Preliminary Info

These are notes typed up Vignesh Jagathese (me!) for a mini-course on perfectoid spaces taught in Fall 2023 at UIC by Kevin Tucker. This will be a gentle and non-terse introduction to perfectoids and their applications to classifying singularities via commutative algebra. This is meant to be an introductory series of lectures, so we'll avoid using the heavy machinery that is common to these sorts of introductions. We will first introduce what perfect rings are in positive characteristic to motivate perfectoid rings, which are a natural mixed characteristic analogue. After this, we'll prove (or sketch a proof of) a number of powerful results that fall out of the theory of perfectoid rings, including the Direct Summand Theorem (Previously, one of the more famous Homological Conjectures). From there we will generalize common positive characteristic invariants to the mixed characteristic setting.

reading these notes is allowed

$$\iff$$

you submit any typos/errors to me at vjagat2 (at) uic (.edu).

Contents

A ***perfectoid*** ring is in some sense a mixed characteristic analogue for perfect rings in positive characteristic. All rings going forward are commutative with 1, and $p > 0$ a fixed prime. Often, we consider $(\mathrm{R}, \mathfrak{m}, \mathfrak{K})$ a local ring with $p \in \mathfrak{m}$ (viewing $p \in \mathbb{Z} \to R$). Thus, $\mathfrak{K}$ must have characteristic p. From here we have two cases:

- R has characteristic p ($\mathbb{F}_p[[x_1, \ldots, x_n]]$, etc.)
- R has characteristic $\neq p$ ($\mathbb{Z}_p[[x_2, \ldots, x_n]]$, etc.). Note that if R is a domain, R necessarily has characteristic 0.

Typically, we'll concern ourselves with complete local rings. Naturally, when proving something about R, we'd like to show that it can be checked locally and/or up to completion. When both of these things are true, it turns out that we are working over a power series ring subject to some relations:

Theorem 0.1. ***(Cohen Structure Theorem)*** $(\mathrm{R}, \mathfrak{m}, \mathfrak{K})$ *is a complete local noetherian ring. Then,*

$$R \cong \mathfrak{K}[[x_1, \ldots, x_n]]/I$$

Or

$$R \cong \Lambda[[x_1, \ldots, x_n]]/I$$

for Λ a Cohen Ring, or a complete DVR with maximal ideal generated by p such that $\Lambda/p\Lambda \cong k$.

For example, when k is a perfect field of characteristic $p > 0$, $W(k)$ (the Witt ring) is a Cohen ring. We can see that $W(\mathbb{F}_p) = \mathbb{Z}_p$.

1 Perfect Rings

We say a ring R is ***perfect*** if R has characteristic p, and the associated ring map $F : R \to R$ mapping $x \mapsto x^p$ is an isomorphism. When F is injective, we can clearly see that R is reduced. Thus sometimes we only focus on the surjectivity; if F is just surjective we say that R is ***semiperfect***. We have the following facts when R is perfect:

- R is reduced.
- If R is Noetherian, then it must be a finite product of prefect fields.

  Proof. As R is reduced, we can quotient by some normal prime and reduce to checking this when R is a domain. Take $a \in R$ nonzero. Consider the ascending chain of ideals $(a) \subseteq (a^{1/p}) \subseteq (a^{1/p^2}) \ldots$. This halts at some N. Thus $a^{1/p^{N+1}} = a^{1/p^N} \cdot t$ for some t. It follows that $a = a^p t'$ after raising to the power of p^{N+1}. Therefore, $a(1 - a^{p-1}t) = 0$, and because $a \neq 0$ and R is a domain, we see that $a(a^{p-2}t) = 1$, so we have an inverse. □

Of course we can take any ring and make it perfect. Here are two ways to do this:

- Let $R_{\text{perf}} = \varinjlim(R \xrightarrow{F} R \xrightarrow{F} R \xrightarrow{F} \dots) = \bigcup_{e>0}(R_{\text{red}}^{1/p^e}) = R_{\text{red}}^{1/p^\infty}$. This is the ***perfection*** of R.
- $R^\flat := \varprojlim(\cdots \xrightarrow{F} R \xrightarrow{F} R)$. This is the ***tilt*** of R. Elements in this ring are sequence $r_\bullet$ such that $r_i^p = r_{i-1}\ \forall i \geq 1, r_i \in R$.

These respectively have inclusions/projections to/from the first coordinate. For instance, if R is reduced, then $R^\flat \hookrightarrow R$ is the subring of elements with p-power roots. For example, if R is reduced, $R^\flat \hookrightarrow R$, and in particular, pth roots are unique. Thus, $R^\flat$ is precisely the subring of R of elements with all of its p-power roots.

More generally, we can even tilt rings that are not equicharacteristic p. For any ring A,

$$A^\flat := (A/(p))^\flat = \varprojlim \left(\dots A/(p) \xrightarrow{F} A/(p) \right)$$

Here, an element $a_\bullet \in A^\flat$ is a sequence such that $a_{n+1}^p \equiv a_n \bmod p$. If A is p-complete and separated, $\exists \sharp : A^\flat \to A$ assigning $a_\bullet \mapsto \lim_{n\to\infty} a_n^{p^n}$. By construction, this limit exists so the map is well defined. Further, one can check that this map is always multiplicative but not additive in general.

In fact, $A^\flat \cong A$ as multiplicative monoids, but not via the $\sharp$ map. We have a map

$$\varprojlim_{x \mapsto x^p} A \to A^\flat$$

As elements in the source are sequences $a_\bullet$ where $a_{n+1}^p = a_n$ on the nose (not mod p), so we have a natural inclusion map. This has an inverse assigning $\alpha \mapsto (\sharp\alpha, \sharp(\alpha^{1/p}), \dots)$.

