Preliminary Info

These are notes typed up Vignesh Jagathese (me!) during my time at the SLMath (formerly MSRI) summer school at the University of Notre Dame on *Commutative Algebra and its Interaction with Algebraic Geometry*. This class took place between May 22nd and June 2nd, 2023 and was organized by

- Steven Cutkosky (University of Missouri)
- Claudia Polini (University of Notre Dame)
- Claudiu Raicu (University of Notre Dame)
- Steven Sam (University of California, San Diego)
- Kevin Tucker (University of Illinois at Chicago)

You can find additional information by clicking this, though the link may be out of date and/or dead by the time you are reading this. There were a number of multi-day lecture series diving deeper into a wide range of topics as well as "one-off" colloquium talks giving a broad overview of the topic. I've assigned 1 chapter each for talks of the former variety, and have compiled my notes into a single chapter, split section by section, for talks of the latter variety. Some of the colloquium talks were quite a bit faster than the long-form lectures, and thus the notes may be a bit lower quality and/or missing some parts. If a word is being defined, it will be written ***like this***, and objects are defined via the $:=$ (or $=:$) notation.

Open Problem 0. A large focus of these lectures was to introduce open problems to the audience. As such, I've highlighted open problems with this formatting.

I may have missed a couple open problems and I've definitely missed a few citations, though I have tried my best to credit people's work. I can also provide assurance that all errors here were caused by me and not by whoever was giving the lecture. As such,

$$\textbf{reading these notes is allowed} \iff \textbf{you submit any typos/errors to me at vjagat2 (at) uic (.edu).}$$

Contents

Chapter 1

Syzygies

These classes were taught by Daniel Erman.

1.1 Lecture 1

These lectures are intended to be a broad overview of the subject, and thus will lack a noteworthy number of proofs, with a focus towards main ideas. For proofs of the statements given, please look at

- Commutative Algebra - David Eisenbud (Chapter 17-20)
- Geometry of Syzygies - David Eisenbud (Chapter 1-4)
- Graded Syzygies - Ireena Peeva

Let k be an algebraically closed field. Syzygies do not necessarily need to be defined over an algebraically closed field, but when applying these constructions to algebraic geometry, the algebraic closure is often needed. Let $S = k[x_0, \ldots, x_n]$ be the polynomial ring associated to $\mathbb{P}^n_k$ (referred to as $\mathbb{P}^n$ when not ambiguous) with the standard grading. Let $m = (x_0, \ldots, x_n)$ be a maximal ideal of S, and let $S_d = (x_0^d, x_0^{d-1}x_1, \ldots, x_n^d)$ be the vector space of homogeneous degree d polynomials over S.

1.1.1 Graded Modules

$I \subseteq S$ is a ***graded ideal*** if it can be generated by homogeneous elements (note that "graded" and homogeneous" may be used interchangeably). For example, $I = (x^2 + y^2, x^3, y^3)$ is graded. Ideals, as readers may know, show up everywhere, and invariants of ideals are absolutely crucial to the study of algebra and algebraic geometry. One such invariant is the ***Hilbert Function*** $HF_{S/I}(d) := \dim_k(S/I)^d$, which is then used to define the ***Hilbert Series*** $HS_{S/I} := \sum_{d \geq 0} HF_{S/I}(d)t^d \in \mathbb{Z}[[t]]$.

Let's work out an explicit example where $I = (x^3 + y^3)$.

d	0	1	2	3	4	...
$\dim(S/I)_d$	1	2	3	3	3	3
basis	1	x, y	x^2, xy, y^2	x^3, x^2y, xy^2	x^4, x^3y, x^2y^2	...

For $d = 3$, y^3 is not a generator, since in S/I there exists a relation $x^3 = -y^3$. Similarly, some generators in $d = 4$ also vanish, and a simple computation shows that the dimension remains 3. It follows that

$$HF_{S/I}(d) = \begin{cases} 1 & d = 0 \\ 2 & d = 1 \\ 3 & d \geq 2 \end{cases}$$

And

$$HS_{S/I} = 1 + 2t + 3t^2 + 3t^3 + 3t^4 + 3t^5 + \dots$$

Theorem 1.1.1. *There exists a polynomial $P_{S/I}(z) \in \mathbb{Q}[z]$, called the* ***Hilbert Polynomial****, where $P_{S/I}(d) = HF_{S/I}(d)$ for sufficiently large $d \gg 0$.*

In fact, $\deg P_{S/I}(z) + 1 = \dim S/I$. One result we (may) get to today is showing that, similarly, the Hilbert Series eventually agrees with a rational function.

All of this naturally extends to where S/I is any S-module M. Recall that we can decompose M by degree; $M = \bigoplus M_j$ where M_j is a k-vector space and $S_j \cdot M_k = M_{j+k}$. Graded S-modules naturally form a category; let M and N be graded S-Modules and $f : M \to N$ an S-module homomorphism. f is ***homogeneous*** (and thus, a morphism of graded modules) if $f(M_e) \subseteq N_e$, i.e. it respects the gradings.

For $d \in \mathbb{Z}$, we can ***twist*** S as follows. Let $S(d)$ be the S-modules where $S(d)_i = S_{i+d}$ (i.e. we shift the grading by d). For example, we can define a graded morphism $S(-3) \to S$ by multiplying by a (homogeneous) degree 3 term, but only by this. Otherwise, the gradings will not agree.

1.1.2 Minimal Free Resolutions

A ***minimal free resolution*** of M is a free complex

$$F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \longleftarrow \dots$$

where for any $i \in \mathbb{N}$,

- F_i is a graded free module
- d_i are all homogeneous maps
- $\operatorname{coker}(d_1) = M$
- $\operatorname{im}(d_{i+1}) = \ker(d_i)$

- $\mathrm{im}(d_i) \subset mF_{i-1}$

The final condition is precisely the minimal condition; if it did not hold, we can do row and column operations on the matrix representing d_i to show we can drop the rank. In particular, every d_i when written as a matrix will have polynomial terms, with no units.

Theorem 1.1.2. ***(Hilbert Syzygy Theorem, 1890)*** *Any minimal free resolution of a finitely generated S-module has length $\leq \dim(S) = n+1$.*

In particular, we can write any minimal free resolution as

$$F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_2 \longleftarrow \dots \longleftarrow F_{n+1} \longleftarrow 0$$

We also have the following (Meta) Theorem:

Theorem 1.1.3. *Theorems about local rings also hold for graded rings, and theorems about regular local rings also hold for the graded polynomial ring.*

For example, consider $S = k[x,y], I = (x^2, xy, y^2)$. We want to write down a minimal free resolution of S/I. We have a natural surjection $S/I \twoheadleftarrow S^1$ with kernel mapping from $S^3(-2)$, defined by a matrix of the generators $[x^2\ xy\ y^2]$. This has a map from $S^2(-3)$ which has a specific form as follows:

$$S/I \twoheadleftarrow S^1 \xleftarrow{\begin{bmatrix} x^2 & xy & y^2 \end{bmatrix}} S^3(-2) \xleftarrow{\begin{bmatrix} y & 0 \\ -x & y \\ 0 & -x \end{bmatrix}} S^2(-3) \longleftarrow 0$$

A ***syzygy*** is a relation on the columns of a matrix. For instance, the above is a syzygy as $y \cdot [x^2] - x \cdot [xy] + 0 \cdot [y^2] = 0$.

1.1.3 Betti Tables

Let $F_\bullet$ be a minimal free resolution of M. $\beta_{ij}(M)$ is the number of degree j generators of F_i, i.e. where $F_i = \bigoplus S(-j)$. $\beta(M)$ is the corresponding table of the form

$$\beta(M) := \begin{pmatrix} \beta_{00} & \beta_{11} & \beta_{22} & \cdots \\ \beta_{01} & \beta_{12} & \beta_{23} & \cdots \\ \beta_{02} & \beta_{13} & \beta_{24} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Let's compute the Betti table for the above examples. Here, $\beta_{00} = 1$ as S^1 0 degree component is generated by 1 elements. $S^3(-2)$ is generated by 3 generators of degree 2, so $\beta_{12} = 3$. $S^2(-3)$ has 2 degree 3, so $\beta_{23} = 2$. Thus,

$$\beta(S/I) = \begin{pmatrix} 1 & - & - \\ - & 3 & 2 \end{pmatrix}$$

1.2 Lecture 2

Let $X \subseteq \mathbb{P}^n$ be a projective variety with defining (homogeneous) ideal $I_X \subset S = k[x_0, \dots, x_n]$. Let $S/I_X \leftarrow F_0 \leftarrow \cdots \leftarrow F_p \leftarrow 0$ be a minimal free resolution of the coordinate ring, where $F_i = \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{ij}(S/I_X)}$. p is a well chosen variable choice, since it denotes the ***projective dimension*** of the module. This can be recovered from the number of columns of the Betti Table.

1.2.1 Recovering Geometry

For today, we'll study the geometry of X that we can see in $F_\bullet$, or even just from the Betti numbers. Here are some questions that one would hope to answer:

- What is the dimension of X?
- What is the (arithmetic) genus of X?
- What is the Hilbert Series of S/I_X?
- Is S/I_X Cohen Macaulay? (Or Gorenstein?)
- Is X a complete intersection of hypersurfaces?
- Which (local/sheaf) cohomology of groups of (S/I_X / $\mathcal{O}_X$) are nonzero?
- What is the Castelnuovo-Mumford Regularity of X?
- Are the deformations of $X \subset \mathbb{P}^n$ unobstructed?

Recall that $HS_{S/I_X} = \sum_{i=1}^p (-1)^i HS_{F_i}$. Notice that $HF_{M\oplus N}(\ell) = \dim(M \oplus N)_\ell = \dim M_\ell + \dim N_\ell = HF_M(\ell) + HF_N(\ell)$. Combining these facts with the fact that $F_i = \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{ij}(S/I_X)}$ yields

$$HS_{S/I_X} = \sum_{i=1}^p (-1)^i HS_{F_i} = \sum_{i=1}^p (-1)^i \sum_{j\in\mathbb{Z}} \beta_{ij}(S/I_X) HS_{S(-j)} =^* \sum_{i=1}^p (-1)^i \sum_{j\in\mathbb{Z}} \beta_{ij}(S/I_X) \frac{t^j}{(1-t)^{n+1}}$$

Where $=^*$ follows from the exercises. It follows that the Betti numbers completely recover the Hilbert Series, therefore recovering the Hilbert polynomial, which we know recovers $\dim(X)$ and the arithmetic genus of X.

To check S/I_X is Cohen Macaulay, the Auslander-Buchsbaum theorem tells us that S/I_X is CM $\iff$ $\text{codim}(S/I_X) = \text{ProjDim}(S/I_X)$. Projective dimension is recovered from the the Betti table, and the codimension is recovered from the dimension of X above. Once you know that S/I_X is Cohen Macaulay, you just need to verify that F_p is rank 1 to confirm that S/I_X is Gorenstein. Equivalently, you can check that the Betti Table is symmetric. Either way, both of these invariants are readily verifiable by looking at the Betti numbers.

1.2.2 Regularity

Theorem 1.2.1. *(Eisenbud, Goto 1984) The following are equivalent for some $r \in \mathbb{N}$ and all i:*

(1) F_i is generated in degree $\leq r+i$ (i.e. $\beta_{ij} = 0$ for $j > r+i$)

(2) $\left(\mathrm{H}^i_m(S/I_X)\right)_j = 0$ for $j \geq r-i+1$

(3) $(S/I_X)_{\geq r}$ has a linear resolution (i.e. the corresponding Betti Table has 1 row)

Here H^i_m denotes local cohomology taken at the maximal ideal $m = (x_1, \ldots, x_n)$. Condition (2) is equivalent (loosely) to the fact $\mathrm{H}^{n-i}(X, \mathcal{O}_X(j)) = 0$ for $j \geq r-i+1, i \geq 2$. We define the ***Castelnuovo-Mumford Regularity*** as the minimal r for which any/all of (1), (2), or (3) hold.

Let C be a smooth genus g curve, and $\mathcal{L}$ a very ample line bundle on C. Let $C \hookrightarrow \mathbb{P}^n$ be an embedding into $\mathbb{P}^n$ via $\mathrm{H}^0(C, \mathcal{L})$. For an example pf this, letting $C = \mathbb{P}^1, n = 3, L = \mathcal{O}(3)$, the twisted cubic is defined by the embedding $\mathbb{P}^1 \to \mathbb{P}^3$ where $[s:t] \mapsto [s^3 : s^2t : st^2 : t^3]$.

Theorem 1.2.2. *If $\deg(\mathcal{L}) \geq 2g+2$, then I_C is generated by quadrics.*

If S/I_C is normal, we say that C is N_0. C satisfies the N_1 condition if it satisfies the N_0 condition and I_C is generated by quadrics (equivalently, F_1 in the free resolution is a bunch of copies of $S(-2)$). Further, it satisfies the N_2 condition if N_1 holds and F_2 is generated entirely in degree 3. We then recursively define N_p conditions this way.

Theorem 1.2.3. *(Green, 1984) If $\deg \mathcal{L} \geq 2g+1+p$, then C satisfies the N_p condition.*

1.3 Lecture 3

1.4 Weighted Projective Space

As before, take $Z \subset \mathbb{P}^n$ to be a projective variety with defining ideal I_Z and minimal free resolution $F_\bullet$. Today, we'll try to loosen the initial assumptions we've been working over and see what breaks. For instance, we can choose $Z \subseteq X \neq \mathbb{P}^n$ for some X. For instance, we can consider

- $\mathbb{P}^n \times \mathbb{P}^m$
- $\mathbb{P}(d_0, \ldots, d_n)$, or ***weighted projective space***. One can define this by $(\mathbb{A}^{n+1} \backslash 0)/\sim$, similar to normal projective space, but we just vary the equivalence. For normal projective space, $a \sim b$ if $a = \lambda b$ for some $\lambda \in k^*$. In this case, we weight the coordinates with respect to the d_i, so

$$(a_0, \ldots, a_n) \sim (\lambda d_0 a_0, \ldots \lambda d_n a_n)$$

  Where $\lambda \in k^*$ as before and $\gcd(d_0, \ldots, d_n) = 1$.

Algebraically, $\mathbb{P}^n \times \mathbb{P}^m$ is defined over the ring $k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ with a $\mathbb{Z}^2$ grading, where $\deg(x_i) = (1,0)$ and $\deg(y_j) = (0,1)$. $\mathbb{P}(d_0, \ldots, d_n)$ is over $k[x_0, \ldots, x_n]$ with the $\mathbb{Z}$ grading $\deg(x_i) = d_i$. Both of these examples have a Nullstellensatz correspondence:

$$Z \subseteq \mathbb{P}(d_0, \ldots, d_n) \text{ closed} \longleftrightarrow P \subseteq S \text{ homogeneous, and } P \not\supseteq m$$

$$Z \subseteq \mathbb{P}^m \times \mathbb{P}^n \text{ closed} \longleftrightarrow P \subseteq S \text{ homogeneous, and } P \not\supseteq (x_0, \ldots, x_n) \cap (y_0, \ldots, y_m)$$

Notice that $(x_0, \ldots, x_n) \cap (y_0, \ldots, y_m)$ is the new irrelevant ideal in the product case, while weighted projective space retains a number of properties from normal projective space (just with a non-standard grading).

Geometrically, a number of interesting questions arise:

(1) Is $Z \subseteq \mathbb{P}^n \times \mathbb{P}^m$ a complete intersection?

(2) Which local/sheaf cohomology groups are zero?

(3) Are there (strong) analogues of Hilbert's Syzygy Theorem, Eisenbud-Goto's Theorem, or Green's N_p theorem?

In non-standard graded cases, we can answer (3) (mostly) in the affirmative, however you need to deform the statements and basic definitions.

Theorem 1.4.1. ***(Symonds)*** *Let $X = \mathbb{P}(d_0, \ldots, d_n)$, i.e. $\deg(x_i) = d_i$ at the level of coordinate rings, and $I \subseteq S$ the defining ideal. let $F_\bullet$ be the minimal free resolution of S/I. Then*

$$\mathrm{H}^i_m(S/I) = 0 \text{ for } j \geq r - i + 1, \forall i \iff F_i \text{ is generated in degree } \leq r + i + \sum_{i=0}^{n} (d_i - 1)$$

The theorem, unlike the Eisenbud-Goto theorem in the standard case, says nothing about truncated resolutions being linear. This is primarily because, well, we'd need to redefine our notion of linear in the case where different variables take on different degrees in the grading. Fortunately, however, we can see that when $d_i = 1\ \forall i$, this equivalence reduces to the first two statements in the Eisenbud-Goto theorem being equivalent.

