. it is $R\Gamma$ of the perfect site of $R\Gamma$) we can also take the perfect prismatic site. Indeed,

$$\Delta_{R/A,\mathrm{perf}} = \mathrm{R}\Gamma((R/A)^{\mathrm{perf}}_{\Delta}, \mathcal{O}_{\Delta})$$

$$R_{\mathrm{perfd}} = \mathrm{R}\Gamma((R/A)^{\mathrm{perf}}_{\Delta}, \overline{\mathcal{O}}_{\Delta})$$

These of course can be taken as the definition. In fact, the equivalence of these definitions implies that $R_{\mathrm{perfd}} \in D^{\geq 0}$. We will use this to prove Andre's flatness lemma: recall that:

Theorem 4.3.2 (Andre's Flatness Lemma)**.** *Suppose S is a perfectoid ring and $g \in S$. Then $\exists S \to S'$ perfectoid so that*

- *S' is p-complete and faithfully flat over S.*
- *g admits a compatible system of p-power roots in S'.*

Proof Sketch. We note that this is a different proof than that of Andre, who actually proved a slightly weaker statement. The statement would only be true up to almost mathematics. This will be true on the nose.

Recall via Bhatt-Scholze that the category of perfectoid rings is equivalent to the category of perfect prisms. We want to enlarge S to have sufficiently many p-power roots; we claim that

$$S' = \left(S[x^{1/p^\infty}]^\wedge/(x-g)\right)_{\mathrm{perfd}}$$

has the desired properties. It's clear that S' has a compatible system of p power roots, so it is sufficient to show that S' is p-complete and faithfully flat over S (and, S' is perfectoid, though it follows from the prior requirements). We show that $\Delta_{T/\mathbb{A}_{\mathrm{inf}},\mathrm{perf}}$ is (p, d) complete and faithfully flat over $\mathbb{A}_{\mathrm{inf}}$.

$$\begin{array}{ccc} \Delta_{T/\mathbb{A}_{\mathrm{inf}},\mathrm{perf}} = \varinjlim & \left(\Delta_{T/\mathbb{A}_{\mathrm{inf}}} \xrightarrow{\varphi} \varphi_*\Delta_{T/\mathbb{A}_{\mathrm{inf}}} \xrightarrow{\varphi} \dots\right)^{\wedge(p,d)} \\ \uparrow & \uparrow \\ \mathbb{A}_{\mathrm{inf}} = \varinjlim(\mathbb{A}_{\mathrm{inf}} & \xrightarrow{\varphi} \varphi_*\mathbb{A}_{\mathrm{inf}} \xrightarrow{\varphi} \dots)^{\wedge(p,d)} \end{array}$$

So it is enough to show that $\Delta_{T/\mathbb{A}_{\mathrm{inf}}}$ is (p,d) complete and faithfully flat over $\mathbb{A}_{\mathrm{inf}}$, which follows from $\overline{\Delta}_{T/\mathbb{A}_{\mathrm{inf}}}$ being p-complete and faithfully flat over S. From here we apply our lemma by giving $\overline{\Delta}_{T/\mathbb{A}_{\mathrm{inf}}}$ a filtration with

$$\mathrm{gr}_i \cong \bigwedge^i \mathbb{L}_{T/S}[-i]\wedge$$

Computing $\widehat{\mathbb{L}_{T/S}}$ with $T = \left(\varinjlim_e S[1/p^e]/(x-g)\right)^\wedge$, we get

$$\widehat{\mathbb{L}_{T/S}} = \left(\varinjlim_e \left(0 \to T \xrightarrow{d(x-g)} T \cdot dx^{1/p^e} \to 0\right)\right)^\wedge$$

With the middle map assigning $1 \mapsto dx$. Well,

$$dx^{1/p^e} = d\left(x^{p^{e+1}}\right)^p = p(x^{1/p^{e+1}})dx^{1/p^{e+1}} = p^2 \cdot \ldots$$

Continuing this process shows that dx^{1/p^e} is infinitely p-divisible. This, $\widehat{\mathbb{L}_{T/S}} \cong T[1]$, and

$$\bigwedge^i \widehat{\mathbb{L}_{T/S}} \cong \bigwedge^i T[i] \cong \Gamma^i T[i]$$

Thus shifting by i yields

$$\bigwedge^i \mathbb{L}_{T/S}[-i]^\wedge \cong \Gamma^i T$$

Which in degree 0 is p-complete and free. Thus, $\overline{\Delta}_{T/\mathbb{A}_{\mathrm{inf}}}$ is p complete and faithfully flat over S. That means, in particular, that $\Delta_{T/\mathbb{A}_{\mathrm{inf}},\mathrm{perf}}$ is a perfect prism, and thus, S' as constructed is perfectoid via the equivalence of categories between perfect prisms and perfectoid rings.