Similarly, if R is perfect and $d_0 \in R$ with R d_0-complete and separated,

$$(R/(d_0))_{\text{perf}} = (R/(d_0))_{\text{red}} = R/\sqrt{d_0} = \frac{R}{\left(d_0, d_0^{1/p}, d_0^{1/p^2}, \dots\right)}$$

Whereas

$$(R/(d_0))^\flat \cong R^\flat \cong R$$

For example, $\mathbb{F}_p[[t]]_{\text{perf}} = \mathbb{F}_p[[t]][t^{1/p^\infty}] =: R$, so taking the perfection can break t-completeness, as R is not complete. However, $\widehat{R} = (R/(t))^\flat$. Such arguments also work over multiple variables/over quotienting out ideals that are not principal.

Bhatt, Iyengar and Ma proved that, while your ring may not be perfect, it admits a faithfully flat extension to a perfect ring. This has a natural mixed characteristic analogue that we'll discuss further later.

Theorem 1.1. ***(Bhatt, Iyengar, Ma)*** *If R is Noetherian of characteristic p, then R is regular $\iff \exists R \to A$ which is faithfully flat where A is a perfect ring.*

Proof. If R is regular, F is a (faithfully) flat morphism, so we can just let $A = R_{\text{perf}}$. This is faithfully flat over R. For the converse, as regularity is checked locally, we can assume that $(\mathrm{R}, \mathfrak{m}, \mathfrak{K})$ is local. Take $x_1, \dots, x_d \in \mathfrak{m}$ a system of parameters. Observe that by previous logic, $\sqrt{x_i A} = \left(x_i^{1/p^\infty}\right) = \varinjlim \left(x_i^{1/p^n}\right)$ and as $\left(x_i^{1/p^n}\right)$ are all principal, so we have the commuting square

$$\begin{tikzcd}
A \arrow[r, "\cdot x_i^{1/p}"] \arrow[d, "1\mapsto x_i"'] & A \arrow[d, "1\mapsto x_i^{1-1/p}\cdot x_i^{1/p}"] \\
(x_i) \arrow[r, hook] & (x_i^{1/p})
\end{tikzcd}$$

so we can rewrite $\varinjlim \left(x_i^{1/p^n}\right)$ as a direct limit

$$\varinjlim \left(A \xrightarrow{\cdot x_i^{1-1/p}} A \xrightarrow{\cdot x_i^{1/p-1/p^2}} \dots \right)$$

Thus, $\varinjlim \left(x_i^{1/p^n}\right)$ is a flat A-module, or equivalently due to the hypothesis, a flat R-module. (non-trivial; this is a result of Hochster and Aberbach). By induction, we show that

$$\left(x_1^{1/p^\infty}, \dots, x_{i-1}^{1/p^\infty}\right) \cdot \left(x_i^{1/p^\infty}\right) = \left(x_1^{1/p^\infty}, \dots, x_{i-1}^{1/p^\infty}\right) \cap \left(x_i^{1/p^\infty}\right)$$

And

$$\mathcal{C}^{(i)} = \bigotimes_{j=1}^{i} \left(\left(x_i^{1/p^\infty}\right) \to A \right)$$

is a flat resolution of $A / \left(x_1^{1/p^\infty}, \dots, x_i^{1/p^\infty}\right)$. The first statement is an element-wise check, and the second follows from the fact that we are taking a tensor product of flat resolutions. From this, we see that $\mathcal{C}_\bullet = \bigotimes_{i=1}^{n} \left((x_i^{1/p^\infty} \to A \right)$ is a flat resolution of $\frac{A}{\left(x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}\right)} = A/\sqrt{\mathfrak{m}A}$ of length precisely d. For some $\ell > d$, consider $\operatorname{Tor}_\ell^A \left(A \otimes_R k, A/\sqrt{\mathfrak{m}A}\right)$. As we can pull the flat resolution through Hom, we see that this Tor $= 0$. However, as A is flat, we see that $0 = \operatorname{Tor}_\ell^A \left(A \otimes_R k, A/\sqrt{\mathfrak{m}A}\right) = \operatorname{Tor}_\ell^R(k, A/\sqrt{\mathfrak{m}A})$ where $A/\sqrt{\mathfrak{m}A}$ is an $\mathfrak{K}$-vector space, so it looks like $k^{\oplus \mathfrak{L}}$ for some large (possibly infinite $\mathfrak{L}$), thus,

$$0 = \operatorname{Tor}_\ell^R(k, A/\sqrt{\mathfrak{m}A}) = \operatorname{Tor}_\ell^R(k, k^{\oplus \mathfrak{L}}) = \operatorname{Tor}_\ell^R(k, k)^{\oplus \mathfrak{L}}$$

$\operatorname{Tor}_\ell^R(k,k)$ vanishes, so $\mathfrak{K}$ has finite projective dimension. It follows that R is regular by Serre's criterion. ☐

2 Perfectoid Rings

We'd like to introduce a class of rings called ***perfectoid*** rings. We begin with some standard examples.

- If R is characteristic p, R is perfectoid $\iff$ R is perfect.
- If A is p-torsion free, A is perfectoid $\iff$ A is p-adically complete and separated and $p = n\varpi^p$ for some unit n, where $F : A/\varpi \to A/p$ is an isomorphism. This means that, up to a unit n, p has a pth root, and when we reduce mod p in the correct way, we get a perfect ring.

For instance, $\mathbb{Z}_p$ is NOT perfectoid, because, while it is p-torsion free, p does not have a pth root up to unit. One would think we could remedy this by adding in pth roots, but even $\mathbb{Z}_p[p^{1/p^\infty}]$ is not perfectoid, as it is not p-complete. $\widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$ is perfectoid, and is one of the easier examples that we can write down.