We'll now define a notion of linearity. Suppose $G_\bullet$ is a graded free complex of S-modules, where $S = k[x_0, \ldots, x_n]$ with the grading $\deg(x_i) = d_i$.

- $G_\bullet$ is ***strongly linear*** if $\exists$ a basis where every entry of differentials is a k-linear combination of the x_i.
- $G_\bullet$ is ***Koszul linear*** if the Betti Table of $G_\bullet$ equals the Betti Table of some strongly linear free complex.

It naturally follows that strong linearity implies Koszul linearity. Here are some examples of linear syzygies:

Syzygy	Strongly Linear?	Koszul Linear?
$S \xleftarrow{x} S(-1)$	Y	Y
$S \xleftarrow{y} S(-2)$	Y	Y
$S \xleftarrow{y-x^2} S(-2)$	N	Y
$S \xleftarrow{x^2} S(-2)$	N	Y
$S \xleftarrow{[x\ y]} S(-1) \oplus S(-2) \xleftarrow{\begin{bmatrix} y \\ -x \end{bmatrix}} S(-3)$	Y	Y
$S \xleftarrow{[x\ y-x^2]} S(-1) \oplus S(-2) \xleftarrow{\begin{bmatrix} x^2-y \\ x \end{bmatrix}} S(-3)$	Y	Y

The last two syzygies are isomorphic via base change. The first one of the two is strongly linear, so the second one must be. The first two are clearly also strongly linear, so the 1st, 2nd, 5th, and 6th are all strongly linear and thus Koszul linear. The 3rd and 4th syzygy are not strongly linear, but they have the same Betti Table as the first/second syzygy, so they are both Koszul Linear.

1.5 Lecture 4

1.5.1 Varieties up to B-Torsion

At the end of the last day (and in the exercises in the following homework session) we discussed the example of two points $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$. This is cut out by the ideal $I_Z = (x_0x_1, x_0y_0, x_1y_1, y_0y_1)$, but up to saturation by the irrelevant ideal $(x_0, x_1) \cap (y_0, y_1)$ one can instead cut out Z via the ideal $J = (x_0y_0, x_1y_1)$. These are "sheaf theoretically" the same, as saturation does not seriously affect the scheme structure. Said more explicitly, we can see that $\mathcal{O}_Z = \widetilde{S/J} = \widetilde{S/I_Z}$; B saturation "gets rid of the noise" and leaves you with a simpler algebraic object that contains the same geometric information. At the level of syzygies, we can see that J induces a much simpler one:

$$S/I_Z \leftarrow S \leftarrow S(-2,0) \oplus S(-1,1)^2 \oplus S(0,-2) \leftarrow S(-2,-1)^2 \oplus S(-1,-2) \leftarrow S(-2,-2) \leftarrow 0$$

$$S/J \leftarrow S \leftarrow S(-1,1)^2 \leftarrow S(-2,-2) \leftarrow 0$$

The moral of this is that B-torsion can sometimes be good; it lets you simplify the situation you are in. We can in fact construct a ***virtual resolution*** of S/I_Z, which is a $\mathbb{Z}^r$-graded free complex of S-modules $F_\bullet$ such that $\widetilde{F}_\bullet \to \widetilde{S/I_Z}$ is a resolution of sheaves. Said simpler, $F_\bullet$ may not be a resolution (due to possible B-Torsion), but, at the level of sheaves, it "geometrically" resolves S/I_Z.

We'd like to govern the length of these virtual resolutions: To do so we recall the following theorem

Theorem 1.5.1. ***(Hilbert Syzygy Theorem)*** *The minimal free resolution of S/I_Z has length $\leq \dim(S) = n+1$.*

It was a conjecture (Erman, Berkesch and Greg Smith) that $\forall Z$, $\exists$ a virtual resolution $V_\bullet$ of S/I_Z with length $\leq \dim X = \dim S - r$. Interestingly, the example of a virtual resolution above achieves this bound. For more examples,

Space	$\mathbb{P}^5$	$\mathbb{P}^3 \times \mathbb{P}^3$	$(\mathbb{P}^1)^{20}$
Hilbert Bound	6	8	40
Virtual Bound	5	6	20

The conjecture above is implied by the following theorem (proved this year!)

Theorem 1.5.2. ***(Hanlon, Hicks, Lazarev 2023)*** *Suppose X is a projective simplicial toric variety and S the cox ring of $k[x_0, \ldots, x_n]$ with a $\mathbb{Z}^r$ grading. Suppose $Z \subset X$ is a toric subvariety. Then S/I_Z has a virtual resolution of length equal to* $\operatorname{codim}_X(Z)$.

It is not clear that this implies the conjecture. As a sketch of how this works, you view Z embedded into the diagonal $\Delta(X \times X)$ and do a (supposedly) standard Fourier-Mukai transform argument that is well beyond the scope of this course. There are, however, two open questions associated to the conjecture (now theorem) above that have not been solved:

Open Problem 1. Can we always choose this (minimal) virtual resolution $V_\bullet$ to be acyclic?

Open Problem 2. Is there a virtual notion of the Auslander-Buchsbaum formula? *(stating that projective dimension is precisely the number of variables minus the depth of the maximal ideal)*

There is a variant of the Eisenbud-Goto Theorem for Virtual Resolutions as well:

Theorem 1.5.3. *The following are equivalent for some $r \in \mathbb{N}$ and all i:*

(1) V_i *is generated in degree* $\leq r + i$ *(i.e.* $\beta_{ij} = 0$ *for* $j > r + i$*)*

(2) $\left(\mathrm{H}^i_B(S/I_X)\right)_j = 0$ *for* $j \geq r - i + 1$

(3) $(S/I_X)_{\geq r}$ *has a quasi-linear resolution (i.e. the corresponding Betti Table has* 1 *row)*

Erman, Berkesch and Greg Smith proved the equivalence of (1) and (2) and the equivalence of (2) and (3) was proven later by Juliette Bruce, Lauren Cranton Heller, and Mahrud Sayrafi. The main differences are that this detects virtual resolutions, local cohomology is taken over a generalized variant of the irrelevant ideal, and finally that we need a new notion of linearity for specifically virtual resolutions.

Open Problem 3. Much of the work done over the first three lectures should be able to be reproved in the language of virtual resolutions, but much of it has not been yet. If you're looking for an open problem in this realm, that would be a great place to start!

Chapter 2

$\mathcal{D}$-Modules

These classes were mostly taught by Anurag Singh, with the last lecture given by Jack Jeffries.

2.1 Lecture 1

2.1.1 Weyl Algebra

Let $D_R^0 := \text{Hom}_R(R, R) \subseteq \text{Hom}_{\mathbb{Z}}(R, R)$ where R is a commutative ring. In general,

$$D_R^k := \left\{\delta \in \text{Hom}_{\mathbb{Z}}(R, R) \mid [\delta, \varphi] \in D_R^{k-1} \ \forall \varphi \in D_R^0\right\}$$

Let $D_R := \bigcup_k D_R^k$ denote the ***Weyl Algebra***. If R is an A-algebra, $D_{R/A} = \{\delta \in D_R \mid \delta \text{ is } A\text{-linear}\}$. For example, if $R = \mathbb{C}[x]$ and $f \in R$, $\frac{\partial}{\partial x}$ ought to be a differential operator. To check this, we compute the bracket.

$$\left[\frac{\partial}{\partial x}, f\right](g) = \left(\frac{\partial}{\partial x} \cdot f - f \cdot \frac{\partial}{\partial x}\right)(g) = g \cdot \frac{\partial f}{\partial x}$$

i.e. $\left[\frac{\partial}{\partial x}, f\right] = \frac{\partial f}{\partial x} \in D_R^0$, so $\frac{\partial}{\partial x} \in D_R^1$ as desired. In particular, for $\partial = \frac{\partial f}{\partial x}$, $[\partial, x] = 1$. From this, we see that $D_{R/\mathbb{C}} = \mathbb{C}\langle x, \partial\rangle / \langle \partial x - x\partial - 1\rangle$. It's important to know that the symbols $\langle$ and $\rangle$ denote "generated" by in a free algebra sense, so $D_{R/\mathbb{C}}$ importantly is NOT commutative. In particular, $A := \mathbb{C}\langle x, y\rangle$ is NOT (left) Noetherian, as $\sum_{k \geq 0} Axy^k$ is NOT a finitely generated ideal. However, we'd like to eventually show that $D_{R/\mathbb{C}}$ is in fact left Noetherian. More generally,

$$D_{\mathbb{C}[x_1,\ldots,x_n]/\mathbb{C}} = \frac{\mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle}{\left\{\begin{smallmatrix} x_ix_j - x_jx_i \\ \partial_i\partial_j - \partial_j\partial_i \\ x_i\partial_j - \partial_jx_i \\ \partial_ix_i - x_i\partial_i - 1 \end{smallmatrix} \middle| 1 \leq i, j \leq n, i \neq j\right\}}$$

2.1.2 $\mathcal{D}$-Modules

We can now consider modules over $D_{R/\mathbb{C}}$. Suppose that M is such a module, but with finite $\mathbb{C}$-rank. x and ∂ are linear operators on M and thus have a notion of trace. As in the standard case, we'd hope that $\text{Tr}(\partial x) = \text{Tr}(x\partial)$, in which case $\text{Tr}[\partial, x] = 0$. Well, $\text{Tr}[\partial, x] = \text{Tr}\,\text{Id} = \text{rank}_{\mathbb{C}} M$ by our hypothesis, so it follows that $\text{Tr}(\partial x) = \text{Tr}(x\partial)$ precisely when $M = 0$, and thus we can conclude that all nonzero $D_{R/\mathbb{C}}$-Modules have infinite $\mathbb{C}$-rank.

We now move on to our goal of proving that $\mathcal{D} := D_{R/\mathbb{C}}$ is left Noetherian. $\mathcal{D}$ has a $\mathbb{C}$-basis of the form $\{x_1^{i_1} \ldots x_n^{i_n} \partial_1^{j_1} \ldots \partial_n^{j_n}\}$. This is known as the Poincare Birkhoff Witt (PBW) basis. For a given basis element, we say it has degree $\sum_{k=1}^{n} i_k + j_k$. For $P \in \mathcal{D}$, $\deg(P)$ is the longest degree of a PBW monomial (i.e. represent P in its PBW basis, and assign the degree of P to be the degree of the largest monomial present).

- $\deg(x^2\partial^3 + 17x^2 + 3\partial) = 5$
- $\deg(\partial x - x\partial) = 0$ (since $\partial x - x\partial$ written in a PBW basis is just 1!)

$\mathcal{D}$ has a ***Bernstein Filtration*** of the form $\mathcal{F}_t := [\mathcal{D}]_{\leq t}$, where it's fairly clear that $[\mathcal{F}_i, \mathcal{F}_j] \subseteq \mathcal{F}_{i+j-2}$. The "$-2$" term is fairly tautological, as $[\partial, x] = 1$ implies that $[\mathcal{F}_1, \mathcal{F}_1] \subseteq \mathcal{F}_0 = \mathcal{F}_{1+1-2}$. We see that $\mathcal{F}_\bullet$ is a filtration of $\mathcal{D}$, and in fact, the $\mathbb{C}$-rank of $\mathcal{F}_t$ is finite $\forall t$ and $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots$. $\mathcal{F}_0 = \mathbb{C}$, and $\bigcup_i \mathcal{F}_i = \mathcal{D}$. One can also verify that $\mathcal{F}_i \cdot \mathcal{F}_j \subseteq \mathcal{F}_{i+j}$. The filtration allows us to consider the following graded ring:

$$\text{gr}(\mathcal{D}) := \mathcal{F}_0 \oplus \frac{\mathcal{F}_1}{\mathcal{F}_0} \oplus \frac{\mathcal{F}_2}{\mathcal{F}_1} \oplus \ldots$$

Notice that if we multiply an element from $\mathcal{F}_i$ with something from $\mathcal{F}_j$, I get something in $\mathcal{F}_{i+j}$, even if I multiply them in the opposite order. The difference of these two must be in $\mathcal{F}_{i+j-2}$ via the logic above, so the difference must be 0. It follows that, surprisingly, $\text{gr}(\mathcal{D})$ is a commutative ring! In particular, When $R = \mathbb{C}[x_1, \ldots, x_n]$, $\text{gr}(\mathcal{D}) = \mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$.

In general a filtration on a D-module M, denoted $\mathcal{G}_\bullet$ is a sequence satisfying the following properties:

- $\text{rank}_{\mathbb{C}} \mathcal{G}_t$ is finite for all $t \geq 0$.
- $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \ldots$
- $\bigcup_{t \geq 0} \mathcal{G}_t = M$
- $\mathcal{F}_i \cdot \mathcal{G}_j \subseteq \mathcal{G}_{i+j}$

It follows for any filtration $\mathcal{G}_\bullet$,

$$\text{gr}(M) := \mathcal{G}_0 \oplus \frac{\mathcal{G}_1}{\mathcal{G}_0} \oplus \frac{\mathcal{G}_2}{\mathcal{G}_1} \oplus \ldots$$

is a graded module over $\text{gr}(\mathcal{D})$.

Lemma 2.1.1. *If M admits a "good" filtration then M is a finitely generated $\mathcal{D}$-module.*

Proof. If $\text{gr}(M) = \sum_{i=1}^{\ell} (\text{gr}(\mathcal{D}))\, \overline{m}_i$, we claim that $M = \sum_{i=1}^{\ell} \mathcal{D} m_i$. If not, take the minimal t such that $\mathcal{G}_t \subsetneq \sum_{i=1}^{\ell} D m_i$. However, $\mathcal{G}_t \subseteq \sum_{i=1}^{\ell} \mathcal{D} m_i + \mathcal{G}_{t-1} = \sum_{i=1}^{\ell} \mathcal{D} m_i$, a contradiction. □

One would hope the converse is true; indeed if M is a finitely generated D-module, say by $m_1, \ldots, m_\ell$. Then, just define $\mathcal{G}_t := \sum_{i=1}^{\ell} \mathcal{F}_t m_i$. One can verify that this is a good filtration. We can also "relatively filter", so to speak, and let $\mathcal{G}_t := \sum_{i=1}^{\ell} \mathcal{F}_{t-k_i} m_i$, if instead of having all $m_i \in \mathcal{G}_0$ we instead want to let each $m_i \in \mathcal{G}_{k_i}$. Such good filtrations are cofinal, meaning that if $\mathcal{G}_\bullet, \mathcal{G}'_\bullet$ are good, there $\exists s$ such that $\mathcal{G}_t \subseteq \mathcal{G}'_{s+t}\ \forall t \geq 0$. Good filtrations are also, in some sense, as fine as possible, largely due to the prior sentence.

Suppose $\mathcal{G}_\bullet$ is a good filtration on M. Then, $\text{gr}(M)$ is a finitely generated **graded** module over $\text{gr}(\mathcal{D})$ (a graded polynomial ring) in the standard sense, enforced by the fact that $\mathcal{F}_i \cdot \mathcal{G}_j \subseteq \mathcal{G}_{i+j}$. We can see that

$$\text{rank}_{\mathbb{C}} \mathcal{G}_t = \text{rank}_{\mathbb{C}} \mathcal{G}_0 + \text{rank}_{\mathbb{C}} \frac{\mathcal{G}_1}{\mathcal{G}_0} + \text{rank}_{\mathbb{C}} \frac{\mathcal{G}_2}{\mathcal{G}_1} + \ldots$$

We thus recover that $H_M(t) := \text{rank}_{\mathbb{C}} \mathcal{G}_t$ is eventually polynomial in t. Notice that $H_M(t)$ is defined independently of the choice of filtration; indeed since good filtrations are as fine as possible, a change in filtration will at worst only change how large of a t is required for $H_M(t)$ to be polynomial. Thus, $H_M(t)$ is, as desired, independent of this choice. We can see that $\deg H_m(t) = \dim(M) = d$ and that the multiplicity $e(M) = c$ if $H_M(t) = \frac{ct^d}{d!} + \{\text{lower order terms}\}$. Observe that

$$H_{\mathcal{D}}(t) = \frac{t^n}{(2n)!} + \{\text{lower order terms}\}$$

$$H_R(t) = \frac{t^n}{n!} + \{\text{lower order terms}\}$$

So $\dim(\mathcal{D}) = 2n, e(\mathcal{D}) = 1$, and $\dim(R) = n, e(R) = 1$.

2.2 Lecture 2

Via the exercises, we've shown that when $R = \mathbb{C}[x]$, on $S = R[1/x]$ we saw that $H_S(t) = 2t + 1$, so $\dim S = 1, e(S) = 2$. It remains to be seen how this generalizes; i.e. if $R = \mathbb{C}[x_1, \ldots, x_n]$ then what is $e(R[1/x_1, \ldots, 1/x_k])$?