□

Chapter 5

Day 5

5.1 Vial, On Proper Splinters

A Noetherian scheme X is a ***derived splinter*** if $\forall$ proper finite surjective morphisms $f : Y \to X$, the map $f^\sharp : \mathcal{O}_X \to Rf_* \mathcal{O}_Y$ splits as an $\mathcal{O}_X$-Module in $\mathrm{D^b}\,\mathrm{Coh}(X)$. As we've seen in prior lectures, the Direct Summand Theorem states that all regular affine schemes are splinters. Here are some other general facts:

- The splinter property is a local property for affine schemes
- $f^\sharp$ splits $\iff \mathrm{Hom}_{\mathcal{O}_X}(f_* \mathcal{O}_X, \mathcal{O}_X) \xrightarrow{- \circ f^\sharp} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$ is surjective.
- For $U \subset X$ open, X being a splinter implies that U is a splinter. This follows from Zariski's Main Theorem; if I take a finite map $V \to U$, via Zariski's Main theorem V extends to a finite cover $\widetilde{V}$ of X. As X is a splinter this splits, and we can restrict the splitting down to a splitting $U \to V$.
- X is a splinter $\iff \mathcal{O}_{X,x}$ is a splinter $\forall x \in X$.
- If X is a splinter then X is normal. This is not too difficult; normality localizes so it is sufficient to show this in the ring case.
- If X is a splinter then X is reduced. This is again not too difficult; At the ring level the surjection $R \twoheadrightarrow R_{\mathrm{red}}$ is a finite map, but can only admita splitting when it is an isomorphism.
- When $X \to \mathrm{Spec}(k)$ for k characteristic 0, X is a splinter $\iff$ X is normal. Further, we have the following theorem:

  Theorem 5.1.1 (Kovács, Bhatt)**.** *When $X \to \mathrm{Spec}(k)$ for k characteristic 0, X is a derived splinter $\iff$ X has rational singularities.*

- In characteristic 0, the slinter property is a local property.
- In the positive characteristic case, so when considering $X \to \mathrm{Spec}(\mathbb{F}_p)$, we have the following theorem:

Theorem 5.1.2 (Hochster, Bhatt)**.** *X is a splinter $\Rightarrow$ X is CM.*

- If R is a splinter, then R is F-rational. Conversely if R is $\mathbb{Q}$-Gorenstein and strongly F-regular, then R is a splinter.
- In positive characteristic, the splinter property is NOT local, i.e. you can have a non-affine scheme that is not a splinter, but for which all of its local rings are splinters. For instance, consider an Elliptic curve over $\mathbb{F}_p$ with the multiplication by p map $[p] : \mathcal{O}_E \to [p]^*\mathcal{O}_E$. $[p]$ does not split, as if there was a splitting s, we would get the diagram
$$\mathrm{H}^1(\mathcal{O}_E) \xrightarrow{\cdot p} \mathrm{H}^1(\mathcal{O}_E) \xrightarrow{s} \mathrm{H}^1(\mathcal{O}_E)$$
that composes to the identity. However, these are all $\mathbb{F}_p$-vector spaces, where multiplying by p yields the 0 map, a contradiction. Conversely, $[p]$ splits on every stalk, implying the property is not a local one.
- There are further obstructions to being a (global) splinter. Say that $X \to \mathrm{Spec}(K)$ is a proper morphism of characteristic p schemes.

Theorem 5.1.3 (Hochster-Huneke, Bhatt)**.** *If X is a splinter, then $\mathrm{H}^i(\mathcal{O}_X) = 0\ \forall i > 0$.*

Proof Sketch. $\exists \pi : Y \to X$ a finite cover such that $\pi^* : \mathrm{H}^i(\mathcal{O}_X) \xrightarrow{0} \mathrm{H}^i(\pi_* \mathcal{O}_Y)$, via a vanishing theorem of Hochster-Huneke. If X were a splinter, that means the 0 map would exhibit a splitting on cohomology, implying that X had 0 cohomology to begin with. □