Consider the standard example of a regular ring in mixed characteristic, $\mathbb{Z}_p[[x_2, \dots, x_n]]$. To make this perfectoid, we first adjoint pth power roots of p and the formal variables, and then complete. We can see that

$$\mathbb{Z}_p[[x_2, \dots, x_n]][\widehat{p^{1/p^\infty}}, x_2^{1/p^\infty}, \dots, x_n^{1/p^\infty}]$$

is perfectoid. A good exercise to do here is to compute $A^\flat$ where A is either of the perfectoid rings mentioned above. When $A = \widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$, notice that

$$A/(p) = \mathbb{Z}_p[p^{1/p^\infty}]/(p) = \mathbb{F}_p[t^{1/p^\infty}]/(t)$$

Which is the quotient of a perfect ring by a single element. Thus,

$$A^\flat = \widehat{\mathbb{F}_p[t^{1/p^\infty}]}$$

completed with respect to t. We'll provide an explicit definition of a perfectoid ring shortly, but we will first need to introduce the notion of a (p-typical) Witt Vector.

3 p-typical Witt Vectors

Let R be any ring. Then the ***Witt Ring*** is defined to be the ring

$$W(R) := \{(a_0, a_1, \dots,) \mid a_i \in R\}$$

With $+, \cdot$ in the ring defined via universal polynomials so that the assignment $w_i : W(R) \to R$ defined by

$$w_i : x_\bullet \mapsto x_0^{p^i} + p x_1^{p^{i-1}} + p^2 x_2^{p^{i-2}} + \cdots + p^i x_i$$

always determine ring maps. In this setting, we see that

$$x_\bullet + y_\bullet = s_\bullet$$

$$x_\bullet \cdot y_\bullet = p_\bullet$$

Where

$$s_0 = x_0 + y_0$$

$$p_0 = x_0 y_0$$

Further, $x_0^p + px_1 + y_0^p + py_1 = s_0^p + ps_1^p$. If R is p-torsion free, we can explicitly solve for s_1:

$$s_1 = x_1 + y_1 - \frac{(x_0+y_0)^p - x_0^p - y_0^p}{p}$$

However, this formula actually works in general, since using the binomial expansion of $(x_0+y_0)^p$ (which works in any ring), we can reduce this to

$$s_1 = x_1 + y_1 - \sum_{0<i<p} \frac{\binom{p}{i}}{p} x_0^i y_0^{p-i}$$

We can continue along to recursively compute all the s_i. Similarly, we can figure out how multiplication works.

$$x_\bullet \cdot y_\bullet = p_\bullet = (x_0 y_0, x_0^p y_1 + x_1 y_0^p + p x_1 y_1, \dots)$$

$W(R)$ has a natural projection map $W(R) \xrightarrow{\pi} R$ via surjectively projecting to the first coordinate. Note that $(W(-), \pi)$ is functorial, i.e. for a ring morphism $R \xrightarrow{\varphi} S$ there exists a unique map $W(R) \xrightarrow{W(\varphi)} W(S)$ such that the following diagram commutes:

$$\begin{tikzcd}
W(R) \arrow[r, dashed, "W(\varphi)"] \arrow[d, "\pi_R"] & W(S) \arrow[d, "\pi_S"] \\
R \arrow[r, "\varphi"] & S
\end{tikzcd}$$

Similarly, the ***truncated Witt vectors*** $W_n(-)$ is functorial with respect to its projection π. Naturally, we can lift the Frobenius to the Witt ring, where $F : W(R) \to F_* W(R)$ maps $(a_0, a_1, \dots,)$ to $(a_0^p, a_1^p, \dots)$ as expected. This is an isomorphism provided that R is perfect, so we can see that R is perfect $\iff$ $W(R)$ is. In fact, when R is perfect, $W(R)$ is the unique p-adically complete, separated, and p-torsion free ring that comes with a surjection $W(R) \twoheadrightarrow R$ with kernel generated by p. For instance, the prototypical example is $W(\mathbb{F}_p) \cong \mathbb{Z}_p$, and when K is a perfect field, $W(K)$ is a mixed characteristic DVR with maximal ideal generated by p with residue field isomorphic to K.

We have some maps associated to Witt vectors. First, we have the shift map $V : W(R) \to W(R)$ sending $(a_0, a_1, \dots,) \mapsto (0, a_0, a_1, \dots)$. This is an additive map, but is notably not multiplicative. It is an easy check that, in positive characteristic, $FV = VF = p$, viewing p as the map $(a_0, a_1, \dots,) \mapsto (0, a_0^p, a_1^p, \dots)$. We also have a Teichmüller map when R is perfect, denoted $[-] : R \cong R^\flat = W(R)^\flat \xrightarrow{\sharp} W(R)$, which assigns $r \mapsto (r, 0, 0, \dots)$. Such a morphism is multiplicative, but not additive. In the case where the Witt vector is of the form $[r]$, we can compute multiplication easily:

$$[r] \cdot s_\bullet = (rs_0, r^p s_1, r^{p^2} s_2, \dots)$$

When R is perfect, we can take an arbitrary vector $r_\bullet \in W(R)$ and break it up as follows:

$$r_\bullet = \sum_{n=0}^{\infty} V^n([r_n]) = [r_0] + p\left[r_1^{1/p}\right] + p^2\left[r_2^{1/p^2}\right] + \dots$$

This decomposition, coupled with the multiplication rule above, makes it quite easy to compute multiplication in $W(R)$ when R is perfect.

4 Structure of Perfectoid Rings

We say A is a ***Perfectoid Ring*** if $A \cong W(B)/(d)$ where B is a perfect ring and $d \in W(B)$ is a distinguished element, i.e. it is of the form $d = (d_0, d_1, \dots)$ where B is d_0-complete and separated and d_1 is unit.

When A is a perfect ring of characteristic p, we can write $A = W(A)/pW(A)$, so A is perfectoid. In general, we will show that the converse holds; if A is characteristic p and perfectoid, it is perfect.