2.2.1 Finite Generation

Lemma 2.2.1. *A submodule of a finitely generated $\mathcal{D}$-Module is itself a finitely generated $\mathcal{D}$-Module.*

Proof. Let M be a finitely generated $\mathcal{D}$-Module with a good filtration $\mathcal{G}_\bullet$. We can then give a submodule $M' \subseteq M$ an induced filtration $\mathcal{G}'_\bullet := M' \cap \mathcal{G}_\bullet$. gr$M'$ is a submodule of gr$M$, so grM' is finitely generated over gr$\mathcal{D}$, as gr$\mathcal{D}$ is Noetherian. Thus, M' is finitely generated over $\mathcal{D}$. □

From this we have the natural corollary:

Lemma 2.2.2. *$\mathcal{D}$ is left-Noetherian.*

In the realm of differential equations, this has a remarkable consequence. If we have a system of differential equations with (possibly infinite size) and polynomial coefficients, the solution set is the same as a finite such system of differential equations. This is a direct application of the Noetherianity of $\mathcal{D}$.

Lemma 2.2.3. *The mapping $M \mapsto grM$ is exact.*

Theorem 2.2.4. ***(Bernstein)*** *Let M be a finitely generated $\mathcal{D}$-Module over $R = \mathbb{C}[x_1, \ldots, x_n]$. Then, $n \leq \dim M \leq 2n$.*

Proof. It is sufficient to check $n \neq \dim(M)$, as the other bound follows from the fact that M is generated by at most $2n$ elements $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$. We first justify the claim that the map $\mathcal{F}_t \to \mathrm{Hom}_{\mathbb{C}}(\mathcal{G}_t, \mathcal{G}_{2t})$ is injective $\forall t \geq 0$. This is certainly true for $t = 0$, but if the claim fails for larger t, there is some minimal t where $\exists$ a nonzero $P \in \mathcal{F}_t$ such that $P\mathcal{G}_t = 0$. If ∂_i occurs in P, then $[P, x_i] \neq 0$ in $\mathcal{F}_{t-1}$. As t was chosen minimally, $\exists m \in \mathcal{G}_{t-1}$ where $[P, x_i]m \neq 0$. $P \circ x_i(m) = 0$ so $x_i m \in \mathcal{G}_t$, and $x_i P(m) = 0$ so $m \in \mathcal{G}_t$, but their difference is precisely $[P, x_i]m \neq 0$, a contradiction. If x_i occurs in P, then repeat the same argument with $[P, \partial_i]$. As $P \neq 0$, either some x_i or ∂_i occurs in P, so the subcases are exhausted. Thus, we can conclude that the claim is true.

$\mathscr{F}_t$ grows like a polynomial of degree $2n$, and both $\mathcal{G}_t$ and $\mathcal{G}_{2t}$ grow like a polynomial of degree $\dim M$ Thus, $\mathrm{Hom}_{\mathbb{C}}(\mathcal{G}_t, \mathcal{G}_{2t})$ grows like a polynomial of degree $2\dim(M)$. Thus by injectivity, comparing dimensions yields $2n \leq 2\dim(M)$, so $n \leq \dim(M)$. □

We say that a finitely generated $\mathcal{D}$-module M is ***holonomic*** if it is dimension either 0 or n (i.e. M achieves its lower dimension bound). This invariant is additive on short exact sequences:

Theorem 2.2.5. *Let*

$$0 \to M' \to M \to M'' \to 0$$

Be a short exact sequence of $\mathcal{D}$-Modules. Then M is finitely generated $\iff$ M', M'' finitely generated, and M is holonomic $\iff$ M', M'' are holonomic. If both of these conditions hold, then $e(M) = e(M') + e(M'')$.

$e(M)$ also limits the length of filtrations of M:

Theorem 2.2.6. *If M is a holonomic $\mathcal{D}$-module then $\ell_{\mathcal{D}}(M) \leq e(M)$, where $\ell_{\mathcal{D}}(M)$ denotes the length of the longest Jordan-Hölder filtration.*

Which has the following corollary:

Lemma 2.2.7. *$e(M) = 1 \Rightarrow M$ is a simple $\mathcal{D}$-Module.*

We have the following neat theorem to check a given $\mathcal{D}$-Module is finitely generated:

Theorem 2.2.8. *Let M be a $\mathcal{D}$-Module with a filtration $\mathcal{G}_\bullet$ such that $\operatorname{rank}_{\mathbb{C}} \mathcal{G}_t \leq ct^m \ \forall t \geq 0$ for some fixed $c \in \mathbb{R}_{>0}$. Then, M is holonomic (and in particular, finitely generated).*

Finite generation can actually be verified at the level of ascending chains of finitely generated submodules; in fact there is the following equivalence:

Lemma 2.2.9. *The following are equivalent:*

- *M is finitely generated.*
- *M satisfies (left) ACC.*
- *M satisfies (left) ACC on finitely generated submodules.*

Being holonomic is preserved by taking localizations.

Theorem 2.2.10. *If M is holonomic and $f \in R$, then M_f is holonomic.*

Proof. We'll provide an outline. Fix a good filtration $\mathcal{G}_\bullet$ on M. We can then define

$$\mathcal{H}_t := \left\{ \frac{m}{f^t} \;\middle|\; m \in \mathcal{G}_{t(1+\deg(f))} \right\}$$

One can check that $\mathcal{H}_0 \subseteq \mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \ldots$ and that $\bigcup_t \mathcal{H}_t = M_f$. All elements of $\mathcal{H}_t$ live in a finite rank space, so $\operatorname{rank}_{\mathbb{C}} \mathcal{H}_t$ is finite. We then check compatibility $\mathcal{F}_i \cdot \mathcal{H}_j \subseteq \mathcal{H}_{i+j}$ (these are all follow your nose proofs for the most part, and are omitted). Thus $\mathcal{H}_\bullet$ is a good filtration. To finish, we bound the rank $\operatorname{rank}_{\mathbb{C}} \mathcal{H}_t \leq c\,(t(1+\deg(f)))^n$, and conclude by a previous theorem. □

2.3 Lecture 3

2.3.1 Local Cohomology

We begin with a theorem characterizing simple $\mathcal{D}$-Modules:

Theorem 2.3.1. *If M is a simple $\mathcal{D}$-Module (i.e. it has no proper left $\mathcal{D}$-submodules) then $|\operatorname{Ass}_R M| = 1$.*

Proof. Let $P = (f_1, \ldots, f_\ell)$ be a maximal element of $\operatorname{Ass}_R M$. Now consider the P-Torsion $\Gamma_P(M) = \{m \in M \mid P^t m = 0 \ \forall t \gg 0\}$. This can be viewed as the kernel of the map $M \mapsto \bigoplus_{i=1}^{\ell} M_{f_i}$, so it is also a $\mathcal{D}$-Module. M is a simple $\mathcal{D}$-Module, so if $\Gamma_P(M)$ is nonzero, then $\Gamma_P(M) = M$, so every element of M is P-torsion; in particular any associated prime of M contains P. However, as P is maximal, all associated primes are contained in P. It follows that $\operatorname{Ass}_R(M) = \{P\}$. □

Holonomic $\mathcal{D}$-Modules are resolved by a composition series, where each quotient is simple. The following lemma then immediately follows from the above theorem:

Lemma 2.3.2. *If M is holonomic, then it has finitely many associated primes.*

From here we began a gentle introduction to local cohomology. I'm going to assume the reader is familiar with some of these basic notions so I'll omit the explanations given. In the context of $\mathcal{D}$-Modules, the upshot is the following theorem of Lyubeznik:

Theorem 2.3.3. *Let $R = \mathbb{C}[x_1, \dots, x_n]$ and $\mathfrak{a} \subset R$ be an ideal. Suppose $k \geq 0$.*

(1) $\mathrm{H}^k_{\mathfrak{a}}(R)$ *is a holonomic $\mathcal{D}$-Module.*

(2) $\mathrm{Ass}_R\, \mathrm{H}^k_{\mathfrak{a}}(R)$ *is finite.*

(3) $\mathrm{InjDim}_R\, \mathrm{H}^k_{\mathfrak{a}}(R) \leq \dim \mathrm{H}^k_{\mathfrak{a}}(R)$

(4) Bass Numbers of $\mathrm{H}^k_{\mathfrak{a}}(R)$ *are finite.*

Oaku, Jakayama, and Walther have shown that $\mathrm{H}^k_{\mathfrak{a}}(R)$ is easily computable (via a computer) over $\mathbb{Q}$.

2.3.2 Positive Characteristic

Suppose p is prime and let $R = \mathbb{F}_p[x]$. Since $\frac{\partial}{\partial x}(x^p) = 0$, we need to be more careful in how we construct our differential operators. Let $\mathcal{D}_t := \frac{1}{t!}\frac{\partial^t}{\partial x^t}$ be the ***divided power operator*** of order t.

$$\mathcal{D}_t(x^m) = \frac{m(m-1)\dots(m-t+1)}{t!}x^{m-t} = \binom{m}{t}x^{m-t}$$

In multiple variables, this generalizes the natural way to the divided power operator $\mathcal{D}_{i,j_i} := \frac{1}{j_i!}\frac{\partial^{j_i}}{\partial x_i^{j_i}}$ of order j_i over x_i. This construction also makes sense in characteristic zero. It turns out, all differential operators over ANY base ring arrives this way.

Theorem 2.3.4. *Suppose A is (any) commutative ring and $R = A[x_1, \dots, x_n]$. Then*

$$\mathcal{D}_{R/A} = \bigoplus_{j_1,\dots,j_n} R\mathcal{D}_{1,j_1}, \dots \mathcal{D}_{n,j_n}$$

In particular, if k is a field of positive characteristic and $R := k[x_1, \dots, x_n]$, then $D_{R/K}$ has a basis $\{x_1^{i_1} \dots x_n^{i_n} D_{1,j_1}, \dots, d_{n,j_n}\}_{i_\ell, j_\ell}$, where each basis element has degree $\sum i_\ell + j_\ell$. From here on out we'll just let $\mathcal{D}$ denote $D_{R/K}$. We can naturally define a filtration $\mathcal{F}_\bullet$ where $\mathcal{F}_t := [\mathcal{D}]_{\leq t}$ as before.
We say that a $\mathcal{D}$-Module M is ***holonomic*** if it admits a filtration $\mathcal{G}_\bullet$ such that $\exists c$ for which $\mathrm{rank}_K \mathcal{G}_t \leq ct^n\ \forall t \gg 0$. This definition is the general form of holonomic for modules over more exotic base rings; the motivation for this definition arises from theorem 2.2.8. Using this definition/theorem, we see that holonomic $\mathcal{D}$-Modules have finite length.

2.4 Lecture 4

Suppose k is a field and $R = k[x_1, \ldots, x_n]$ (or $k[[x_1, \ldots, x_n]]$. Let $\mathcal{D} := D_{R/k}$. Then, $\forall f \in R, \ell_{\mathcal{D}}(R_f) < \infty$.

Proven by:

	char 0	char p
Polynomial Ring	Bernstien (1972)	Bograd (1995)
Power Series Ring	Bjork (1979)	Lyubeznik (1997)

In any of these cases, the Cech Complex with respect to a sequence $f_1, \ldots, f_m$ is a complex of finite length $\mathcal{D}$-modules. Hence, each $\mathrm{H}^k_{(f_1,\ldots,f_m)}(R)$ is a finite length $\mathcal{D}$-module, so $\operatorname{Ass} \mathrm{H}^k_{(f_1,\ldots,f_m}(R)$ is finite. In any of these cases, we have the following chain in R_f:

$$\mathcal{D}\frac{1}{f} \subseteq \mathcal{D}\frac{1}{f^2} \subseteq \mathcal{D}\frac{1}{f^3} \subseteq \ldots$$

Because R_f has finite length, the above chain stabilizes. Thus, $\exists t$ such that $\mathcal{D}\frac{1}{f^t} = R_f$. The question now remains, what t?

2.4.1 Principal Generation in Positive Characteristic

Consider the example $R = k[x, y, w, z]$ and $f = wx - yz$. In characteristic 0, $\mathcal{D}\frac{1}{f} \subsetneq \mathcal{D}\frac{1}{f^2} = R_f$. This behavior cannot happen in positive characteristic, though. We have the following general statement:

Theorem 2.4.1. ***(Alvarez, Blickle, Lyubeznik)*** *Let k be a field of characteristic $p > 0$ and $R = k[x_1, \ldots, x_n]$. Let $f \in R$. Then, $\mathcal{D}\frac{1}{f} = R_f$.*

Proof. For simplicity, let's assume that k is perfect and $R = k[x]$. $D_{R/k}$ is generated over R by the divided power operators $\mathcal{D}_t := \frac{1}{t!}\frac{\partial^t}{\partial x^t}$, where $\mathcal{D}_t \cdot x^m = \binom{m}{t} x^{m-t}$. Well,

$$\mathcal{D}_1 f^p g = \frac{\partial}{\partial x} f^p g = f^p \frac{\partial g}{\partial x} + p(\ldots) = f^p \frac{\partial g}{\partial x}$$

Thus $\mathcal{D}_1$ is nice, since f^p can be pulled out (i.e. it is R^p linear). Similar arguments show that $\mathcal{D}_2, \ldots, \mathcal{D}_{p-1}$ are also R^p linear, but $\mathcal{D}_p$ is not. More generally, we can see that if $t < p^e$ then $\mathcal{D}_t$ is R^{p^e}-linear. Furthermore, R has a R^p basis of the form $1, x, x^2, \ldots, x^{p-1}$ (be warned; the field being perfect is required for this to be true). We noted that $\mathcal{D}_{p-1}$ is R^p linear, so let's see what happens when we apply the operator $\mathcal{D}_{p-1} \circ x^{p-3}$ to this basis:

$$1 \mapsto 0$$

$$x \mapsto 0$$

$$x^2 \mapsto 1$$

$$x^3 \mapsto 0$$
$$\dots$$
$$x^{p-1} \mapsto 0$$

It follows that $\mathcal{D}_{p-1} \circ x^{p-3}$ can see specifically x^2, and we can extend this to a similar argument about $\mathcal{D}_{p-1} \circ x^{p-\ell}$. Thus, we see that

$$k\langle x, \mathcal{D}_{p-1}\rangle = \mathrm{Hom}_{R^p}(R, R)$$

Now let's extend to n variables, defining

$$\mathcal{D}^{(e)} = R\langle \mathcal{D}_{i,t} \mid 1 \leq i \leq n, t < p^e\rangle = \mathrm{Hom}_{R^{p^e}}(R, R)$$

Where the second equality naturally follows from identical logic to the 1-variable case. Let's now see, for $f \in R$, what $\mathcal{D}^{(e)}(f)$ looks like. Let $\{\mu_i\}$ be a monomial basis for R over R^{p^e}. If $f = \sum_i c^{p^e}\mu_i$, then $\mathcal{D}^{(e)}(f) \subset (\underline{c_i}^{p^e}) = (\underline{c_i})^{[p^e]}$. Given a fixed μ_i, $\exists P \in \mathcal{D}^{(e)}$ where $\mu_j = \delta_{ij}$. Thus, $P(f) = c_i^{p^e}$. Thus we've recovered everything, so we can conclude that $\mathcal{D}^{(e)}(f) = (\underline{c_i}^{p^e})$.

We're not quite done yet, though. For simplicity, let $I_e(f) := (\underline{c_i})$. One can readily see that $I_{e+1}(f^p) = I_e(f)$; as $f^p = \sum_i c_i^{p^{e+1}} \mu_i^p$ where μ_i^p are part of a monomial basis for R over R^{p+1}. Also, observe that we have the following descending chain:

$$I_1(f^{p-1}) \supseteq I_2(f^{p^2-1}) \supseteq \dots$$

To show this holds, it is sufficient to show that

$$I_e(f^{p^e-1}) \supseteq I_{e+1}(f^{p^{e+1}-1})$$

Well, from the previous logic we know that $I_e(f^{p^e-1}) = I_{e+1}(f^{p^{e+1}-p})$, so it is sufficient to check that

$$\left(I_{e+1}(f^{p^{e+1}-p})\right)^{[p^{e+1}]} \supseteq \left(I_{e+1}(f^{p^{e+1}-1})\right)^{[p^{e+1}]}$$

Which immediately follows from the fact that

$$\left(I_{e+1}(f^{p^{e+1}-p})\right)^{[p^{e+1}]} = \mathcal{D}^{(e+1)}(f^{p^{e+1}-p}) \supseteq \mathcal{D}^{(e+1)}(f^{p^{e+1}-1}) = \left(I_{e+1}(f^{p^{e+1}-1})\right)^{[p^{e+1}]}$$

So now we have the chain above as desired. We now need to verify that this chain in fact stabilizes. Consider the ideal $I_e(f^{p^e-1}) = (\underline{c_i})$ and write $f^{p^e-1} = \sum c_i^{p^e}\mu_i$, where μ_i is an R-basis over R^{p^e}. Thus,

$$(p^e - 1)\deg(f) \geq p^e \deg(c_i) \Rightarrow \deg(f) > \deg(c_i)$$

Thus, the decreasing chain is completely determined by intersecting the chain by things that are of lower degree than f, where f is a fixed polynomial. It follows that the degree growth along the chain is bounded above, so it must stabilize.