- $\mathrm{H}^i(X, \mathcal{L}) = 0\ \forall i > 0$ and ample line bundles $\mathcal{L}$. The result has been extended to show this even true for semiample line bundles, and further due to Kawamata-Viehweg Vanishing, $\mathrm{H}^i(X, \mathcal{L}^{-1}) = 0\ \forall\ \mathcal{L}$ big and semi ample and $\forall i < \dim X$.
- Via the theorems of Hochster, Huneke and Bhatt, it follows that X is a splinter implies that $\mathrm{H}^0(\omega_X) = 0$, and thus $\omega_X \ncong \mathcal{O}_X$. As a result of the speaker and Krah, X being a splinter implies that ω_X is not torsion (i.e. the Wiel divisor it corresponds to has no torsion).
- If X is a globally F-regular F-finite scheme, then X is a splinter. Schwede and Smith proved that in the $\mathbb{Q}$-Gorenstein case, a globally F-regular F-finite scheme has big $-K_X$.
- It is currently an open question to whether or not X being a splinter implies that X is globally F-regular. In the $\mathbb{Q}$-Gorenstein case, this conjecture also asserts that $-K_X$ is big.

We have numerous examples:

- A proper curve C over an algebraically closed field $k = \bar{k}$ is a splinter $\iff$ $C \cong \mathbb{P}^1$.

- The author and Krah proved that blowups of $\mathbb{P}^2$ at ≤ 5 points is a splinter.
- Take 9 distinct points on a smooth cubic curve in $\mathbb{P}^2_{\mathbb{F}_p}$. Then the blowup at those 9 points is NOT a splinter. It is an open question whether or not if, we take a blowup of $\mathbb{P}^2$ at 9 very general points when $-K_X$ is not big, whether or not it is a splinter.
- If X is a proper surface over an algebraically closed field, then X being a splinter implies it is rational.
- We expect when X is a proper splinter over an algebraically closed field, then X is SRC. This is known for curves but is difficult otherwise.
- If X is a (not necessarily smooth) projective variety, but still over an algebraically closed field, when X is a proper splinter we know:
  - $\mathrm{H}^0(\Omega^1_X) = 0$
  - $\mathrm{H}^0(\omega_X^{\otimes n}) = 0 \; \forall n > 0$.
  - $\pi_1^N(X, x)$, the Nori fundamental group classifying G-torsors over X for G group schemes, is trivial.

Lemma 5.1.4. *(Lifting Lemma) If $\pi : Y \to X$ is a finite surjective morphism of Noetherian schemes, and either*

- *π is étale*
- *$\pi^! \mathcal{O}_X \cong \mathcal{O}_Y$ and either $\mathrm{H}^0(\mathcal{O}_X) = \mathrm{H}^0(\mathcal{O}_Y)$ or $\mathrm{H}^0(\mathcal{O}_Y)$ is a field*

Then if X is a splinter, Y is a splinter.

Theorem 5.1.5 (Vial, Krah)**.** *Say X, Y are Gorenstein proper over a characteristic p field K and X is a splinter. Then if X and Y are derived equivalent (i.e. $\mathrm{D^b}(X) \cong \mathrm{D^b}(Y)$) and $-K_X$ is big, then Y is a splinter.*

Theorem 5.1.6 (Vial, Krah)**.** *If X, Y are Gorenstein and proper over a characteristic p field and X is a globally F regular splinter, then if X and Y are derived equivalent, then Y is globally F-regular.*

5.2 Witaszek, p-adic Riemann-Hilbert Correspondence

5.2.1 Functor Construction

Let R be a finitely presented and flat over a DVR $\mathcal{O}_C$ of mixed characteristic. We use this notation under the assumption that $\mathcal{O}_C$ is a ring of integers of a p-adic field C; in practice we will compute examples where $C = \mathbb{Q}_p$ and thus, $\mathcal{O}_C = \mathbb{Z}_p$. Note that we do not assume R is Noetherian, so specifying finitely presented (as opposed to finitely generated) is pertinent. Further suppose there is a perverse t-structure on $\mathrm{D^b}\,\mathrm{Coh}(\widehat{R})$. Recall this yields

$$K \in^p D^{\leq 0} \iff \mathrm{R}\,\Gamma_x(K) \in D^{\leq -\dim \overline{\{x\}}+1}$$

$$K \in^p D^{\geq 0} \iff \mathrm{R}\Gamma_x(K) \in D^{\geq -\dim \overline{\{x\}}+1}$$

$\forall x \in \mathrm{Spec}(R/p)$. Let $\mathbb{Q}_p \subset C$ be a perfectoid extension. These results will hold in this generality, but for simplicity we will express them in the specific case where $\mathcal{O}_C = \mathbb{Z}_p[p^{1/p^\infty}]^\wedge$. We will introduce the notation $\mathrm{D^b}\,\mathrm{Coh}(\widehat{R})_a$ to denote the bounded derived category of almost coherent $\widehat{R}$-modules, i.e. those that are coherent up to sufficiently large p quotient.