This definition has some small caveats we need to check. First, is this B uniquely determined? Indeed it is, in fact, $B = A^\flat$. To check this, observe that

$$A/(p) = W(B)/(p, d) = B/(d_0)$$

As B is perfect. As B is perfect, d has a decomposition

$$[d_0] + [d_1^{1/p}]p + \dots$$

And by construction, we can write $d = [d_0] + up$ for u a unit. As B is perfect, B/d_0 is semi-perfect, so $(B/d_0)^\flat = \widehat{B}^{d_0}$, but B is by construction d_0 complete, so this is just B. It follows that

$$A^\flat = \varprojlim (A/p)^\flat = \varprojlim (B/d_0)^\flat = \varprojlim B = B$$

We also observe that the canonical surjection $W(B) \twoheadrightarrow A$ (which determines the isomorphism $W(B)/(d) \cong A$ is uniquely determined. In fact, this is called ***Fontaine's Map***.

$$\theta : W(A^\flat) \to A$$

assigns

$$\left[(a_0, a_1, \dots) = [a_0] + [a_1^{1/p}]p + \dots\right] \mapsto \left[a_0^\sharp + \left(a_1^{1/p}\right)^\sharp \cdot p + \dots\right]$$

Where the decomposition exists because $A^\flat$ is perfect. A is p-complete, so such a map is well defined, and indeed, a surjection. It is nontrivial, however, that this is a ring map. It suffices to check this is additive and multiplicative modulo p^n for each n. Take

$$\Theta(a) = a_0^\sharp + \left(a_1^{1/p}\right)^\sharp \cdot p + \dots + \left(a_n^{1/p^n}\right)^\sharp \cdot p^n$$

modulo p^{n+1}. This can be written modulo p^{n+1} as

$$\left(\left(a_0^{1/p^n}\right)^\sharp\right)^{p^n} + \left(\left(a_1^{1/p^n}\right)^\sharp\right)^{p^{n-1}} \cdot p + \dots + \left(a_n^{1/p^n}\right)^\sharp \cdot p^n = w_n\left(\left(a_0^{1/p^n}\right)^\sharp, \dots, \left(a_n^{1/p^n}\right)^\sharp\right)$$

For w_n the nth universal Witt Polynomial. We have the diagram

$$\begin{tikzcd}
W(A) \arrow[r, two heads, "w_n"] \arrow[d] & A \arrow[d, two heads] \\
W(A/(p)) \arrow[r, "\overline{w_n}"] & A/(p^{n+1})
\end{tikzcd}$$

Where $\overline{w_n}$ assigns $(\overline{x_0}, \dots,) \mapsto \overline{w(x_0, \dots)}$, where the over line denotes reducing modulo p on the left, and modulo p^{n+1} on the right. We have a sequence of morphisms

$$W(A^\flat) \xrightarrow{W(F^{-n}} W(A^\flat) \to W(A/p) \xrightarrow{\overline{w_n}} W(A/p^{n+1})$$

Which are all ring maps, assigning

$$a_\bullet \mapsto \left(a_\bullet^{1/p^n}\right) \mapsto \left(a_\bullet^{1/p^n}\right) \bmod p \mapsto \Theta(a_\bullet) \bmod p^{n+1}$$

So this is a ring map as desired. This leads us to an alternative definition of being perfectoid. A is a perfectoid ring $\iff$ the following conditions hold:

- A is p-adically complete
- $A/(p)$ is semi-perfect
- the kernel of θ is principal
- There is a pth root of p in A, up to unit (i.e. $\exists u$ unit and $\exists \varpi$ such that $p = u \cdot \varpi^p$).

If A is p-torsion free, it is perfectoid $\iff$ it is p-complete and there is a pth root of p up to unit; this is because being p-torsion free immediately gives you the other conditions above. Further, a result of Bhatt, Morrow, and Scholze that $\exists \varpi, u \in A$ with u unit such that $p = u\varpi^p$ and $F : A/\varpi \to A/(p)$ is an isomorphism. The forward case is straightforward, via the above definition. The converse direction, however, is a bit non-trivial.

Replacing ϖ with a multiple, we can assume that ϖ has a system of p-power roots. Once we have such a ϖ we know that it lies in the image of the $\sharp$ map, i.e. $\exists d_0 \in A^\flat$ with $\theta([d_0]) = d_0^\sharp = \varpi^p$. Because $F : A/\varpi \to A/(p)$ is an isomorphism, we see that $d_0 = (\varpi^\flat)^p$ where $\varpi^\flat := d_0^{1/p}$. We claim that this generates the kernel of the natural map $A^\flat \to A/(p)$ (which is obtained by taking the Fontaine map $W(A^\flat) \to A$ and reducing it mod p).

We've already shown that, when R is of characteristic p, being perfect implies you are perfectoid. We'll now show the converse; to begin, we'll need to first check that perfectoid rings are reduced. Using the alternative definition above, we know the mod p reduction (which in characteristic p is itself) is semi-perfect, so checking R is reduced is not only necessary, but sufficient.

Lemma 4.1. *Suppose $W(A^\flat)/(d) = A$ is perfectoid. Then,*

(1) $\overline{A} := A/A[p]$ for $A[p] = \{a \in A \mid pa = 0\}$ is perfectoid, and by construction, p-torsion free.

(2) $A/\sqrt{(p)}$ and $\overline{A}/\sqrt{(p)}$ are perfect rings of characteristic p, hence perfectoid.

(3) A can be interpreted as the gluing of $\overline{A}$ and $A/\sqrt{(p)}$ along the common subscheme $\overline{A}/\sqrt{(p)}$, i.e.

$$A = \overline{A} \times_{\overline{A}/\sqrt{(p)}} A/\sqrt{(p)}$$

In other words, we have a short exact sequence

$$0 \to A \to \overline{A} \oplus A/\sqrt{p} \to \overline{A}/\sqrt{(p)} \to 0$$

A natural Corollary of having the short exact sequence is that perfectoid rings are reduced, which would establish the criterion we started with. The proof of the lemma is a bit of a mess, and will be punted, so instead we just verify the corollary follows from this.