We now move forward with yet another claim. We would like to show that

$$\left\{ I_{e+1}(f^{p^{e+1}-p}) = \right\} I_e(f^{p^e-1}) \subseteq I_{e+1}(f^{p^{e+1}-1})$$

Equivalently,

$$\mathcal{D}^{(e+1)}(f^{p^{e+1}-p}) \subseteq \mathcal{D}^{(e+1)}(f^{p^{e+1}-1})$$

So $\exists P \in \mathcal{D}^{(e+1)}$ such that $P(f^{p^{e+1}-1}) = f^{p^{e+1}-p}$. As P is $R^{p^{e+1}}$-linear, divide both sides by $f^{p^{e+1}}$ to get $P\left(\frac{1}{f}\right) = \frac{1}{f^p}$. Repeating this process, we can construct Q such that $Q\left(\frac{1}{f^p}\right) = \frac{1}{f^{p^2}}$, and so on. It follows that $\mathcal{D} \cdot \frac{1}{f} = R_f$. □

2.5 Lecture 5

2.5.1 $\mathcal{D}$-Modules over $\mathbb{Z}$

The goal for today is to extend results to $R = \mathbb{Z}[x_1, \ldots, x_n]$ and show that $\operatorname{Ass} \mathrm{H}^k_{(f_1,\ldots,f_m)}(R)$ is finite in this case as well. We can't hope for a finite length statement because $\mathbb{Z}$ itself does not have finite length. However, we'll show that we can still show that R has local cohomology modules with finitely many associated primes.

Theorem 2.5.1. ***(Bhatt, Blickly, Lyubeznik, Singh, Zhang)*** *Let $R = \mathbb{Z}[x_1, \ldots, x_n]$ and $\mathfrak{a} \subseteq R$. Then each $\mathrm{H}^k_{\mathfrak{a}}(R)$ has finitely many associated prime ideals.*

We need to introduce some machinery first. If R is a commutative ring and $f \in R$, then we have an associated Koszul Complex $0 \to R \xrightarrow{f} R \to 0$ and Cech Complex $0 \to R \to R_f \to 0$. These form a natural commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \xrightarrow{\cdot f} & R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \cdot f} & & \\ 0 & \longrightarrow & R & \longrightarrow & R_f & \longrightarrow & 0 \end{array}$$

Constructing these complexes over $f_1, \ldots f_n$, then taking their tensor product, induces a natural correspondence between Koszul Cohomology and Local Cohomology. In the case above, $0 :_R f = \to \Gamma_{(f)}R = \mathrm{H}^0_{(f)}(R)$ and $R/fR \to R_f/R = \mathrm{H}^1_{(f)}(R)$. Taking the $\mathcal{D}$-Module span of the image, we can see that the theorem from yesterday tells us that we have a surjection on first cohomology. In general, we have

$$K^\bullet((f_1, \ldots, f_n), R) = \bigotimes K^\bullet((f_i), R) \to \bigotimes C^\bullet((f_i), R) = C^\bullet((f_1, \ldots, f_n), R)$$

Implying the following strengthening of yesterday's theorem 2.4.1:

Theorem 2.5.2. *For $\underline{f} \in R$, the image of $\mathrm{H}^k(\underline{f}, R)$ in $\mathrm{H}^k_{(\underline{f})}(R)$ generates $\mathrm{H}^k_{(\underline{f})}(R)$ as a $D_{R/k}$ Module.*

This implication is not straightforward, but requires machinery to rigorize that is beyond the scope of this course. We now provide a key lemma that shows a similar style of correspondence holds over $\mathbb{Z}$:

Lemma 2.5.3. *Let $R = \mathbb{Z}[x_1, \dots, x_n]$. For $\underline{f} \in R$, fix $k \geq 0$. If a prime integer p is a nonzerodivisor on $\mathrm{H}^k(\underline{f}, R)$, then p is a nonzerodivisor on $\mathrm{H}^k_{(\underline{f})}(R)$.*

Proof. Take the short exact sequence $0 \to R \to R \to R/pR \to 0$. This induces the following sequence on Koszul Cohomology:

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & \mathrm{H}^{k-1}(\underline{f}, R) & \longrightarrow & \mathrm{H}^{k-1}(\underline{f}, R/pR) & \longrightarrow & \mathrm{H}^k(\underline{f}, R) & \xrightarrow{p} & \mathrm{H}^k(\underline{f}, R) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & \mathrm{H}^{k-1}_{(\underline{f})}(R) & \longrightarrow & \mathrm{H}^{k-1}_{(\underline{f})}(R/pR) & \longrightarrow & \mathrm{H}^k_{(\underline{f})}(R) & \xrightarrow{p} & \mathrm{H}^k_{(\underline{f})}(R) & \longrightarrow & \dots
\end{array}$$

With the vertical maps existing by the previous argument. p is a nonzerodivisor, so the map $\cdot p$ is injective. Thus, the map preceding it (by exactness) has zero image. Putting this in and labelling some other maps yields

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & \mathrm{H}^{k-1}(\underline{f}, R) & \xrightarrow{\pi} & \mathrm{H}^{k-1}(\underline{f}, R/pR) & \xrightarrow{0} & \mathrm{H}^k(\underline{f}, R) & \xrightarrow{p} & \mathrm{H}^k(\underline{f}, R) & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & \mathrm{H}^{k-1}_{(\underline{f})}(R) & \xrightarrow{\pi'} & \mathrm{H}^{k-1}_{(\underline{f})}(R/pR) & \longrightarrow & \mathrm{H}^k_{(\underline{f})}(R) & \xrightarrow{p} & \mathrm{H}^k_{(\underline{f})}(R) & \longrightarrow & \dots
\end{array}$$

Here, π has full image, as the succeeding map has full kernel. R/pR is a polynomial ring over the field $\mathbb{Z}/p\mathbb{Z}$. By Theorem 2.5.2 we see that the image of α determines a generating set of the target $\mathrm{H}^{k-1}_{(\underline{f})}(R/pR)$ as a $D_{(R/pR)/(\mathbb{Z}/p\mathbb{Z})}$-Module. Because the diagram commutes, $\mathrm{im}(\alpha) = \mathrm{im}(\alpha \circ \pi) = \mathrm{im}(\pi' \circ \alpha')$. However, $\mathrm{im}(\pi' \circ \alpha') \subseteq \mathrm{im}(\pi')$. Thus $\mathrm{im}(\pi')$ generates $\mathrm{H}^{k-1}_{(\underline{f})}(R/pR)$ as a $D_{(R/pR)/(\mathbb{Z}/p\mathbb{Z})}$-Module. $\mathrm{im}(\pi')$ is $D_{R/\mathbb{Z}}$ linear, and moreover is killed by p as it lands in $\mathrm{H}^{k-1}_{(\underline{f})}(R/pR)$. Hence, it is $D_{R/\mathbb{Z}}/pD_{R/\mathbb{Z}}$ linear, so it is $D_{(R/pR)/(\mathbb{Z}/p\mathbb{Z})}$ linear. Therefore, π' in fact attains full image, so it is surjective, implying by exactness that $\cdot p$ is injective on local cohomology. □

We are now ready to prove theorem 2.5.1:

Proof. As $\mathbb{Z} \subset R$, we have a natural morphism $\mathrm{Spec}(\mathbb{Z}) \to \mathrm{Spec}(R)$, inducing a morphism on modules $\mathrm{Ass}_R \mathrm{H}^k_{\mathfrak{a}}(R) \to \mathrm{Ass}_{\mathbb{Z}} \mathrm{H}^k_{\mathfrak{a}}(R)$ sending $P \mapsto P \cap \mathbb{Z}$. $\mathrm{Ass}_R \mathrm{H}^k(\underline{f}, R)$ is finite, so it follows that all but finitely many prime integers are injective on $\mathrm{H}^k(\underline{f}, R)$, and thus all but finitely many prime integers are injective on $\mathrm{H}^k_{\mathfrak{a}}(R)$ via the previous lemma.

Thus, $\mathrm{Ass}_{\mathbb{Z}} \mathrm{H}^k_{\mathfrak{a}}(R)$ is finite. It is thus sufficient to show that the preimage of each element along the morphism $\mathrm{Ass}_R \mathrm{H}^k_{\mathfrak{a}}(R) \to \mathrm{Ass}_{\mathbb{Z}} \mathrm{H}^k_{\mathfrak{a}}(R)$ is finite. This is because $\mathrm{Ass}_R \mathrm{H}^k_{\mathfrak{a}}(R)$ can be written as the finite union over finitely many finite preimages.

Recall that all primes on $\mathbb{Z}$ are either 0 or p. Thus, we only have two separate cases for us to consider. Suppose that $(0) \in \mathrm{Ass}_{\mathbb{Z}} \mathrm{H}^k_{\mathfrak{a}}(R)$. Then its preimage are ideals P such that $P \cap \mathbb{Z} = 0$. Thus we need to only consider primes that contain no integers. Thus the preimage of (0) is the contraction of the associated primes of $\mathrm{H}^k_{\mathfrak{a}}(R \otimes_{\mathbb{Z}} \mathbb{Q})$. $R \otimes_{\mathbb{Z}} \mathbb{Q}$ is a polynomial ring over a characteristic 0 field, so by a result of a previous lecture $\mathrm{H}^k_{\mathfrak{a}}(R \otimes_{\mathbb{Z}} \mathbb{Q})$ is finite.

Now Suppose that $(p) \in \mathrm{Ass}_{\mathbb{Z}} \mathrm{H}^k_{\mathfrak{a}}(R)$. The preimage of (p) corresponds to

$$\mathrm{Ass}_R \ker \left(\mathrm{H}^k_{\mathfrak{a}}(R) \xrightarrow{\cdot p} \mathrm{H}^k_{\mathfrak{a}}(R) \right)$$

From the exact sequence used in lemma 2.5.3, we identify the kernel with the preceding image in the sequence. Thus the preimage of (p) can be identified with

$$\mathrm{Ass}_R \operatorname{im} \left(\mathrm{H}^{k-1}_{\mathfrak{a}}(R/pR) \to \mathrm{H}^k_{\mathfrak{a}}(R) \right)$$

$\mathrm{H}^{k-1}_{\mathfrak{a}}(R/pR)$ is a finite length $D_{(R/pR)/(\mathbb{Z}/p\mathbb{Z})}$-Module, and thus has finitely many composition factors over $D_{(R/pR)/(\mathbb{Z}/p\mathbb{Z})}$, denoted by $M_1, \dots, M_\ell$, where each M_i has 1 associated prime. It follows that $\mathrm{H}^{k-1}_{\mathfrak{a}}(R/pR)$ has ℓ, and in particular, finitely many, associated primes. Thus, its image under a $D_{R/\mathbb{Z}}$-Module homomorphism will necessarily also have finitely many associated primes. □

It's known that $\mathrm{H}^k_{\mathfrak{a}}(R)$ can have infinitely many associated primes in general, even when R is a hypersurface. For instance, take

$$R = \frac{\mathbb{Z}[x_1, x_2, x_3, y_1, y_2, y_3]}{\sum x_i y_i}$$

One can verify that $\mathrm{H}^3_{(\underline{x})}(R)$ has p-torsion for any prime p. To see this, consider $(x_1y_1)^p + (x_2y_2)^p + (x_3y_3)^p$; it is 0 mod p, so $\frac{1}{p}(x_1y_1)^p + (x_2y_2)^p + (x_3y_3)^p$ lives in R. In particular,

$$\eta_p := \frac{(x_1y_1)^p + (x_2y_2)^p + (x_3y_3)^p}{p(x_1x_2x_3)^p}$$

lives in $\frac{R_{x_1x_2x_3}}{R_{x_1x_2} + R_{x_1x_3} + R_{x_2x_3}}$. One can easily see here that $p\eta_p = 0$, but in the problem set we will show that $\eta_p \neq 0$. This implies that every prime ideal (p) is associated

Open Problem 4. It is currently unknown whether or not there is an algorithm to determine whether $\mathrm{H}^k_{\mathfrak{a}}(R)$ vanishes. Such an algorithm works when we are working over $\mathbb{Q}$ (as we'll see in a later talk by Uli Walther), but when R is a polynomial over $\mathbb{Z}$ it is currently open.

2.6 Lecture 6 - Jeffries

Suppose k is a field and R is a k-algebra. We can associate to this the following differential information:

- $D^0_{R/k} = \mathrm{Hom}_R(R,R)$
- $D^i_{R/k} = \left\{\delta \in \mathrm{Hom}_k(R,R) \mid [\delta, r] \in D^{r-1}_{R/k}\ \forall r \in D^0_{R/k}\right\}$
- A noncommutative ring $\mathcal{D} := D_{R/k} = \bigcup_i D^i_{R/k}$ where $R \subseteq D_{R/k} \subseteq \mathrm{Hom}_k(R,R)$.

In characteristic 0 and over a polynomial ring, Recall that $\mathcal{D}$ is finitely generated over R by $D^1_{R/k}$, and is left and right Noetherian. Also R is a simple $\mathcal{D}$-Module, and every nonzero $\mathcal{D}$-module is dimension $\geq n$. Furthermore, $\mathrm{H}^i_I(R)$ has finitely many associated primes.

In characteristic p, we could still write $\mathcal{D} = R\langle D_{i,t} \mid t \in \mathbb{N}\rangle$, but we had to take the divided power operators to avoid some characteristic p issues. However, $\mathcal{D}$ is no longer finitely generated over R and is not left/right Noetherian. However, the other statements above from characteristic 0 still hold.

2.6.1 $\mathcal{D}$-Modules of Varieties

What about over k-algebras that are not just polynomial rings? In this case we can ask what $\mathcal{D}$ looks like, and if it has any of the properties discussed above.

Lemma 2.6.1. ***(Smith-Stafford)*** *Let $S = k[x_1, \ldots, x_n]$ and $R = S/I$. Then*

$$D_{R/k} = \frac{\{\partial \in D_{S/k} \mid \delta(I) \subseteq I\}}{ID_{S/k}}$$

We can write $(I :_{D_{S/k}} I) = \{\partial \in D_{S/k} \mid \delta(I) \subseteq I\}$. For example, if $I = (xy) \subseteq \mathbb{C}[x,y]$, then

$$(I :_{D_{S/k}} I) = \bigoplus_{\substack{c>0 \Rightarrow a>0 \\ d>0 \Rightarrow b>0}} \mathbb{C}x^a y^b \partial_x^c \partial_y^d$$

Then we see that

$$D_{R/k} = \mathbb{C} \bigoplus_{a>0} \mathbb{C}x^a \partial_x^c \oplus \bigoplus_{b>0} \mathbb{C}y^b \partial_y^d$$

If we let $S = \mathbb{C}[x,y,z], I = (x^3+y^3+z^3)$, then we can compute $D_{R/k}$. Let $E = x\partial_x + y\partial_y + z\partial_z$ and $H_{xy} = x^2\partial_y - y^2\partial_x$, similarly defining H_{xz}, H_{yz}. Further, define $A_z = \frac{1}{z}\left(12H^2_{xy} + 3xyE^2 + xyE\right)$, and similarly, with A_x, A_y. Then,

$$\mathcal{D} = R \oplus R \cdot \{E, H_{xy}, H_{yz}, H_{xz}\} \oplus R \cdot \left\{E^2, EH_{xy}, EH_{yz}, EH_{xz}, A_x, A_y, A_z\right\} \ldots$$

2.6.2 $\mathcal{D}$-Modules of G-Invariant Subvarieties

If a group G acts on $S = k[\underline{x}]$, it follows that G acts on $D_{S/k}$ via conjugation.

Theorem 2.6.2. ***(Kantor)*** *If k is characteristic 0 and G is a finite group acting by degree preserving automorphisms with no pseudoreflections, then*

$$D_{S^G/k} \cong (D_{S/k})^G$$

For instance, let $S = \mathbb{C}[x, y]$ with $G = \{\pm 1\}$, acting the standard way. -1 acts trivially on monomials with even degree, so $S^G = \mathbb{C}[x^2, xy, y^2]$. Thus,

$$D_{S^G/k} = \bigoplus_{a,b,c,d \text{ even}} x^a y^b \partial_x^c \partial_y^d = D_{S/k}^G$$

For many actions of infinite reductive groups, we at the very least get a surjection $(D_{S/k})^G \twoheadrightarrow D_{S^G/k}$, via work of Schwarz. In these cases, $D_{R/k}$ has a known description when R is toric or determinental (in characteristic 0).