Theorem 5.2.1 (Bhatt-Lurie)**.** *There exists a Riemann-Hilbert Functor*

$$RH : \mathrm{D^b}_{cons}(R[1/p], \mathbb{Z}_p) \xrightarrow{almost} \mathrm{D^b}\,\mathrm{Coh}(\widehat{R})_a$$

such that

- $RH(\mathbb{Z}_p) = R_{perfd}$
- $RH(\mathbb{D}-) = \mathbb{D}RH(-)$
- *it is t-left exact.*
- *it commutes with proper pushforwards.*

Globally, we are considering proper schemes (i.e. where $\mathrm{D^b}\,\mathrm{Coh}(X^\wedge) \cong \mathrm{D^b}\,\mathrm{Coh}(X)$) and will thus get the following functor:

Theorem 5.2.2 (Bhatt-Lurie)**.** *There exists a Riemann-Hilbert Functor*

$$RH : \mathrm{D^b}_{cons}(X[1/p], \mathbb{Z}_p) \to \mathrm{D^b}\,\mathrm{Coh}(X)_a$$

Satisfying the properties above.

When X is proper and smooth, a key corollary of this theorem is that

$$\mathrm{RH}(\mathbb{Q}_p) = \mathcal{O}_{X[1/p]} \oplus \Omega_{X[1/p]}[1] \oplus \cdots \oplus \Omega^d_{X[1/p]}[d]$$

Recovering $\mathrm{Gr}_\bullet\mathrm{DR}$, i.e. Jakub's favorite functor.

Theorem 5.2.3 (Bhatt, Bhatt-Lurie)**.** *Let* $(\mathrm{R}, \mathfrak{m})$ *be a mixed characteristic ring that is essentially of finite type over* $\mathcal{O}_C$*, defined as above. Then* $\mathrm{H}^i_{\mathfrak{m}}(R^+)$ *is p-almost* 0 *for all* $i < \dim(R)$*.*

Proof Sketch. As R^+ is perfectoid, $\mathrm{RH}(R^+, \mathbb{Z}_p) = \widehat{R^+}$. This is perverse, so if we trace back through the basic properties of t structures, we recover that $\mathrm{R}\Gamma_{\mathfrak{m}}(R^+) \in D^{\geq d}$, which is precisely what we are after. □

5.2.2 Equicharacteristic Case

Let X be a variety over k. We intend to define the ***Intersection Complex*** in each possible characteristic case. let $k = \mathbb{C}$ first. $\mathrm{IC}_{X,\mathbb{C}}$ as follows:

$$\mathrm{IC}_{X,\mathbb{C}} := \mathrm{im}\,(j_!\mathbb{C}[d] \to j_*\mathbb{C}[d])$$

where $j : U \hookrightarrow X$ is an inclusion of a smooth open affine subset $U \subset X$ into X. As the notation would suggest, the intersection complex is invariant of choice of j and U. As $j_*, j_!$ are both t-exact, it is easy to check that $j_!\mathbb{C}[d], j_*\mathbb{C}[d]$, and thus $\mathrm{IC}_{X,\mathbb{C}}$, is perverse. When X is smooth, $\mathrm{IC}_{X,\mathbb{C}} = \mathbb{C}[d]$ on the nose. Even if X is smooth outside of a closed point $x \in X$, taking $j : X\backslash\{x\} \hookrightarrow X$ yields $\mathrm{IC}_{X,\mathbb{C}} = \tau^{\leq -d}Rj_*\mathbb{C}[d]$, which is only supported in degree $\leq -d$. Taking Cohomology of this complex yields the powerful ***Intersection Cohomology*** invariant, a construction of Goresky and McPherson

$$\mathrm{IH}^i(X^{an}, \mathbb{C}) := \mathbb{H}^i(X^{an}, \mathrm{IC}_{X,\mathbb{C}}[-d]$$

When X is smooth, due to the above result this just recovers singular cohomology. Ironically, singular cohomology is not a very good invariant when X is singular; it fails Poincare Duality, for instance. Intersection cohomology proves to be the "correct" cohomological invariant in this regard, satisfying Poincare duality as expected:

$$\mathrm{IH}^i(X, \mathbb{C}) \cong \mathrm{IH}^{2d-i}(X, \mathbb{C})^*$$

The key fact for why this works is that $\mathrm{IC}_{X,\mathbb{C}}$ is in a sense simple, i.e. there aren't any quotients or subojects of it supported on any closed subset of X that is strictly of lower dimension.