Proof. The lemma tells us that every perfectoid A lies inside the product of a p torsion free ring and a perfect ring. This lets us reduce to p-torsion free case. Suppose that $A = W(A^\flat)/(d)$ is perfectoid and p-torsion free, with $p = u\varpi^p$, and say $\exists a$ such that $a^p = 0$. As $\ker(A/p \xrightarrow{F} A/p) = (\varpi)/(p)$, we see that $a = a'\varpi$ for some a'. However, we also have that $0 = u^{-1}a^p = p(a')^p$, so $0 = a^p = (a')^p p u^{-1}$, so $(a')^p p = 0$. As A is p-torsion free, $(a')^p = 0$. This tells us that a' is also a multiple of ϖ, so $a \in \varpi^2$. Continuing this logic, we see that $a \in \bigcap(\varpi^n) = \bigcap(p^n) = 0$. Thus, $a = 0$. □

5 Regular Rings Admit Faithfully Flat Perfect(oid) Extensions

Earlier we proved the following theorem:

Theorem 5.1. ***(Bhatt, Iyengar, Ma)*** *Suppose that R is Noetherian of characteristic $p > 0$. Then R is regular $\iff \exists R \to A$ such that A is perfect and faithfully flat over R.*

Bhatt Iyengar and Ma also proved an analog in arbitrary characteristic:

Theorem 5.2. *If R is Noetherian with p lying within the Jacobson radical (denoted $\mathcal{J}(R)$), then R is regular $\iff \exists R \to A$ where A is perfectoid and faithfully flat over R.*

In the characteristic p case, as we saw, the forward direction is straightforward; just let $R = R_{\text{perf}}$. The reverse direction is very similar for both results, so we'll appeal to the proof already given. Notably, however, the forward case of the mixed characteristic statement has some content.

Proof. (Forward case, 2nd theorem) Take $R \to \prod_{\mathfrak{m} \in \text{MaxSpec}(R)} R_{\mathfrak{m}}$ is a surjective morphism on $\text{MaxSpec}(R)$, so it is faithfully flat. Thus, it is sufficient to check this result when R is local, as long as we verify that products of perfectoid rings are local. Further, completion on a local Noetherian ring is faithfully flat, so we can reduce to the case where $R = (\mathrm{R}, \mathfrak{m}, \mathfrak{K})$ is complete local. Further, take the perfect closure $K/\mathfrak{K}$. Via a fundamental result in EGA, we know that there exists a faithfully flat extension of R with residue field K, so we can pass to this extension. Thus, we say that R is complete local with perfect residue field. Thus we have 3 cases:

- If R is characteristic p, we've tackled this already. In this case, Cohen Structure theorem says that $R \cong \mathfrak{K}[[x_1, \dots, x_d]]$.
- In the unrammified case, $R \cong W[[x_2, \dots, x_d]]$ for $W = W(k)$.
- If there is some ramification, $R \cong W[[x_1, \dots, x_d]]/(p - f)$ where $f \in (x_1, \dots, x_d)^2$ and p does not divide f.

In both the latter cases, we can do the computation explicitly. In the unrammified case, we can look at $R[p^{1/p^\infty}, x_2^{1/p^\infty}, \dots, x_d^{1/p^\infty}]$. This is a colimit of adjoining $1/p^n$ roots for each n; for each element in the colimit these correspond to free extensions, and tensors pass through colimits. It follows that the colimit retains flatness. This is not perfectoid, but if we take the p-completion, we should get a perfectoid ring; we just need to verify that p-completion does not break flatness. Let

$$A = [p^{1/p^\infty}, \widehat{x_2^{1/p^\infty}, \dots, x_d^{1/p^\infty}}]_p$$

This is the extension we are after.

Similarly, in the ramified case, consider $A = R[\widehat{x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}}]_p/(p - f)$. This is perfectoid and faithfully flat over R; faithful flatness follows immediately via the proof in the previous case. Thus, we just need to check that A is perfectoid. Well, since R is p-torsion free, we see from faithful flatness, we can conclude that A is p-torsion free. Thus we just need to show that A/p is semi-perfect and $p = \varpi^p u$, where ϖ generates the kernel of Frobenius on A/p. Well, A/p kills the quotient and kills the completion, so you are just left with

$$A/p = \mathfrak{K}[[x_1, \ldots, x_d]][x_1^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]/(f)$$

Thus

$$A^\flat = \widehat{A/p_f}$$

With surjection $W(A^\flat) \twoheadrightarrow A$ given by the θ map. This allows us to choose $\widetilde{f}$ in the preimage of $f \in A$, where $\widetilde{f} = [\widetilde{f} \bmod p] + pv$, which in turn recovers ϖ such that $p = \varpi^p u$. We can also do more with this lift; a direct check shows us that ϖ and u satisfy the desired properties.

□

We initially spent some time discussing examples of Perfectoid rings:

- $\widehat{\mathbb{Z}_p[p^{1/p^\infty}]}_p$
- $\mathbb{Z}_p[[p^{1/p^\infty}, x_1^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]]$
- $\widehat{W(k)[[x_2, \ldots, x_d]][p^{1/p^\infty}, x_2^{1/p^\infty}, \ldots, x_d x_2^{1/p^\infty}]}_p$
- $\widehat{R^+}$, when $(\mathrm{R}, \mathfrak{m}, \mathfrak{K})$ is a complete local domain with $\mathfrak{K}$ perfect.

This last example is quite important; in characteristic p it reduced to checking that R^+ is perfect. Well, it's a domain, so F is injective. Furthermore, the algebraic closure of $\mathrm{Frac}(R)$ is a field of characteristic p, and taking a pth root is the solution to a monic equation over R, so R^+ has pth roots. It follows that F is also surjective, and thus an isomorphism. We'll now discuss a very powerful theorem about perfectoid rings:

Theorem 5.3. *Suppose $R \to S$ is a ring map where R is perfectoid and S is p-complete. Then either the map is finitely presented OR is an integral map up to p-completion.*

Further, $\exists! S_{\text{perfd}}$ which is the "smallest" perfectoid ring over S. By smallest, we mean that for any perfectoid ring A with $f : S \to A$ a ring map, $\exists! \overline{f} : S_{\text{perfd}} \to A$ that makes the expected diagram commute. It can be checked that taking the perectoidization preserves being p-torsion free.