Theorem 2.6.3. ***(Paul Smith)*** *Suppose k is a perfect field of characteristic p and R is finitely generated over k. Then,*

$$D_{R/k} = \bigcup_{e \in \mathbb{N}} \operatorname{Hom}_{R^{p^e}}(R, R)$$

It is a conjecture of Mukai that,

Open Problem 5. if R is a finitely generated $\mathbb{C}$-algebra, then $D_{R/k} = R\langle D^1_{R/k}\rangle$ implies that R is smooth. This would imply the Zariski-Lipman conjecture if true.

2.6.3 More Finiteness Properties

When $R = k[\underline{x}]$, we have the following:

- R is simple over $\mathcal{D}$.
- **(Bernstein's Inequality)** All $\mathcal{D}$-Modules have dimension $\geq \dim(R)$.
- All Local Cohomology modules have finitely many associated primes.

When is the last condition true?

- All rings of invariants in characteristic 0
- Modules with F-finite Representation Type (FFRT) property in characteristic p.
- $\dim(R) \leq 3$ in general.

Yet this is NOT true in full generality. We can also ask where Bernstein's inequality holds:

- Invariant rings of finite groups in characteristic 0.
- FFRT + Strongly F-regular in characteristic p
- Coordinate rings of $\mathbb{P}^n \times \mathbb{P}^n$ in characteristic 0.

Chapter 3

Linkages

These classes were mostly taught by Bernd Ulrich, with the last lecture given by Juan Migliore.

3.1 Lecture 1

We'll start with some general theory. For example, consider the twisted cubic $C \subseteq \mathbb{P}^3$, parameterized by the degree 3 monomials in 2 variables $[s : t] \mapsto [s^3 : s^2t : st^2 : t^3]$. This has defining ideal $(\Delta_1, \Delta_2, \Delta_3)$, where Δ_i are determinants of the 2×2 minors of the matrix $\left[\begin{smallmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{smallmatrix}\right]$ (here Δ_i corresponds to the 2×2 minor constructed by removing the ith column). Restricting our attention to (Δ_1, Δ_2), this is a regular sequence and the ideal lies in $I \cap (x_2, x_3)$. Thus the complete intersection $V(\Delta_1, \Delta_2) = C \cup V(x_2, x_3)$, where $\cup$ is a scheme theoretic union. We say that C (a curve) and $V(x_2, x_3)$ (a line) are ***linked*** by a complete intersection.

We now give the general definition. Suppose that R is Noetherian and $I, J \subsetneq R$ are proper ideals of R. We say that I and J are ***linked*** if $\exists$ a regular sequence $\underline{\alpha} = \alpha_1, \ldots, \alpha_g$ such that $J = (\underline{\alpha}) : I$ and $I = (\underline{\alpha}) : J$. If two ideals are linked we write $I \sim J$, and if we want to specify the regular sequence linking them, we write $I \sim_{\underline{\alpha}} J$. For instance, $I \sim (x_2, x_3)$ in the previous example. One can see from the definition that, given $I \sim J$ with respect to the regular sequence $\underline{\alpha}$, $(\underline{\alpha}) \subset I \cap J$.

3.1.1 Unmixedness

From now on, we assume that R is Cohen-Macaulay, so every ideal generated by regular sequence $\underline{\alpha} = \alpha_1, \ldots, \alpha_n$ is unmixed of height n. In other words, all associated primes of $(\underline{\alpha})$ have height exactly n.

Lemma 3.1.1. *Suppose R is Cohen Macaulay and $I, J \subseteq R$. Further, $I \sim J$ with respect to the sequence $\underline{\alpha} = \alpha_1, \ldots, \alpha_g$ Then,*

$$\mathrm{Ass}_R(R/(\underline{\alpha})) = \mathrm{Ass}_R(R/I) \cup \mathrm{Ass}_R(R/J)$$

In particular, I and J are themselves unmixed of height g.

Proof. We first prove the $\subset$ direction. Choose $P \in \mathrm{Ass}_R(R/(\underline{\alpha}))$. Then, $\mathrm{ht}(P) = g$. Now notice that $IJ \subset (\underline{\alpha})$ via the colon condition. As $(\underline{\alpha}) \subset P$, we see that $IJ \subset P$. As P is prime, one of I or J is contained in P; without loss of generality let $I \subset P$. Well, $g \leq \mathrm{ht}(I)P) = g$, so I and P have the same height. Thus, P is a minimal prime over I, and in particular, $P \in \mathrm{Ass}_R(R/I)$. If $J \subset P$, repeat this argument to prove that $P \in \mathrm{Ass}_R(R/J)$. Regardless, $P \in \mathrm{Ass}_R(R/I) \cup \mathrm{Ass}_R(R/J)$.

Now for the $\supset$ direction. Choose $P \in \mathrm{Ass}_R(R/I)$ then $\exists a \in R \backslash I$ such that $Pa \in I$. Thus, $PaJ \subset IJ(\underline{\alpha})$, but $a \notin I = (\underline{\alpha} : J)$, so $aJ \notin (\underline{\alpha})$. Thus $\exists b \in aJ \backslash (\underline{\alpha})$ and $pb \in (\underline{\alpha})$, so $P \subset Q$ for some $Q \in \mathrm{Ass}(R/(\underline{\alpha})$. Well, $g \leq \mathrm{ht}(P) \leq \mathrm{ht}(Q) = g$, so P and Q have the same height, so $P = Q$. It follows that $P \in \mathrm{Ass}(R/(\underline{\alpha})$. Repeat this proof for $P \in \mathrm{Ass}_R(R/J)$, identical to the above via symmetry, to conclude the desired result. $\square$

We can strengthen the linkage as follows. We say that I and J are ***geometrically linked*** if $I \sim J$ and $\mathrm{ht}(I + J) > g = \mathrm{ht}(I) = \mathrm{ht}(J)$.

Lemma 3.1.2. *Suppose $I \sim_{\underline{\alpha}} J$. Then the following are equivalent:*

(1) $I \sim_{\underline{\alpha}} J$ is a geometric linkage.

(2) I and J have no associated primes in common.

(3) $I_P = (\underline{\alpha})_P \; \forall P \in \mathrm{Ass}_R(R/I)$ (i.e. I is generically a complete intersection).

(4) $I \cap J = (\underline{\alpha})$

This will be proved in the exercises. We now introduce a new definition. If $I, J \subsetneq R$ we say that I and J are in the same ***linkage class*** if $\exists I_0, \ldots, I_n$ such that

$$I = I_0 \sim I_1 \sim \cdots \sim I_n = J$$

I is ***licci*** if I and a complete intersection belong to the same linkage class.

3.1.2 Duality

From now on let's assume that R is regular and local (though we can get away most of the time by just assuming that R is Gorenstein). Let $I \subseteq R$ be an ideal of height g, $S = R/I$, and $F_\bullet$ a minimal free resolution of S/I. As we've seen in the talks on syzygies, we have the following result:

Theorem 3.1.3. *S is Cohen Macaulay* $\iff \mathrm{ProjDim}_R(S) = g$.

We now want to associate a ***canonical module*** (often called a dualizing module) to S. Denote ω_S, we can explicitly define this as $\omega_S := \mathrm{Ext}^g_R(S, R)$. I'll assume here that you know what a dualizing module is and some basic properties of it; if not, you can read my notes from Wenliang Zhang's commutative algebra class at UIC (2022). If you want a deeper introduction with some exercises, please consult Bruntz and Herzog's *Cohen Macaulay Rings*, specifically chapter 2. We will review some facts, though. Recall that a finite S-Module M is Maximal Cohen Macaulay (MCM) if $\dim M = \dim S$ and $\operatorname{ProjDim}_R M = g$. The primary theorem of importance is the following:

Theorem 3.1.4. *Assume that S is Cohen Macaulay. Then $\operatorname{Hom}_S(-, \omega_S)$ is a duality functor on the category of MCM S-Modules.*

We'll denote the functor $\operatorname{Hom}_R(-, R)$ acting on R-modules by $(-)^*$. We have the following proposition:

Lemma 3.1.5. *Assume S is CM. Then $(F_\bullet)^*(-g)$ is a minimal free R-resolution of ω_S. In particular, ω_S is generated by $\operatorname{rank}_R F_g$ elements.*

This number of generators is denoted $r(S)$ and is the ***type*** of S.

Proof. $\operatorname{Ext}^i_R(S, R) = 0 \ \forall i \leq g$. Dualizing the resolution $0 \to F_\bullet \to S \to 0$ yields the resolution $0 \to (F_\bullet)^* \to \operatorname{Ext}^g_R(S, R) \to 0$, where $\operatorname{Ext}^g_R(S, R)$ is precisely ω_S. By minimality, the surjection $F_g^* \twoheadrightarrow \omega_S$ tells us that ω_S is generated by precisely $\operatorname{rank}_R F_g^* = \operatorname{rank}_R F_g$ elements. □

This implies the following lemma:

Lemma 3.1.6. *Assuming S is CM, the following are equivalent:*

- $\omega_S \cong S$
- $F^*(-g) \cong F$
- $r(S) = 1$

If S satisfies any/all of these conditions, we say S is ***Gorenstein***.

3.2 Lecture 2

To keep things simple, let R be a regular local ring and I an ideal of height g. let $S = R/I$ and let $\omega_S = \operatorname{Ext}^g_R(S, R)$ be the canonical module of S. Further, let $F_\bullet$ be a minimal free R-resolution of S.

3.2.1 Koszul Complexes

Let R be any commutative ring and $F = R^{\oplus n}$ a finite free module with basis $e_1, \ldots, e_n$. We let

$$\bigwedge^\bullet F := \left(\bigoplus_{i \geq 0} F^{\otimes i} \right) \Big/ \langle x \otimes x \mid x \in F \rangle$$

Be the ***Koszul complex*** of F. The ith component of F is precisely the ith alternating power of F (hence the notation) where

$$\bigwedge^i(F) = R^{\binom{n}{i}} = \bigoplus_{v_1 < \cdots < v_i} Re_{v_1} \wedge \cdots \wedge Re_{v_i}$$

One can define the differentials of the Koszul Complex with respect to a sequence $\overline{a} = \{a_1, \ldots, a_n\} \subset R$. $\partial_i : \bigwedge^i F \to \bigwedge^{i-1} F$ where

$$e_{v_1} \wedge \cdots \wedge Re_{v_i} \mapsto \sum_{j=1}^{i} (-1)^{j+1} a_{v_j} e_{v_1} \wedge \ldots \widehat{e_{v_j}} \cdots \wedge Re_{v_i}$$

We thus denote the Koszul Complex as $K_\bullet = K(\underline{a})$, and seek to understand its homology. We can see that $H_0(K_\bullet) = R/(\underline{a})$, but the question remains; when is this the only homology we see?

Theorem 3.2.1. *Assume R is Noetherian and $\underline{a} \subseteq \mathcal{J}(R) := \bigcap_{m \in \mathrm{MaxSpec}(R)} m$ (the Jacobson radical of R). Then $K_\bullet$ is acyclic $\iff$ $\underline{a}$ forms a regular sequence over R.*

As a natural corollary of this, we can see that if R is a regular local ring and $\underline{a}$ is a regular sequence, then $R/(\underline{a})$ is Gorenstein. Thus, when your rings are local, all complete intersections are Gorenstein. Now, we go back to linkages:

Theorem 3.2.2. ***(Peskine - Szpiro)*** *Over a regular local ring R and $I, J \subsetneq R$ where $I \sim_{\underline{a}} J$,*

(a) R/I is CM $\iff$ R/J is CM.

(b) $\omega_{R/J} \cong I/(\underline{a})$ (and symmetrically, $\omega_{R/I} \cong J/(\underline{a})$)

Proof. We prove (a) first. We may replace R by $R/(\underline{a})$ to assume that $g = 0$. We may have lost regularity, but our new R is still Gorenstein by the above corollary. Thus, $\omega_R \cong R$. Thus we just need to show that if R/I is CM, then $R/(0 : I)$ is CM (here we see that $J = (0 : I)$ as we've passed to the quotient and I and J are linked). Since $g = 0$, R/I is a MCM R-Module. Dualizing the short exact sequence $0 \to I \to R \to R/I \to 0$ yields

$$0 \to \mathrm{Hom}_R(R/I, R) \to \mathrm{Hom}(R, R) \to \mathrm{Hom}_R(I, R) \to 0$$

$\mathrm{Hom}_R(R/I, R) = (0 : I)$ and $\mathrm{Hom}_R(I, R) = R/(0 : I)$. Thus, $R/(0 : I)$ is at the tail end of an exact sequence of MCMs, and we can conclude that $R/(0 : I)$ is indeed CM. We now prove (b).

$$\omega_{R/I} = \mathrm{Ext}^g_R(R/I, R) \cong \mathrm{Hom}(R/I, R/(a_1, \ldots, a_g) = (\underline{a}) : I/(\underline{a}) = J/(\underline{a})$$

We can conclude the opposite case by symmetry. □

It's worth noting that we don't need the assumption that R is regular and $\underline{a}$ is a regular sequence, just that the corresponding quotient $R/(\underline{a})$ is Gorenstein. This is the generality in which Peskine and Szpiro proved this theorem. As a consequence, we see that $r(R/J) = \mu(I/(\underline{a}) \geq \mu(I) - g$. This difference is also considered to be the ***complete intersection defect*** of I, denoted $d(I)$. Thus, $r(R/J) = d(I)$ and $r(R/I) = d(J)$. In particular, R/I is Gorenstein when $r(R/I) = 1$, so $\mu(J) \leq g + 1$. In this case we say that J is ***almost a complete intersection***. Being almost a complete intersection and being Gorenstein are complementary properties; Kunz has proved that rings satisfying both these conditions are indeed complete intersections.

3.2.2 Mapping Cones

Suppose that R is a commutative ring and $U_\bullet : E_\bullet \to F_\bullet$ is a morphism of complexes concentrated in degrees ≥ 0.

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & F_3 & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 \\
 & & & & {\scriptstyle U_2}\uparrow & & {\scriptstyle U_1}\uparrow & & {\scriptstyle U_0}\uparrow \\
 & & \dots & \longrightarrow & E_2 & \longrightarrow & E_1 & \longrightarrow & E_0
\end{array}$$

The ***Mapping Cone*** is the complex generated by taking diagonal direct sums as follows:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & F_3 & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 \\
 & & & & {\scriptstyle U_2}\uparrow & & {\scriptstyle U_1}\uparrow & & {\scriptstyle U_0}\uparrow \\
 & & \dots & \longrightarrow & E_2 & \longrightarrow & E_1 & \longrightarrow & E_0
\end{array}$$

Yielding the complex $C(U_\bullet) := \{\cdots \to E_1 \oplus F_2 \to E_0 \oplus F_1 \to F_0\}$.

Lemma 3.2.3. *If $0 \to M \xrightarrow{\varphi} N \to K \to 0$ is exact and $E_\bullet, F_\bullet$ are free resolutions of M and N respectively, then $C(U_\bullet)$ is a free resolution of K.*

Now let's go back to the standard setting. R is a regular local ring, $I \sim_{\underline{a}} J$ with $\underline{a} = a_1, \dots, a_g$ a sequence as before, R/I is CM, and $F_\bullet$ is a minimal free resolution of R/I. $K_\bullet = K_\bullet(\underline{a})$, and $(-)^* = \mathrm{Hom}_R(-, R)$. We have a map of complexes

$$\begin{array}{ccccccccccc}
F_g & \longrightarrow & \dots & \longrightarrow & F_1 & \longrightarrow & R & \longrightarrow & R/I & \longrightarrow & 0 \\
{\scriptstyle U_g}\uparrow & & & & {\scriptstyle U_1}\uparrow & & \Updownarrow & & {\scriptstyle \pi}\uparrow & & \\
K_g = R & \longrightarrow & \dots & \longrightarrow & K_1 & \longrightarrow & R & \longrightarrow & R/(\underline{a}) & \longrightarrow & 0
\end{array}$$

Where the dashed maps are recursively defined from the identity map $R \Leftrightarrow R$. We then have a map of complexes $U_\bullet : K_\bullet \to F_\bullet$.