As one would expect, the positive characteristic version of the intersection complex (and thus intersection cohomology) is defined similarly.

$$\mathrm{IC}_{X,\mathbb{F}_p} := \mathrm{im}\left(j_!\mathbb{F}_p[d] \to j_*\mathbb{F}_p[d]\right)$$

As it turns out, intersection cohomology in positive characteristic can be used to detect commutative algebraic info.

5.2.3 Applications to Test Ideals

Let $(R, \mathfrak{m})$ be a Noetherian local F-finite ring. We define the $+$-rational test ideal to be

$$\tau_+(\omega_R) := \bigcap_{\substack{R \subset S \\ \text{finite}}} \mathrm{im}(\mathrm{Tr} : \omega_S \to \omega_R)$$

When R is CM, $\tau_+(\omega_R) = \omega_R \iff R$ is $+$-rational, as one would expect. Such a construction also has a closed form; in particular in the CM case if you p-complete, then we have the following useful identity:

$$\tau_+(\omega_R)^{\wedge p} = \mathrm{im}(\mathrm{H}^d_{\mathfrak{m}}(R) \to \mathrm{H}^d_{\mathfrak{m}}(R^+))^\vee$$

where $(-)^\vee$ denotes Matlis Duality. We also have a powerful connection between this test ideal and the intersection complex.

Theorem 5.2.4.
$$\tau_+(\omega_R) = \operatorname{im}\left(\mathrm{H}^d_{\mathfrak{m}}(R) \to \mathrm{H}^d_{\mathfrak{m}}(IC_{R,\mathbb{F}_p}[-d])\right)^\vee$$

As a corollary, recall that as derived functors, $\mathbb{D}(-) \cong \mathrm{R}\,\Gamma_{\mathfrak{m}}(-)^\vee$. Thus we can replace terms in the above theorem with their perverse variants:

$$\tau_+(\omega_R) = \mathbb{D}\operatorname{im}\left({}^p\,\mathrm{H}^d(R) \to^p \mathrm{H}^d(\mathrm{RH}\left(\mathrm{IC}_{R,\mathbb{F}_p}[-d]\right))\right)$$

Duality commutes with taking images contravariantly, so we can conclude that

$$\tau_+(\omega_R) = \operatorname{im}\left(\mathbb{D}^p\,\mathrm{H}^d(R) \leftarrow \mathbb{D}^p\,\mathrm{H}^d(\mathrm{RH}\left(\mathrm{IC}_{R,\mathbb{F}_p}[-d]\right))\right)$$

This variant of the alteration test ideal is quite beneficial; for starters it shows that $\tau_+(\omega_R)$ localizes in positive characteristic. There is a key idea we will use in addition to this however: Let $\mathscr{K} \to \mathscr{L}$ be an injection of perverse $\mathbb{F}_p$-sheaves. Via the Riemann-Hilbert Correspondence, $\mathrm{RH}(\mathscr{K}) \to \mathrm{RH}(\mathscr{L})$ is an injection of perverse Noetherian sheaves, and as $\mathrm{R}\,\Gamma_{\mathfrak{m}}$ is t-left exact, passing it through this functor preserves this injectivity. Well, $\mathrm{R}\,\Gamma_{\mathfrak{m}}(\mathrm{RH}(\mathscr{K})) \cong \mathrm{H}^d_{\mathfrak{m}}\,\mathrm{RH}(\mathscr{K}[-d])$ and similarly for $\mathscr{L}$. We'll use this to sketch the beginnings of a proof of the equality relating the alteration test ideal and $\mathrm{H}^d_{\mathfrak{m}}(R) \to \mathrm{H}^d_{\mathfrak{m}}(R^+)$.