6 Big Cohen Macaulay Modules (BCMs) Exist

Lemma 6.1. ***([A refined] Andre's Flatness Lemma)*** *Let A be a perfectoid ring with $g \in A$. Then $\exists A \to A'$ where A' is perfectoid and faithfully flat mod p^n $\forall n \geq 1$, so that g has a compatible system of p power roots in A'.*

This ends up being quite powerful, and was a key ingredient of Andre's proof of the direct summand conjecture. Essentially, once can extend a ring in an almost faithfully flat way to include p power roots of a given element, in a way that preserves homological properties and being perfectoid.

We'll use this lemma to sketch the proof that BCMs exist.

Proof. Let R be a ring of mixed characteristic over an algebraically closed residue field k. Write $R = S/Q$ where $S = W(k)[[z_1, \dots, z_n]]$ and $p \notin Q$. Let $c = \mathrm{ht}(Q)$. Via prime avoidance, we can choose $f_1, \dots, f_c \in Q$ so that $p, f_1, \dots, f_c$ are a regular sequence in S, and then extend it to a full system of parameters $p, f_1, \dots, f_c, x_2, \dots, x_d$ on S. As a sanity check, note that d is chosen so that $c + d = \dim(S)$. This tells us that $p, x_2, \dots, x_d$ is a system of parameters of $S/(f_1, \dots, f_c)$. As S is Cohen Macaulay, $S/(f_1, \dots, f_c)$ is Cohen Macaulay, so $p, x_2, \dots, x_d$ are a regular sequence. Thus, via dimension reasons, they are also a system of parameters for $S/Q = R$. Choose $g \notin Q$ so that $gQ \subseteq \sqrt{(f_1, \dots, f_c)}$. Such an element exists because Q is a minimal prime over $(f_1 \dots, f_c)$, and is thus an associated prime of $S/(f_1, \dots, f_c)$. Q can then be written as $\mathrm{Ann}_{S/(f_1,\dots,f_c)}(g)$ for some g; just choose this g. Now let

$$S_\infty := S[p^{1/p^\infty}, \widehat{z_1^{1/p^\infty}, \dots,} z_n^{1/p^\infty}]_p$$

This is perfectoid by construction, though it is lacking some roots. By the flatness lemma, there exists a perfectoid extension $S_\infty \to S'_\infty$ so that S'_∞ is a faithfully flat S_∞ mod p^n for any $n \geq 1$, and so that $g, f_1, \dots, f_c$ all have p power roots. One can see that S_∞ is p-torsion free (S_∞ is in fact a flat S-algebra, so since p is a nonzero divisor on S, one can see that it is still a nonzero divisor over the base change. Further, we can see that S'_∞ is p separated; it follows from these two facts that S_∞ must also be p-torsion free. As $p, f_1, \dots, f_c, x_2, \dots, x_d$ are a regular sequence in S, they are also a regular sequence in S_∞. As we can take arbitrary powers of a regular sequence and maintain regularity, it follows that for any choice of e, $p, f_1^{1/p^e}, \dots, f_c^{1/p^e}, x_2, \dots, x_d$ is a regular sequence over S'_∞. Thus, through a bit of work we can show that $p, x_2, \dots, x_d$ are a regular sequence on

$$T := S'_\infty/(\widehat{f_1^{1/p^\infty}, \dots} f_c^{1/p^\infty})_p$$

T is p-torsion free, and modulo p we've killed a radical ideal. We can use this to deduce that, if we are in the case where $Q = (f_1, \dots, f_c)$, then T is the BCM we are looking for. In general, consider the extension of rings

$$T \to T' := \mathrm{Hom}_T((g^{1/p^\infty}), T)$$

It is not clear that T' is a ring; indeed it is a ring under composition. This is because $(g^{1/p^\infty})(g^{1/p^\infty}) = (g^{1/p^\infty})$, so in fact, the image of any map in T' lands back in (g^{1/p^∞}). Thus, you can compose like normal.

In practice, ideals of the form $I^2 = I$ form the basis for ***almost mathematics***. In our case, raising $0^0 = 1$, and $1/p^\infty \approx 0$, so we can view (g^{1/p^∞}) as essentially being the unit ideal. Thus, T' is almost like $\mathrm{Hom}_T(T,T) = T$, so the ring extension isn't supposed to be a particularly large extension. This almost notion is rigorous, though the beyond the scope of the seminar; in a rigorous sense, $T \to T'$ is an almost isomorphism. Further, $T'/(p, x_2, \dots, x_d)T'$ is not (g^{1/p^∞}) almost zero, same as T. However, how trivial this extension may seem, this tweaking allows T' to be killed by Q. The fact that $gQ \subseteq \sqrt{(f_1, \dots, f_c)}$, then $(g^{1/p^\infty})Q \subseteq \sqrt{(f_1, \dots, f_c)} = (f_1^{1/p^\infty}, \dots, f_c^{1/p^\infty})$. Therefore, under the composition $S \to T \to T'$, $Q \mapsto 0$, so T' is an R-algebra!

The final step is referred to Gabber's trick. Let $B = W^{-1}\prod_{\mathbb{N}} T'$, where W is the multiplicative set generated by $(g, g^{1/p}, g^{1/p^2}, \dots)$. T' naturally maps into B, so B is a T'-algebra, and thus an R-algebra. We want to check that $p, x_2, \dots, x_d$ is a regular sequence (not even just an almost regular sequence) on B; this is because we've in essence inverted all the "almost" elements. We also need to check that $B/\mathfrak{m}B \neq 0$, as if it were, B would be trivial by Nakayama. From here we take the $\mathfrak{m}$ completion of B, denoted $\widehat{B}$, and can get an almost perfectoid BCM. $\square$

If $(\mathrm{R}, \mathfrak{m}, \mathfrak{K})$ is a complete local ring and $\{B_\lambda\}$ is a set of perfectoid BCM R-algebra (or equivalently, perfectoid BCM R^+ algebras) then a paper by Ma and Schwede says that we can find a perfectoid BCM R-algebra (equivalently, R^+ algebra) dominating all of them. Further, if this set of algebras form a directed system, we can do it over the colimit $\varinjlim B_\lambda$.