Theorem 3.2.4. *$C(U_\bullet)^*[-g-1]$ is a free resolution of R/J.*

Proof. We have the short exact sequence

$$0 \to J/(\underline{a}) \to R/(\underline{a}) \to R/J \to 0$$

From previous work we know that $J/(\underline{a}) = \omega_{R/I}$ with resolution $F_\bullet^*[-g]$ and $R/(\underline{a})$ is Gorenstein so it is isomorphic to $\omega_{R/(\underline{a})}$, with complex $K_\bullet^*[-g]$. Thus we have a map on complexes $U_\bullet^*[-g]$, and from what we know about mapping cones, $C(U_\bullet^*[-g])$ is a free resolution of of R/J. There is a natural isomorphism $C(U_\bullet^*[-g]) \cong C(U_\bullet)^*[-g-1]$ which you can get by following your nose, completing the proof. $\square$

3.3 Lecture 3

Theorem 3.3.1. *R regular local, I an unmixed ideal where* $\operatorname{ht}(I) = g$, *$\underline{a} = \{a_1, \dots, a_g\} \subset I$ a regular sequence, and $J = (\underline{a}) : I \neq R$. Then $I \sim_{\underline{a}} J$.*

The "meat" of this theorem is that we do not need to also verify that symmetric colon statement; i.e. we can recover $I = (\underline{a}) : J$.

Proof. We always know that $I \subset (\underline{a}) : (\underline{a} : I) = (\underline{a}) : J$. We just need to show equality. Localizing at any associated prime of I and factoring out $(\underline{a})$, we may assume that the new ambient ring is 0-dimensional and still at least Gorenstein. In this setting, we'd like to show that $I \subset 0 : (0 : I)$ is in fact an equality.

As R is Gorenstein, dualizing with respect to R is the same as dualizing with respect to ω_R; we can thus view $(-)^*$ as the functor $\operatorname{Hom}_R(-, \omega_R)$. We've also seen that $R/(0 : I) = I^*$. Dualizing again, we recover that $(R/(0 : I))^* = I^{**}$, so $I^{**} = (0 : (0 : I))$, so by duality we get that $I = (0 : (0 : I))$. This duality works because we are working over an Artinian ring; these modules are all of finite length, so in fact they have the same length. Thus containment is synonymous with equality. □

Lemma 3.3.2. *(Wiebe, Buchweitz) $I = (\underline{b}) \supset \underline{a}$. Where $\underline{b} = b_1, \dots, b_g$ and $\underline{a} = a_1, \dots, a_g$ form regular sequences on R. Furthermore, $[\,\underline{a}\,] = [\,\underline{b}\,]\,A$ and $J = (\underline{a}) : I$. Then, $J = (\underline{a} : \det A)$.*

Proof. We know that I and J are linked by the previous theorem. We resolve I via the Koszul Complex $K_\bullet(\underline{b})$;

$$0 \to \bigwedge^g R^g \to \cdots \to \bigwedge^2 R^g \to R^g \xrightarrow{\underline{b}} R$$

We can also resolve $K_\bullet(\underline{a})$:

$$0 \to \bigwedge^g R^g \to \cdots \to \bigwedge^2 R^g \to R^g \xrightarrow{\underline{a}} R$$

Where the diagrams are linked together by the following commutative diagram:

$$\begin{array}{ccc} R^g & \xrightarrow{\underline{b}} & R \\ {\scriptstyle A}\big\uparrow & & \big\Updownarrow \\ R^g & \xrightarrow{\underline{a}} & R \end{array}$$

It follows that we can extend A back by taking wedge powers, and in particular

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigwedge^g R^g & \longrightarrow \ \dots \ \longrightarrow & \bigwedge^2 R^g & \longrightarrow & R^g & \xrightarrow{\underline{b}} & R \\ & & {\scriptstyle \bigwedge^g A = \det(A)}\big\uparrow & & {\scriptstyle \bigwedge^2 A}\big\uparrow & & {\scriptstyle A}\big\uparrow & & \big\Updownarrow \\ 0 & \longrightarrow & \bigwedge^g R^g & \longrightarrow \ \dots \ \longrightarrow & \bigwedge^2 R^g & \longrightarrow & R^g & \xrightarrow{\underline{a}} & R \end{array}$$

It immediately follows that $J = (\underline{a} : \det A)$. □

Lemma 3.3.3. *Suppose that k is a characteristic 0 field and $R = k[x_1, \ldots, x_g]$, with $\underline{a} = a_1, \ldots, a_g$ a regular sequence consisting of homogeneous elements. define $S = R/(\underline{a})$; S is Gorenstein so the Socle* $\operatorname{Soc}(S) := 0 :_S (\underline{x})S$ *is 1 dimensional.*

With these hypotheses, $\operatorname{Soc}(S)$ *is generated by* $\det(A)$*, i.e.* $\operatorname{Soc}(S) = \left|\frac{\partial a_i}{\partial x_j}\right| S$.

Proof. Take $\underline{b} = \underline{x}$ and $d_i = \deg(a_i)$. From the previous result it follows that

$$\begin{bmatrix} d_1 a_1 & \ldots & d_g a_g \end{bmatrix} = \underbrace{\begin{bmatrix} x_1 & \ldots & x_g \end{bmatrix}}_{\underline{b}} \underbrace{\begin{bmatrix} \frac{\partial a_j}{\partial x_i} \end{bmatrix}}_{A}$$

So

$$(\underline{a}) :_R (\underline{x}) = \left(\underline{a}, \left|\frac{\partial a_j}{\partial x_i}\right|\right)$$

□

3.3.1 Classes of Linkages

Theorem 3.3.4. ***[Apeiy, Gaeta]*** *Let R be a regular local ring and $I \subsetneq R$ an ideal of height 2. Then I is licci $\iff$ R/I is CM.*

We already know that CM is a necessary condition, even with lighter hypotheses. It is sufficient then to check the reverse case in this instance.

Proof. Induct on $n = \mu(I) \geq 2$ (recall that $\mu(I)$ denotes the minimal number of generators of I). The $n = 2$ case is clear, so we let $n \geq 3$. We want to construct a link; let $\underline{a} = a_1, a_2$ be a regular sequence inside I (we can take this because I is of height 2). Without loss of generality, we can let a_1, a_2 be members of a minimal generating set of I. By hypothesis, I requires at least 3 generators. Taking $J = (\underline{a}) : I \sim_{\underline{a}} I$ (linked by the theorem) we see that R/J is CM as $r(R/J) = \mu(I/(\underline{a})) = n - 2$ by hypothesis. Thus R/J has projective dimension 2 as an R-module, so the minimal free R-resolution of R/J is

$$0 \to R^{n-2} \to R^{n-1} \to R \to 0$$

Thus J is generated by $n - 1$ elements, so J is licci by the inductive hypothesis. It follows that, because I is linked to J, I is licci. □

We can also study other classes of licci ideals I, but we will omit the proofs as they are too long:

- **(Watanabe)** If $\operatorname{ht}(I) = 3$, I is licci $\iff$ R/I is Gorenstein.
- If $\operatorname{ht}(I) = 1$ or 3, I is licci $\iff$ R/I is CM and $d(I) = 1$.

Open Problem 6. (Weyman's Conjecture) if $\operatorname{ht}(I) = 3$, R/I is CM and either $r(I) \leq 2$ and $d(I) \leq 4$, or $r(I) \leq 4$ and $d(I) \leq 2$. A student of Eisenbud has possibly proven this conjecture, but the work has not been published just yet.

- **(Ulrich)** If $I \sim J$ is a geometric linkage of licci ideals, then $I + J$ is a licci ideal.
- **(Kustin-Miller)** if X, Y are generic matrices of indeterminates of size $1 \times n$ and $n \times (n-1)$ respectively. Then $I := I_1(XY) + I_{n-1}(Y)$ is licci.

3.3.2 Minimal Representatives of Linkage Classes

Theorem 3.3.5. ***(Huneke, Ulrich)*** *$R' = k[x_1, \ldots, x_d] \supset I'$, a homogeneous ideal of height g with minimal homogeneous resolution*

$$0 \to \bigoplus_i R'(-n_{gi}) \to \cdots \to \bigoplus_i R'(-n_{1i} \to 0$$

And R'/I' is Cohen Macaulay. If $\max(n_{gi}) \leq (g-1)\min(n_{1i})$, then for $R = R'_{(x_1,\ldots,x_d)}$, $I = I'R$, whenever J is an ideal in the even linkage class (i.e. linked by an even number of generators) of I, $\mu(J) \geq \mu(I)$ and $r(R/J) \geq r(R/I)$.

In particular, it follows that I cannot be licci.

This is saying that, as long as these shifts don't go "too fast", we will always get the smallest representative of the linkage class.

3.3.3 Finer Invariants of Linkage Classes

Suppose $R = k[[x_1, \ldots, x_n]]$ where k is a field, and $S = R/I$. S is a complete local noetherian. We'd like to consider infinitesimal first order deformations μ as follows:

$$\begin{array}{ccc} D & \longrightarrow & S = D/\varepsilon D \\ {\scriptstyle \text{flat}}\uparrow{\scriptstyle \eta} & & \uparrow \\ k[\varepsilon] = k[t]/(t^2) & \twoheadrightarrow & k \end{array}$$

There is a bijective correspondence $\{\eta\} \leftrightarrow \mathrm{Hom}_S(S/I^2, S)$. We can then take higher order deformations similarly:

$$\begin{array}{ccccccc} \ldots & \longrightarrow & D_2 & \longrightarrow & D & \longrightarrow & S = D/\varepsilon D \\ & & {\scriptstyle \text{flat}}\uparrow{\scriptstyle \eta_2} & & {\scriptstyle \text{flat}}\uparrow{\scriptstyle \eta} & & \uparrow \\ \ldots & \longrightarrow & k[t]/(t^3) & \longrightarrow & k[\varepsilon] = k[t]/(t^2) & \twoheadrightarrow & k \end{array}$$

Obstructions for lifting infinitesimal deformations lie in $T^2(S/R)$. Passing to the $\varprojlim$, we get the diagram

$$\begin{array}{ccc} T & \longrightarrow & S = T/tT \\ {\scriptstyle \text{flat}}\uparrow & & \uparrow \\ k[[t]] & \longrightarrow & k \end{array}$$

Where t is T-regular. Now let $(\mathrm{R}, \mathfrak{m})$ be a regular local ring, $I \subset R$ is an ideal of height g, and $S = R/I$ is a Cohen Macaulay ring of dimension d. As before, let $F_\bullet$ be a minimal free R-resolution of S.

$$\mathrm{Tor}_i^R(S, \omega_S) \cong \mathrm{Ext}_R^{g-i}(S, S)$$

When $i = g - 1$, this is just $\mathrm{Hom}_S(S/I^2, S)$. For $i = 1$, we get $I/I^2 \otimes_S \omega_S$, otherwise known as the ***twisted conormal module***. If I is a generically complete intersection (i.e. if one localizes at any minimal prime P, I_P is a complete intersection) then

$$T^2(S/R) \cong \mathrm{Ext}_S^1(I/I^2, S)$$

Theorem 3.3.6. ***(Buchweitz, Ulrich)*** *The following are invariants of the linkage class of I:*

- $\mathrm{H}_{\mathfrak{m}}^j\left(\mathrm{Tor}_i^R(S, \omega_S)\right)$ $\forall j \neq \dim(S)$.
- *The depth of* $\mathrm{Tor}_i^R(S, \omega_S)$.
- $T^2(S/R)$ *for generic complete intersections.*
- $\mathrm{Ext}_R^\bullet(S, S)$ *is strictly graded commutative, and invariant of the linkage class.*

3.4 Lecture 4 - Migliore

3.4.1 Linkage Classes of Curves

If you have a Licci ideal I, you can link "down" in some sense to an ideal you understand, such as a complete intersection. Working backwards lets you understand what I looks like. In the early days of this theory, the hope was that this works in general, i.e. if you have a curve, can you link down to another, easier to understand curve and use that to try and understand the original curve. Harris did not believe such a think was possible, but Lazarsfeld and Rao proved that indeed for a general curve, you cannot link down for anything simpler. The class of general curves (smooth, reasonable degree) is fairly strict, and thus this paper provided a relatively strict structure on their linkage classes. In today's talk, we'd like to investigate which classes of curves have this property (hereafter referred to as the ***LR Property***) in general.

This talk will be relatively geometric, so we'll provide a sort of dictionary below:

Algebra	Geometry
Height	Codimension
Minimal Primes	Components
Cohen Macaulay (CM)	Arithmetically Cohen Macaulay (ACM)
Gorenstein	Arithmetically Gorenstein
Unmixed	Equidimensional

Let $V \subseteq \mathbb{P}^n$ be a closed subscheme with a saturated ideal $I_V \subset R = k[x_0, \dots, x_n]$. We say that V is ***Arithmetically Cohen Macaulay (ACM)*** if R/I_V is CM. Now let $\mathcal{I}_V$ be the ideal sheaf associated to V (one can view this as the sheafification of I_V). Let $r = \dim(V)$ and assume that $1 \leq r \leq n-2$. Now choose $1 \leq i \leq r$. We define the ith ***Hartshorne-Rao Module***, or ***Deficiency Module*** of V as follows:

$$(M^i)(V) := \bigoplus_{t \in \mathbb{Z}} \mathrm{H}^i(\mathbb{P}^n, \mathcal{I}_V(t))$$

We recall/state the following facts that will be helpful later:

- V is ACM $\iff$ the minimal free resolution has the form

  $$0 \to F_c \to F_{c-1} \to \cdots \to F_1 \to I_V \to 0$$

  Where $c = n - r$. Equivalently, it is of the form

  $$0 \to F_c \to F_{c-1} \to \cdots \to F_1 \to R \to R/I_V \to 0$$

- We can characterize (M^i) algebraically:

  $$(M^i)(V) = \left(\mathrm{Ext}_R^{n-i+1}(R/I_V, R)(-n-1)\right)^{\vee} = \mathrm{H}^{i+1}_{\mathfrak{m}}(I_V) = \mathrm{H}^i_{\mathfrak{m}}(R/I_V)$$

- In codimension 2, being Gorenstein and a Complete Intersection are equivalent.

- For a general line L, we have the short exact sequence

  $$0 \to \mathcal{I}_V(t-1) \xrightarrow{\cdot L} \mathcal{I}_V(t) \to \mathcal{I}_{(V \cap H)|_H}(t) \to 0$$

Lemma 3.4.1. *Let $V \subseteq \mathbb{P}^n$, $\dim(V) = r \geq 0$.*

(a) If $r = 0$, then V is ACM.

(b) If $r \geq 1$, V is ACM $\iff$ $(M^i)(V) = 0 \; \forall 1 \leq i \leq r$.

(c) if V is ACM, V is equidimensional.

(d) if V is ACM and of dimension ≥ 1, V is connected.

For example, the union of a line and a point if not ACM, but all lines in $\mathbb{P}^3$ are ACM, as are complete intersections (in any codimension). determinental varieties are also ACM.

3.4.2 Linkages of Subvarieties

For $i = 1, 2$, $V_i, X \subseteq \mathbb{P}^n$, we can say that V_1 and V_2 are ***linked***, i.e. $V_1 \sim_X V_2$, if $I_X : I_{V_1} = I_{V_2}$ and $I_X : I_{V_2} = I_{V_1}$ for some arithmetically Gorenstein X. Furthermore, we say that V_1 and V_2 are ***geometrically linked*** if V_1 and V_2 have no common components.

Theorem 3.4.2. ***(Hartshorne, Schenzel)*** *Let $V_1, V_2 \subseteq \mathbb{P}^n$ be of dimension r, where $V_1 \sim_X V_2$, where X is arithmetically Gorenstein with minimal free resolution $0 \to R(-t) \to \cdots \to I_X \to 0$. Then,*

$$(M^i)(V)^\vee(n+1-t) = (M^{r-i+1})(V_2)$$

As a corollary of Hartshorne and Schenzel's Theorem, for even links the collection $(M^i)(V)$ just gets shifted; no duals. For example, when $r = 1$ (i.e. when we are talking about curves) then we are only considering (M^1) and comparing them explicitly. In that case, we see that linked curves in an even number of steps have the same Hartshorne-Rao modules (up to twisting one of them).

> **Open Problem 7.** It is currently an open question as to whether the converse holds; i.e. if V_1, V_2 are curves such that $(M^1)(V_1) = (M^1)(V_2)(\delta)$ for some δ, then are V_1 and V_2 linked? This is true in $\mathbb{P}^3$ due to Rao, but it is not true for complete intersection links.

3.4.3 Glicci Ideals

Lemma 3.4.3. *Being ACM is preserved under geometrically liaison. Furthermore, if V is in the **Gorenstein liaison class** of a complete intersection (otherwise referred to as V being **glicci**), then V is ACM.*

> **Open Problem 8.** It is currently open whether the converse of the second statement holds, i.e. does being ACM mean you are glicci?

We have the following progress:

- A theorem of Gaeta proves that this holds in codimension 2.
- Huneke and Ulrich proved that ACM cannot imply licci.
- Migliore and Nagel proved that this conjecture holds up to deformation.
- Migliore and Nagel proved also that if you step in to $\mathbb{P}^{n+1}$ at some point in the proof (i.e. you need to embed in something a bit bigger) then the conjecture holds.
- Kleppe, Migliore, and Miro-Rorg-Nagel-Peterson proved that Standard determinental varieties are glicci.
- Gorla proved that most determinental ideals are glicci.