Proof Sketch. We first introduce some notation that we will use later. Let $\pi : \mathrm{Spec}(R^+) \to \mathrm{Spec}(R)$ and $U \subset \mathrm{Spec}(R)$ open. Further, let $U^+ \subset \mathrm{Spec}(R^+)$ be the open set such that $\pi(U^+) = U$. We define
$$\mathbb{F}_{p,U^+} := (\pi|_{U^+})_*\,\mathbb{F}_p$$
and its global variant
$$\mathbb{F}_{p,X^+} := \mathbb{F}_{p,R^+} = \pi_*\mathbb{F}_p$$
To be the pullbacks of $\mathbb{F}_p$ along π. Recall the notation from before that $j : U \to \mathrm{Spec}(R)$ is an open inclusion. From this we get the sequence of maps

$$\mathrm{H}^d_{\mathfrak{m}}(R) \to \mathrm{H}^d_{\mathfrak{m}}\left(\mathrm{RH}\left(\mathrm{IC}_{R,\mathbb{F}_p}\right)\right) \to \mathrm{H}^d\left(\mathrm{RH}(j_*\mathbb{F}_p)\right) \to \mathrm{H}^d(\mathrm{RH}(j_*\mathbb{F}_{p,U^+}))$$

It turns out that the last term in this is just $\mathrm{H}^d_{\mathfrak{m}}(R^+)$. We claim this map is injective precisely when $j_*\mathbb{F}_p[d] \to j_*\mathbb{F}_{p,U^+}$ is injective. We can show this using the fact that j_* is t-exact and that the intersection complex $\mathrm{IC}_{U,\mathbb{F}_p}$ is a simple object; we will leave these details out. □

5.2.4 Mixed Characteristic Case

We define the ***Alteration Test Ideal*** for any ring R admitting a dualizing complex ω_R as follows:
$$\tau_{\mathrm{alt}}(\omega_R) := \bigcap_{\substack{g:Y\to\mathrm{Spec}(R)\\ \text{an alteration}}} \operatorname{im}\left(\mathrm{Tr} : g_*\omega_Y \to \omega_R\right)$$

Theorem 5.2.5. *When* $(\mathrm{R}, \mathfrak{m})$ *is local,* $\tau_{alt}(R) = \tau_+(R)$.

We will leave this as a (fairly hard) exercise. Do note that this is reminicent of the fact that alteration rationality is equivalent to $+$-rationality in positive and mixed characteristic. We will now move entirely into the mixed characteristic settinng. Let R be a ring of mixed characteristic $(0, p)$ that is of finite type over a DVR $\mathcal{O}_C$ (again, we can assume that this is just $\mathbb{Z}_p$ in our case for examples; the genrealizations from this case are quite straightforward). Define

$$\mathcal{O}_{C_\infty} := \mathbb{Z}_p\left[p^{1/p^\infty}\right]^{\wedge p}$$
$$R_\infty = R \otimes_{\mathcal{O}_C} \mathcal{O}_{C_\infty}$$

Theorem 5.2.6 (Big 7 author collaboration)**.** *Up to p-almost mathematics,*

$$\tau_{alt}(\omega_{R_\infty}) = \operatorname{im}\left(\omega_{R_\infty/\mathcal{O}_{C_\infty}} \leftarrow \mathbb{D}^p\, \mathrm{H}^d\left(RH\left(IC_{R_\infty,\mathcal{O}_C[-d]}\right)\right)\right)$$

We would of course like the descend this back to R in a way reminicent of the equicharacteristic case; perhaps by a trace map. The issue is that $R \to R_\infty$, though integral, is definitely not of finite type. Thus, we don't get a trace map for free. The solution to this is to define a "weak" trace map $t : \mathcal{O}_{C_\infty} \to \mathcal{O}_C$ via p-completing. We only need to consider where to send p^{a/p^e} for a fixed a with $(a, p) = 1$ and e, as these form a basis for the source. In this case, t will map $p^{a/p^e} \mapsto 1 \iff a = p^e - 1$ and 0 otherwise. Using this, we define the ***almost Alteration Test Ideal*** as follows:

$$\tau^a_{\text{alt}}(\omega_R) = \bigcup_{e>0} \bigcap_{\substack{g:Y\to\text{Spec}(R)\\ extrmanalteration}} \operatorname{im}\left(g_*\omega_Y \xrightarrow{\text{Tr}\circ(-\cdot p^{1/p^e})} \omega_R\right)$$

Where it can be seen that

Theorem 5.2.7 (Big 7 author collaboration)**.** $\tau^a_{alt}(\omega_R) = t\left(\tau_{alt}(\omega_{R_\infty})\right)$

And further, this construction both localizes and can be computed via computing the image of a distinguished alteration of R. This implies the corollary that when R is $\mathbb{Q}$-Gorenstein, the locus of primes where R is a p-almost splinter is open.