7 A Sketch Of the Direct Summand Theorem

We proved that Big Cohen Macaulay (BCM) modules exist over complete local domains and stated a slight generalization from Ma and Schwede. We'll now sketch a proof of the most notable corollary of this result, the direct summand theorem.

Theorem 7.1. ***(Direct Summand Theorem)*** *Suppose that $(\mathrm{R}, \mathfrak{m}, \mathfrak{K})$ is a regular local ring. Then, R is a splinter, i.e. $\forall R \subseteq S$ module finite extensions, there exists an R-module splitting $S \to R$.*

This is equivalent to checking that $R \to R^+$ is a pure extension. Some facts about splinters:

- all splinters are normal, but it is not true in general that the converse holds (outside of the case where $\mathbb{Q} \subseteq R$, which is funnily enough proved in Atiyah Macdonald).
- If $\widehat{R}$ is a splinter then R is a splinter, however, the converse does not hold in general unless $R \to \widehat{R}$ is a regular map (i.e. when R is a Grothendieck ring or less generally excellent).
- If $p \in \mathfrak{m}$, Then splinters are Cohen Macaulay. This follows from a far more general result of Bhatt that $\widehat{R^+}$ is BCM.

Proof. (Sketch) Without loss of generality we can assume that R is complete, as it is sufficient to check that $\widehat{R}$ is a splinter to verify that R is a splinter. Let $R \subseteq S$ be a module finite extension, where without loss of generality, S is also complete and local. We can do this because S is always complete with respect to the $\mathfrak{m}$-adic topology on R, so we can use the Chinese Remainder Theorem to break S up into products of localizations at finitely primes at R.

Let $x_1, \ldots, x_d$ be a regular system of parameters. $\dim(R) = \dim(S)$, via module finiteness, so we can conclude that this is also a regular system of parameters of S. Via Andre's Theorem, $\exists$ a BCM S-algebra B such that $x_1, \ldots, x_d$ form a regular sequence on B. We claim that $R \to B$ is then faithfully flat; to see this we can use the local criterion[1]:

$$\operatorname{Tor}_1^R(\mathfrak{K}, B) = \mathrm{H}_1(\underline{x}, B) = 0$$

The first equality is tautological and the second follows from the fact that $\underline{x}$ is a regular sequence. Flatness implies purity, and over complete local rings, for any map of R-modules $R \to M$, splitting and purity are equivalent. It follows that $R \to B$ splits. $\square$

8 Test Ideals in Mixed Characteristic

Going forward, we only consider $(\mathrm{R}, \mathfrak{m}, \mathfrak{K})$ where R is a complete local domain of dimension d. R can be either characteristic p or mixed characteristic. Along those lines, we assume that $\mathfrak{K}$ is perfect. Let B be a BCM R^+-algebra. R is ***B*-BCM regular** with respect to B if R is normal $\mathbb{Q}$-Gorenstein and $R \to B$ splits. Let $\tau_B(R) := \operatorname{im}(\operatorname{Hom}(B, R) \to R)$ denote the B test ideal. If evaluation is surjective (i.e. R splits) we easily see that $\tau_B(R) = R$, and moreover, that this condition holds $\iff$ R is BCM B-regular. We say that R is ***BCM regular*** if R is BCM B-Regular $\forall$ perfectoid BCM R^+ algebras. We can construct a more general test ideal

$$\tau_{\mathcal{B}}(R) = \bigcap_{R^+ \subseteq B \text{ BCM}} \tau_B(R)$$

By the extension of Andre's result by Ma and Schwede, we see that $\exists B \in R$ such that $\tau_{\mathcal{B}}(R) = \tau_B(R)$. In fact we can choose this B to be any BCM R^+ algebra that is "sufficiently large". We can also define a similar notion but one that is analogous to F-rationality. Say that B is a BCM R^+-algebra as before. We say that R is **BCM *B*-rational** provided that R is CM and $\mathrm{H}^d_{\mathfrak{m}} R \to \mathrm{H}^d_{\mathfrak{m}} B$ is injective. This is equivalent (via taking the Matlis Dual) to enforcing that R is CM and $\operatorname{Hom}_R(B, \omega_R) \twoheadrightarrow \omega_R$ (note that R has a canonical module because R is complete local and Cohen Macaulay). This suggests the notion of a BCM B-rational test ideal:

$$\tau_B(\omega_R) = \operatorname{im}\left(\operatorname{Hom}_R(B, \omega_R) \to \omega_R\right)$$

It is clear from our discussion above that $\tau_B(\omega_R) = \omega_R$ is equivalent to R being BCM B-rational. Similarly, R is ***BCM rational*** in full generality if it is BCM B-rational for all choices of B, and we can define

$$\tau_{\mathcal{B}}(\omega_R) = \bigcap_{R^+ \subseteq B \text{ BCM}} \operatorname{im}\left(\operatorname{Hom}_R(B, \omega_R) \to \omega_R\right) = \tau_B(\omega_R)$$

---

[1] This is technically only usable when B is finitely generated, or more generally $\mathfrak{m}$-adically separated. B is essentially never the former, but it is the latter. We won't prove this here, though.

For B sufficiently large, as before. One would expect that BCM regularity implies BCM rationality, with equivalence when R is Gorenstein. This is true, and in fact, if you are BCM regular then you are BCM R^+ regular, which is equivalent to $R \to R^+$ being a splinter and R being normal $\mathbb{Q}$-Gorenstein. The converse of this statement, that being BCM R^+ regular implies that you are BCM regular, is known in characteristic p but is unknown in general.