We can also ask about the structure of a given liaison class. There is an Annals paper by Ulrich and Huneke discussing the specifics of this.

Chapter 4

Symbolic Powers

These classes were taught by Alexandra Seceleanu and Eloisa Grifo.

4.1 Lecture 1 - Seceleanu

This first lecture will focus on the geometric aspects and motivations of symbolic powers of ideals, with the second lecture focusing more on recent strides in the field.

Let R be a commutative ring and I a radical ideal of R. For geometry, the standard example here is $R = k[x_1, \dots, x_d]$ where $X = V(I)$. Let $\min_R(I)$ denote the set of minimal primes of I and $\operatorname{Ass}_R(I)$ the set of associated primes to R/I (the R subscript will be omitted when clear). Any ideal can be decomposed as the intersection of the minimal primes that contain it, or equivalently, any vanishing locus $V(I)$ is a union of vanishing loci of its minimal primes.

Before discussing symbolic powers, we first state some results about taking powers of ideals the "ordinary" way. Recall that $I^b \subseteq I^a \iff a \leq b$. Furthermore, $I^a I^b = I^{a+b}$ as expected. One can see that $\min(I^n) = \min(I)$, but I^n can have different associated primes than I. For instance, consider $I = (xy, xz, yz) \subseteq k[x, y, z]$. We can find the minimal primes by finding primes who's intersection determine I:

$$I = (x, y) \cap (x, z) \cap (y, z)$$

We can see that

$$I^2 = (x, y)^2 \cap (x, z)^2 \cap (y, z)^2 \cap \underline{(x^2, y^2, z^2)}$$

This change of decomposition is important, in particular $m \in \operatorname{Ass}(I^2)$ but $m \notin \operatorname{Ass}(I)$.

We define the n**th symbolic power** of I as follows in two equivalent ways:

$$I^{(n)} := \bigcap_{P \in \min(I)} I^n R_P \cap R = \left\{ f \in R \;\middle|\; \exists s \notin \bigcup_{P \in \min(I)} P \text{ such that } sf \in I^n \right\}$$

The first is more tautological but the second is more workable in practice. It's important to note here that there are two schools of thought; some prefer to define symbolic powers as the intersection over associated primes, but there are inherent hassles regarding this (i.e. $I^1 \neq I^{(1)}$ in general). We will use the minimal prime definition here. We have the following properties:

- $I^{(0)} = I^0 = R$
- $I^{(1)} = I$ when I is radical.
- $I^{(n)} \supseteq I^n$ in general.
- $I^{(b)} \subseteq I^{(a)} \iff a \leq b$.
- $I^{(a)} I^{(b)} \subseteq I^{(a+b)}$ (this implies that the family of symbolic powers form a graded family)
- $\text{Ass}(I^{(n)}) = \text{Ass}(I)$ when I is radical.
- $I^{(n)}$ can also be computed by removing the embedded primes of I (when I is radical).

Considering the example $I = (xy, xz, yz)$ as above, we know that (x^2, y^2, z^2) is an embedded component, so we can just remove that from the decomposition to get symbolic power:

$$I^2 = (x,y)^2 \cap (x,z)^2 \cap (y,z)^2$$

4.1.1 Computations

Letting $I = P_1 \cap \cdots \cap P_r$ where P_i are all primes, then $I^{(n)} = P_1^{(n)} \cap \cdots \cap P_r^{(n)}$. If I is in fact generated by a regular sequence, we get an even better characterization; $I^{(n)} = I^n$ for any n! Combining these two characterizations yields nice results; for instance the example I above had a decomposition into prime ideals each generated by regular sequences, so we only had to take the power of the ideals as opposed to the symbolic powers.

Theorem 4.1.1. ***(Zariski-Nagata, Eisenbud-Hochster)*** *Let $R = k[x_1, \ldots, x_d]$ and I be a radical ideal of R. Then,*

$$I^{(n)} = \bigcap_{I \subseteq m \in \text{MaxSpec}(R)} m^n$$

Geometrically, we can view m^n as parameterizing functions at the point $V(m) \in V(I)$ to order n.

This theorem actually suggests a much deeper problem referred to as the ***interpolation problem***.

4.1.2 The Interpolation Problem

Given a finite set of points $X \subseteq \mathbb{A}^d_k$ (or $\mathbb{P}^d_k$) find the $I(X)^{(n)}$, or the functions vanishing at these points at order at least n. This question is quite hard in general, but we can ask the following related question; what is the smallest degree of an element of $I(X)^{(n)}$? In other words, what is the smallest degree polynomial vanishing at all the points with order n? We'll refer to this polynomial as $\alpha(I(X)^{(n)})$.

Related to this interpolation problem is a conjecture of Nagata:

Open Problem 9. Suppose that X is a set of $r \geq 10$ very general points in $\mathbb{P}^2_k$, where k is a characteristic 0 field. Now consider $\lim_{n\to\infty} \frac{\alpha(I(X)^{(n)})}{n}$, defined to be the ***Waldschmidt constant*** of I. Nagata conjectured that the value of this limit ought to be $\sqrt{r}$. This is known when r is a perfect square.

Open Problem 10. Jarrobino extended this conjecture to higher dimensions, i.e. if we are instead working over $\mathbb{P}^d_k$, we'd expect the Waldschmidt constant above to be $\sqrt[d]{r}$.

4.1.3 The Containment Problem

Given I, we would like to find pairs (a, b) such that $I^{(a)} \subseteq I^b$.

Theorem 4.1.2. ***(Swanson)*** *Such pairs (a, b) exist $\iff \exists c = c(I)$ such that $I^{(cn)} \subseteq I^n$ for any $n \in \mathbb{N}$. In other words, pairs exist if and only if a is a linear function of b.*

This presents a natural question; what is c?

Theorem 4.1.3. ***(Ein-Lazarsfeld-Smith, Hochster-Huneke, Ma-Schwede, Murayama)*** *Suppose R is a regular ring and $h = \max(\text{ht}(P) \mid P \in \text{Ass}(I))$. Then $h = c$ as above.*

This is deeply related to the following statement in characteristic p:

Theorem 4.1.4. *If R is of characteristic p and $q = p^e$ for any e, then $I^{(hq)} \subseteq I^{[q]}$, with h as above.*

4.1.4 Introduction to Harbourne's Conjecture

This h is also related to the following conjecture of Harbourne:

Open Problem 11. (Harbourne's Conjecture) For $I \subseteq k[x_1, \ldots, x_d]$, $I^{(hnh+1} \subseteq I^n$ $\forall n \geq 1$.

In some sense, it is known that $I^{(a)} \subseteq I^{(b)}$ for $a = hb$, but it is known for $hb - h + 1 \leq a \leq hb$. This is clear for $n = 1$, but not known in general. In fact, the conjecture is false for $n = h = 2$ (i.e. $I^{(3)} \subseteq I^2$). We'll discuss this conjecture more in the second lecture.

Theorem 4.1.5. ***(Dumnicki - Szernberg - Tutaj - Gasinska)*** *Letting* $I = (x(y^3 - z^3), y(z^3 - x^3), z(x^3 - y^3)) \subseteq \mathbb{C}[x, y, z]$*, then* $I^{(3)} \subsetneq I^2$*.*

For this example, $V(I)$ corresponds to the 12 points in $\mathbb{P}^2_{\mathbb{C}}$ in the intersection of 9 lines. We can actually classify examples of this nature that come from reflection groups.

- 1 infinite family $I_{m,d}$ where $m, d \geq 3$,
- 5 sporadic examples:
  - 2 in $\mathbb{P}^2$ (Klein, Wiman)
  - 1 in $\mathbb{P}^3$ (Drabkin, Seceleanu)
  - 1 in $\mathbb{P}^4$ (Drabkin, Seceleanu)
  - 1 in $\mathbb{P}^5$ (Drabkin, Seceleanu)

4.2 Lecture 2 - Grifo

This lecture will cover a wide variety of applications of taking symbolic powers.

4.2.1 The Differential Zariski-Nagata Theorem

We first introduce a differential version of the Zariski-Nagata Theorem above:

Theorem 4.2.1. *Suppose* $R = k[x_1, \ldots, x_d]$ *for* k *a perfect field. Let* $I \subseteq R$ *be a radical ideal. Then,*

$$I^{(n)} = \left\{ f \in R \mid \partial(f) \in I \; \forall \partial \in D^{n-1}_{R/k} \right\}$$

Before you ask, this is not a type (See the case where $n = 1$ to check this for yourself). We'll refer to $I^{\langle n \rangle} := \left\{ f \in R \mid \partial(f) \in I \; \forall \partial \in D^{n-1}_{R/k} \right\}$ as the nth differential power. There are natural ways to extend this theorem via slightly changing the initial assumptions. For instance, can we replace k by $\mathbb{Z}$? The answer is actually no; consider $R = \mathbb{Z}[x]$ and $m = (2, x)$. Then, $m^{(n)} = m^n$. Notice that $\partial(2) = 2\partial(1) \in m$, so hitting 2 with any differential operator lands in m. Thus, $2 \in m^{\langle n \rangle}$ for any n. However, $2 \notin m^{(2)}$.
We'll now introduce p-derivations to show when a statement like this does hold. Suppose $p \in ZZ$ (or more generally, $p \in R$ regular). A ***p*-derivation** on R is a map $\delta : R \to R$ such that:

- $\delta(1) = 0$.
- $\delta(ab) = a^p \delta(b) + \delta(a) b^p + p\delta(a)\delta(b)$
- $\delta(a + b) = \delta(a) + \delta(b) + \frac{a^p + b^p - (a+b)^p}{p}$

The upshot of this is that $\Phi(x) := x^p \delta(x)$ is a ring homomorphism (check this!).

Theorem 4.2.2. ***(De Stefani - Grifo - Jeffries, 2020)*** *Let $A = \mathbb{Z}$ (or any complete unramified DVR with perfect residue field) and $R = A[x_1, \ldots, x_d]$, with $Q \subseteq R$ a prime ideal, and $p \in Q$. Then,*

$$Q^{(n)} = \left\{ f \in R \mid (\delta^a \circ \partial) f \in Q,\ \partial \in D^b_{R/A},\ a + b < n \right\} =: Q^{\langle n \rangle_{mixed}}$$

Where $Q^{\langle n \rangle_{mixed}}$ is the nth ***mixed differential power***.

Now let's analyze changing another assumption of the differential Zariski Nagata theorem. What if the base field is not perfect? For instance if we let $k = \mathbb{F}_p(t)$ and we look at $R = k[x]$, Let $Q = (x^p - t)$. Then $Q^{(2)} = Q^2$ However, $\mathcal{D}^1 = R \oplus R\frac{\partial}{\partial x}$. However, $\frac{\partial}{\partial x}(x^p - t) = 0$, so $\mathcal{D}^1 \cdot (x^p - t)$ is contained in every single differential power, which means the theorem does not hold. We can, however, weaken the theorem as follows:

Theorem 4.2.3. ***(De Stefani - Grifo - Jeffries, 2020)*** *Let $R = A[x_1, \ldots, x_d]$ and Q prime,*

- *If A is any field or completely unramified DVR and $Q \cap A = (0)$, then $Q^{(n)} = Q^{\langle n \rangle_{\mathbb{Z}}}$*
- *If $Q \ni p$, $Q^{(n)} = Q^{\langle n \rangle_{\mathbb{Z},mixed}}$*

Here the corresponding derivations need to $\mathbb{Z}$-linear, hence the notation.

4.2.2 Harbourne's Conjecture, revisited

Let's go back to the c, h of the theorem of Swanson. We'll refer to this as the ***big height*** of I, and h refers to this for the rest of the talk. These theorems are over regular rings, but we can ask what happens if we introduce singularities. This suggests the following open problem:

Open Problem 12. If R is a complete noetherian local domain, is there $d = d(R)$ (independent of choice of ideal) such that $Q^{(dn)} \subset Q^n\ \forall n \geq 1$, for any prime ideal Q.

This is known to hold when R has an isolated singularity (proved by Huneke and others) but it is wide open in general.

This being said, let's return to the setting where R is regular.We previously discussed Harbourne's conjecture, which as stated above, is generally not true. The conjecture is in fact true in characteristic p precisely when the choices of n are a multiple of p. We can also say some things about other settings in which Harbourne's conjecture holds.

Theorem 4.2.4. ***(Grifo-Huneke, 2019)*** *If R/I is F-pure (equivalently, F-split) then I satisfies Harbourne's conjecture.*

Being F-pure is quite a broad condition. The following ideals are examples of things that satisfy this condition.

- When I is a square-free monomial ideal
- $I = I_t(X) = t \times t$ minors of a generic X.
- R/I is a Veronese Ring, i.e. $R/I \cong k[\text{monomials of degree } d]$.

There is currently a variant of Harbourne's Conjecture that remains open, referred to as the Stable Harbourne's Conjecture (though he is known not to like this name that much).

> **Open Problem 13.** For fixed I, $I^{(hn-h+1)} \subseteq I^n$ for all $n \gg 0$.

This conjecture actually holds for ALL known counterexamples of the "standard" Harbourne's conjecture, so it is absolutely wide open (though it is quite hard). We have the following nice simplification:

Theorem 4.2.5. ***(Grifo, 2019)*** *If $I^{(hn-h)} \subseteq I^n$ for some n, then the Stable Harbourne's Conjecture holds.*

Via Bocci and Harbourne, we define the ***resurgence*** $\rho(I) := \sup\left\{\frac{a}{b} \mid I^{(a)} \not\subseteq I^b\right\}$. Over regular rings, It is immediate that $1 \leq \rho(I) \leq h$. This number is computable, and is in some sense the "worst" case of failure. In fact, if $\rho(I) < h$, then the Stable Harbourne Conjecture holds for I. This is because $I^{(hn-c)} \subseteq I^n$ is implied by the fact that $\rho(I) = \frac{hn-c}{n}$, which holds $\iff n > \frac{c}{\rho(I)-h}$. When taking n sufficient large, it is sufficient to show that this lower bound is finite. When is this finite? When $\rho(I) \neq h$!

Theorem 4.2.6. ***(Grifo - Huneke - Mukundan, 2021)*** *Let $(R, \mathfrak{m})$ be a local ring and $I^{(n)} = (I^n : m^\infty)$ (i.e. we can multiply into the symbolic power from the power via some number of elements in the maximal ideal). If $I^{(hn-h+1)} \subset mI^n$ for some n, then $\rho(I) < h$.*

Removing the m is currently open, i.e.

> **Open Problem 14.** $I^{(hn-h+1)} \subset I^n$ for some n implies $\rho(I) < h$.

Chapter 5

Colloquium Talks

These are notes from colloquium talks given during the summer school. Unfortunately, my computer died before Gulio Cavligia's talk, so I have no notes from that.

5.1 Multiplicities in Commutative Algebra - Montaño

5.1.1 Some History

Multiplicities have their roots in intersection theory.

Theorem 5.1.1. ***(Bezout's Theorem)*** *Two plane curves C and D in $\mathbb{P}^2_{\mathbb{C}}$ intersect in* $\deg(C) \cdot \deg(D)$ *points counted with their intersection multiplicity.*

$$\deg(C) \cdot \deg(D) = \sum_{P \in C \cap D} \textit{mult}(P; C, D)$$

Bezout's theorem actually only yields a $\geq$ bound for the above equation in general, but equality where these curves intersect transversely. This yields the following questions:

What should mult$(P; C, D)$ be for any given point?

A big goal of modern intersection theory was to extend Bezout's Theorem to intersections of varieties $X, Y \subseteq \mathbb{P}^n$. The degree of a variety has been well understood in the geometric perspective (points remaining when we cut down by $\dim(X) - 1$ hyperplane sections) as well as from the algebraic perspective (via looking at the leading coefficient of the Hilbert polynomial) for over a century. That being said, the notion of multiplicity was not well defined up until the mid 1950s.