Theorem 8.1. *Let R be a ring of characteristic p. Then BCM Regularity is equivalent to being $\mathbb{Q}$-Gorenstein and (strongly) F-regular. Along those lines,*

$$\tau_{\mathcal{B}}(R) = \tau(R) = \bigcap_{I \subseteq R} (I : I^*)$$

Theorem 8.2. *Let R be a ring of characteristic p. Then BCM Rationality is equivalent to being F-rational. Along those lines,*

$$\tau_B(\omega_R) = \tau_{R^+}(\omega_R) = \tau(\omega_R) = \left(\mathrm{H}^{\dim(R)}_{\mathfrak{m}} R \middle/ 0^*_{\mathrm{H}^{\dim(R)}_{\mathfrak{m}} R} \right)^{\vee} = \tau_S(\omega_R)$$

For S sufficiently large.

Suppose that R is $\mathbb{Q}$-Gorenstein, (possibly) normal, and Cohen Macaulay. We have the following dictionary across multiple characteristic choices.

equicharacteristic p	mixed characteristic	equicharacteristic 0
F-regularity	BCM Regular	KLT
$\tau(R)$	$\tau_{\mathcal{B}}(R)$	$\mathcal{J}(R)$
F-rationality	BCM rational	Rational
$\tau(\omega_R)$	$\tau_{\mathcal{B}}(\omega_R)$	$\mathcal{J}(\omega_R)$

Where $\mathcal{J}(R)$ is the multiplier ideal, and $\mathcal{J}(\omega_R)$ is the multiplier submodule. We now consider some examples:

- Take the Fermat hypersurface $R = \frac{\mathbb{Z}_p[[y_2,\dots,y_n]]}{(p^m + y_2^m + \cdots + y_n^m)}$ for $p \gg 0$ and $m < n$.
- $R = \frac{\mathbb{Z}_p[[x,z]]}{(x^2 + p^2 z + z^3)}$ for $p > 5$ is never strongly F-regular, as it is not even normal. This is a rational double point (RDP) in mixed characteristic
- Any $\mathbb{Q}$-Gorenstein direct summand of a regular ring is BCM regular as a direct consequence of the direct summand theorem.
- Every log regular ring is isomorphic (via a theorem of Kato) to a ring that looks like

  $$\frac{W(k)[[\mathcal{M}]]}{(p - f)}$$

  For $\mathcal{M}$ some monoid and $f \in I_{\mathcal{M}} + (p^2)$, for $I_{\mathcal{M}}$ the ideal of monomials inside the monoid.

9 Multiplicities in Mixed Characteristic

We'll now add a bit of additional setup to state the next result. Via a structure theorem of Cohen Gabber, for $(\mathrm{R}, \mathfrak{m}, \mathfrak{K})$ local, $\exists (A, \mathfrak{m}_A, K)$ such that $A \hookrightarrow R$ is module finite and generically separable. If we are in characteristic p, $A = k[[x_1. \ldots, x_d]]$, and if we are in mixed characteristic, $A = W(k)[[x_2, \ldots, x_d]]$, for $x_1 = p$. $A_\infty := A[\widehat{x_1^{1/p^\infty}, \ldots_d^{1/p^\infty}}]$ is perfectoid and agrees with A_{perf} in characteristic 0. Set

$$R^{A_\infty}_{\text{perfd}} := (R \otimes_A A_\infty)_{\text{perfd}}$$

$R \otimes_A A_\infty$ is a finite A_∞-module, hence module finite over a perfectoid ring. Thus the perfectoidization is a ring in earnest (no derived nonsense), so $R^{A_\infty}_{\text{perfd}}$ is a ring.

We now define a notion of normalized length (due to Faltings) for $\mathfrak{m}_A$-power torsion A_∞-Modules. For such a module M, define $\lambda_\infty(M) \in \mathbb{R}_{\geq 0}$ as follows:

- If M is finitely presented, M is defined after taking only finitely many roots over $A_e = A\left[x_1^{1/p^e}, \ldots, x_d^{1/p^e}\right]$, with corresponding module M_e such that $M = M_e \otimes_{A_e} A_\infty$. Then,
$$\lambda_\infty(M) := \frac{1}{p^{ed}} \ell_{A_e}(M_e)$$
- if M is finitely generated. If it is not finitely generated, the presentation is infinitely generated. In this case,
$$\lambda_\infty(M) := \inf\left\{\lambda_\infty(M') \mid M' \twoheadrightarrow M, M' \text{ finitely presented}\right\}$$
- For Arbitrary M,
$$\lambda_\infty(M) = \sup\left\{\lambda_\infty(M') \mid M' \subseteq M, M' \text{ finitely generated}\right\}$$

One needs to check that $\lambda_\infty(-)$ is well defined and, like length functions should be, additive on short exact sequences.

Theorem 9.1. *Take the setup above when R is characteristic p. Choose $I \subset R$ where $\ell_R(R/I)$ is finite. Then*

$$\lambda_\infty\left(R_{perf}/IR_{perf}\right) = \mathrm{HK}(R; I)$$

The Hilbert Kunz multiplicity of R. Further, taking $I_\infty := \{x \in R_{perf} \mid R \to R_{perf} \text{ not split}\}$, then

$$\lambda_\infty(R_{perf}/I_\infty) = s(R)$$

The F-signature of R.

Suppose R is mixed characteristic with system of parameters $\underline{x}$. Define

$$e_{\mathrm{HK}}^{\underline{x}}(I) := \lambda_\infty\left(R_{\mathrm{perfd}}^{A_\infty}\Big/IR_{\mathrm{perfd}}^{A_\infty}\right)$$

And similarly setting $I_\infty := \{x \in R_{\mathrm{perfd}} \mid R \to R_{\mathrm{perfd}} \text{ not split}\}$, we define

$$S_{\mathrm{perfd}}^{\underline{x}}(R) = \lambda_\infty\left(R_{\mathrm{perfd}}^{A_\infty}/I_\infty R_{\mathrm{perfd}}^{A_\infty}\right)$$

These are the mixed characteristic analogues of HK multiplicity and F-signature. Proving even basic facts about these is quite hard.

Theorem 9.2. *When R is $\mathbb{Q}$-Gorenstein,*

$$S_{perfd}^{\underline{x}}(R) > 0 \iff \textit{BCM regular}$$