For $X, Y \subseteq \mathbb{P}^n$, we say they ***meet properly*** if for every irreducible component $Z \subseteq X \cap Y$, $\operatorname{codim}_{\mathbb{P}^n}(Z) = \operatorname{codim}_{\mathbb{P}^n}(X) + \operatorname{codim}_{\mathbb{P}^n}(Y)$. An early guess for mult$(Z; X, Y)$ was the length of the Artinian local ring of Z, $\mathcal{O}_{Z,z}$, where $z \in Z$ was the generic point. This worked well for $n \leq 3$, but the definition was ill formed for $n = 4$:

Lemma 5.1.2. *Let X be the image of the Veronese Embedding $\mathbb{P}^2 \to \mathbb{P}^4$ and $Y \subseteq \mathbb{P}^4$ be the 2-dimensional hyperplane $(0 : x_2 : x_3 : 0 : x_5)$. X and Y intersect at a single point, so the intersection multiplicity must be 4. That being said, $\ell(\mathcal{O}_{P,p}) = 5$.*

In 1951, we introduced Hilbert Samuel Multiplicity: Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and I an $\mathfrak{m}$-primary ideal. The Hilbert polynomial of $\mathcal{G}(I, R) = \bigoplus_{n \geq 0} I^n / I^{n+1}$ can be written as:

$$\ell(I^n / I^{n+1}) = e_0(I)\binom{n+d-1}{d-1} - e_1(I)\binom{n+d-2}{d-2} + \cdots + (-1)^{d-1}e_{d-1}(I)$$

$e_0(I)$ is the Hilbert Samuel Multiplicity of I:

$$e_0(I) := \lim_{n\to\infty} \frac{(d-1)!}{n^{d-1}}\ell(I^n / I^{n+1}) = \lim_{n\to\infty} \ell(R/I^n)$$

Intersection Theory has been quite rigorized in recent years, primarily due to Fulton's book on Intersection Theory (1984). This leads us to a generalized version of Bezout's Theorem:

Theorem 5.1.3. *(Generalized Bezout) Let $X, Y \subseteq \mathbb{P}^n$ be subvarieties meeting properly and $Z_1, \ldots, Z_r$ the irreducible components of $X \cap Y$. There exists a notion of intersection multiplicity such that*

$$[X][Y] = \sum_{i=1}^{r} mult(Z_i; X, Y)[Z_i]$$

The product $[X][Y]$ takes place in the ***Chow Ring*** $A^*(\mathbb{P}^n)$, and the class $[X]$ generalized the notion of degree as it encodes the idea of moving X continuously (i.e. X is defined up to rational equivalence). $\text{mult}(Z_i; X, Y)$ is precisely Serre's Intersection Multiplicity.

We now arrive to our first open problem. Let $(R, \mathfrak{m}, \mathfrak{K})$ be a regular local ring and P, Q be prime ideals such that $\sqrt{P+Q} = \mathfrak{m}$. Set

$$\chi(R/P, R/Q) := \sum_{i=1}^{\infty}(-1)^i \ell\left(\text{Tor}_i^R(R/P, R/Q)\right)$$

We know that $\dim(R/P) + \dim(R/Q) \leq \dim(R)$ via Serre. We also know that $\chi(R/P, R/Q) \geq 0$; this was initially a conjecture for many years but was proved by Gabber in 1995. If $\dim(R/P) + \dim(R/Q) < \dim(R)$ (i.e. the inequality is strict) in 1985 it was proved that $\chi(R/P, R/Q) = 0$. The converse of this, including in mixed characteristic, is open:

$$\dim(R/P) + \dim(R/Q) = \dim(R) \Rightarrow \chi(R/P, R/Q) > 0$$

Norcott and Rees in 1954 defined an ideal I to be a reduction of I if $J \subseteq I$ and $I^{n+1} = JI^n$ for $n \gg 0$. To denote this we write $J \subset_{red} I$. If J is a reduction of I¡ it is a relatively simple computation to show that J and I are the same up to radical and have the same integral closure. Their motivation came from the following observation about $\mathfrak{m}$ primary ideals: $J \subset_{red} I \Rightarrow J^n \supseteq J^n \supset J^n I^m = I^{n+m}$ when $m, n \geq 0$. Thus,

$$\ell(R/I^n) \leq \ell(R/J^n) \leq \ell(R/I^{n+m})$$

Fixing m and letting $n \to \infty$, we see that $e_0(I) = e_0(J)$. The converse was far more hard to show and surprising:

Theorem 5.1.4. ***(Rees)*** *If $(R, \mathfrak{m})$ is analytically unramified and $J \subseteq I$ are $\mathfrak{m}$-primary ideals such that $e_0(I) = e_0(J)$, then J is a reduction of I.*

This theorem is in some sense the model for the development of multiplicities in commutative algebra.

5.1.2 Hilbert Samuel Multiplicities

Nowadays, Multiplicities are extremely wide and far reaching.

- Hilbert Samuel (HS) Multiplicity
- Generalized Multiplicities (i.e. multiplicities outside the $\mathfrak{m}$-primary case)
- Multigraded multiplicities
- Multiplicities in Positive Characteristic (Hilbert-Kunz Multiplicities
- Multiplicities of multi-graded ideals.

We'll now discuss HS Multiplicity. We have a motivating conjecture by Lech:

Open Problem 15. (Lech's conjecture) If $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a flat extension of local rings, then $e_0(\mathfrak{m}) \leq "_0(\mathfrak{n})$. Nagata shows in 1956 that when R is unmixed, $e_0(\mathfrak{m}) = 1 \iff R$ is regular.

Thus, Lech's conjecture can be reformed as "singularities do not get better in flat extensions". We have made some progress since, and know Lech's conjecture is true in the following cases:

- $\dim(R) \leq 2$
- $S/\mathfrak{m}S$ is a complete intersection
- $\mathcal{G}(\mathfrak{m}, R)$ is a complete intersection
- $\dim(R) = 3$ and R contains a field
- R is the localization at the homogeneous maximal ideal of a standard graded ring over a perfect field.

When $(R, \mathfrak{m}, \mathfrak{K})$ is CM, we know that

- $e_0(I) - e_1(I) - \ell(R/I) \leq 0$ (1960)
- $e_1(I) \geq 0, e_2(I) \geq 0$, but $e_3(I)$ may be negative (1963)
- $e(\mathfrak{m}) \geq \mu(\mathfrak{m}) - \dim(R) + 1$ **(Abhyankar's Inequality)**
- R has minimal multiplicity (i.e. equality above) implies that $\mathcal{G}(\mathfrak{m}, R)$ is CM.

- if R has almost minimal multiplicity (i.e. at most 1 off) then the depth of $\mathcal{G}(\mathfrak{m}) \geq \dim(R) - 1$.
- Extensions of notions of minimal and almost minimal multiplicity for $\mathfrak{m}$-primary ideals.

One can also extend this multiplicity theory to "good" filtrations (i.e. the corresponding Rees-Algebra is Noetherian). Let $\overline{(-)}$ denote integral closure. Then,

$$\ell(R/\overline{I^{n+1}}) = \overline{e_0}(I)\binom{n+d}{d} - \overline{e_1}(I)\binom{n+d-1}{d-1} + \cdots + (-1)^{d-1}\overline{e_d}(I)$$

Where $\overline{e_i}(I)$ are the ***normal Hilbert coefficients*** of I. In some sense, these are more well behaved than the usual Hilbert coefficients. For instance, we know that $\overline{e_i}(I) \geq 0$ if $i \geq 3$,

Open Problem 16. This holds for all i (recall that even $e_3(I)$ can be negative).

5.1.3 Generalized Multiplicities

When I is not $\mathfrak{m}$-primary, we can use $\mathrm{H}^0_{\mathfrak{m}}$ to obtain

$$j(I) := \lim_{n\to\infty} \frac{(d-1)!}{n^{d-1}} \ell(\mathrm{H}^0_{\mathfrak{m}}(I^n/I^{n+1}))$$

which is the j multiplicity of I. We can also define

$$\varepsilon(I) := \limsup_{n\to\infty} \frac{(d)!}{n^d} \ell(\mathrm{H}^0_{\mathfrak{m}}(R/I^n))$$

which is the ε multiplicity of I. Cutosky proved the limit exists when R is analytically unramified. We have some properties:

- $j(I) \in \mathbb{Z}_{\geq 0}$
- $\varepsilon(I)$ is far harder to grasp; it can even be irrational!
- $j(I) > 0 \iff \ell(I) := \dim(\mathcal{F}(I)) = d$
- $\varepsilon(I) \leq j(I)$
- if I is $\mathfrak{m}$-primary, these both coincide with $e(I)$.

$j(I)$ has significant applications to intersection theory, serves as a numerical condition for integral dependence, relates to the depth of blowup algebras, and can compute generalized Hilbert Coefficients akin to the HS multiplicity case. ε-multiplicity is useful for an analogue or Rees's Theorem that does not require an $\mathfrak{m}$-primary hypothesis.

Open Problem 17. Unlike HS multiplicity, these generalized multiplicities $j(I), \varepsilon(I)$ are hard to compute. That being said, $e(I)$ has a number of properties that extend to $j(I)$ and $\varepsilon(I)$, but as of now it not known if many other properties extend. This is a natural open problem that one can work on.

Open Problem 18. One can also try to determine nice computational tools, or even work through some computations, for generalized multiplicities of other classes of ideals (like, for instance, binomial ideals).

5.1.4 Multigraded Multiplicities

Suppose k is a field. Let $R = k[x_{1,0}, \ldots, x_{1,m_1}] \otimes \ldots \otimes k[x_{p,0}, \ldots, x_{p,m_p}]$. $I \subset R$ is a ***multihomogeneous ideal*** if it is homogeneous in each block of variables. One can study the multiplicites of these ideals in a relative sense; keeping track of the relative degrees in each block of variables. Unfortunately, most definitions at this point are far too hard to type up, so I've given up.

If you're reading this, please take away the following lesson: NEVER try to live-LATEX a beamer/slide talk. These notes were taken over the course of 35 minutes and covered only 24/34 slides in the talk; I can't imagine ever trying to type 4+ pages of LATEX live over the course of a bit more than half an hour ever again.

5.2 Infinite Resolutions - Eisenbud

5.2.1 Resolutions of Group Algebras

Let G be a finite group, k a field of characteristic p, and $k[G]$ be the corresponding group algebra. We can ask ourselves, what is the representation theory of $k[G]$ like? We can resolve k as follows:

$$k \leftarrow k[G] \leftarrow F_\bullet$$

We can then take the Group Cohomology $\mathrm{H}^*(G, M) = \mathrm{Ext}^*_{k[G]}(k, M)$ which helps us understand the representation theory of $k[G]$; be warned though, the resolutions as above are often infinite.

Theorem 5.2.1. ***(Tate, 1958)*** *Suppose $G = \mathbb{Z}/p \oplus \cdots \oplus \mathbb{Z}/p$ and k is a field of characteristic p. Suppose we have n copies of $\mathbb{Z}/P$; let g_i $(1 \leq i \leq n)$ denote a generator of the ith copy of $\mathbb{Z}/p$. Then*

$$k[G] = k[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$$

Let $k[G] = R$. We have the Koszul Complex

$$0 \leftarrow k \leftarrow R \xleftarrow{(x_1, \ldots, x_n)} R^n \leftarrow \bigwedge^2 R^n \leftarrow \cdots \leftarrow \bigwedge^n R^n \leftarrow 0$$

After localizing at $(x_1 \dots x_n)$, we see that any f_j in the maximal ideal of R is of the form $f_j = \sum x_i h_{ij}$. Notice that f_j is determined by a column vector that passes through $R^n \to R$, multiplied by the row vector $(x_1, \dots, x_n)$. Moving back further, we have the map $d : \bigwedge^2 R^n \to R^n$ satisfying the identity $d(fg) = (df)g + (-1)^{\deg(f)} f(dg)$. Tate's idea was to tensor $\bigwedge^2 R^n$ with the free algebra R^g, and extend this notion to the entire wedge algebra $\bigwedge R^n$. Letting $D(R^g)$ be the ***divided power algebra*** (which is isomorphic to $(\mathrm{Sym}_R(R^g))^*$) we get the complex

$$
\begin{array}{ccccccccccccc}
0 & \longleftarrow & k & \longleftarrow & R & \underset{(x_1,\dots,x_n)}{\longleftarrow} & R^n & \underset{d}{\longleftarrow} & \bigwedge^2 R^n & \longleftarrow & \bigwedge^3 R^n & \longleftarrow & \bigwedge^4 R^n & \cdots \\
 & & & & & & & \nwarrow & & & & & & \\
 & & & & & & & & \oplus & & \oplus & & \oplus & \cdots \\
 & & & & & & & & & & & & & \\
 & & & & & & & & R^g & \longleftarrow & R^n \otimes R^g & \longleftarrow & \bigwedge^2 R^n \otimes R^g & \cdots \\
 & & & & & & & & & & & \nwarrow & & \\
 & & & & & & & & & & & & \oplus & \cdots \\
 & & & & & & & & & & & & D_2(R^g) & \cdots \\
 & & & & & & & & & & & & & \ddots
\end{array}
$$

One may think that we would need to keep expanding out. The beauty of this is that Tate showed you actually do not need to lift much.

Theorem 5.2.2. ***(Tate)*** *If R is a complete intersection, then the minimal resolution of k is the free commutative differential algebra on R^n and R^g.*

Outside of the space of local intersections the picture is not as nice (but still workable!)

$$
\begin{array}{ccccccc}
k & \longleftarrow & R^n & \longleftarrow & \bigwedge^2 R^n & \longleftarrow & \bigwedge^R R^n \\
 & & & \nwarrow & & & \\
 & & & & \oplus & & \oplus \\
 & & & & R^g & \longleftarrow & \bigwedge^n R^n \otimes D(R^g) \otimes AltR^h
\end{array}
$$

Outside of the complete intersection case, you need to add additional variables to account for the noise. We have results bounding how many variables we need to add, but suffice to say this is hard to pull off consistently. In the local case, we have the following result due to Gallikson:

Theorem 5.2.3. $(\mathrm{R}, \mathfrak{m}, \mathfrak{K})$ *is a local ring. Then the minimal R-free resolution of $\mathfrak{K}$ is a free commutative differential graded algebra (DGA) in finitely many variables.*

5.2.2 A Problem of Serre and Kaplansky

Now suppose that $I, J \subset (R, \mathfrak{m})$ where $\sqrt{I+J} = \mathfrak{m}$. We define the multiplicity $\text{mult}(R/I, R/J) := \sum_{i \geq 0}(-1)^i \ell\left(\text{Tor}_i^R(R/I, R/J)\right)$.

Open Problem 19. (Serre, Kaplansky) If $(R, \mathfrak{m}, \mathfrak{K})$ is a local ring and $F_\bullet$ is the minimal resolution of k, then

$$\sum(\text{rank} F_i)t^i = \sum \dim_k \text{Tor}_i^R(\mathfrak{K}, \mathfrak{K})t^i$$

is a rational function.

There is a lot of literature pertaining to this, with it being proved true in various cases. Oddly enough, a counterexample was found via a topologist, who claimed that they had a counter example for quite some time. Thus, while this is not true in general, we can still ask when this condition holds.

We define the ***socle*** of a module M over a local ring $(R, \mathfrak{m})$ as all the elements annihilated by the maximal ideal. In particular, we say that $\text{Soc}(M) := (0 :_M \mathfrak{m})$. We can ask, when taking a resolution of a module which syzygies admit a Socle summand? A theorem of Eisenbud (and others) shows that if $\dim_k \frac{\mathfrak{m}}{(\mathfrak{m}I):(I:\mathfrak{m})} \geq 2$, the 7th syzygy of any finitely generated R-module has a summand isomorphic to $\mathfrak{K}$.

5.3 Differential Graded Algebras - Miller

We'll abbreviate these to DGAs when clear.

5.3.1 Motivating Examples

Let R be a commutative ring (local or possibly graded when necessary). Before defining a DGA we provide some useful examples:

- A Koszul complex α on $f_1, f_2 \in R$:

$$0 \to R \xrightarrow{\begin{bmatrix} -f_2 \\ f_1 \end{bmatrix}} R^2 \xrightarrow{[f_1\ f_2]} R \to 0$$

  Which we can instead view as

$$0 \to \bigwedge^2 F \xrightarrow{\begin{bmatrix} -f_2 \\ f_1 \end{bmatrix}} \bigwedge^1 F \xrightarrow{[f_1\ f_2]} \bigwedge^0 F \to 0$$

  With bases $\{e_1e_2 = e_1 \wedge e_2\}$, $\{e_1, e_2\}$, and $\{e\}$ respectively.

  This yields, among other things, a well defined differential map ∂. On the other hand,note that $\bigoplus F_i = \bigoplus \bigwedge^i F = \bigwedge F$, which is the ***exterior algebra*** $R\langle e_1, e_2\rangle$. This gives us a ring. The ring and differential interact as follows:

$$\partial(e_1e_2) = \partial(e_1)e_2 - e_1\partial(e_2)$$

- This example is from Algebraic Topology. Suppose X is a tooplogical space and $C^\bullet(X)$ a chain co-complex used to compute cohomology. The cup product turns $C^\bullet(X)$ into a DGA.

We're now ready to define a DGA. Let R be a commutative ring. A ***Differential Graded Algebra (DGA)*** over R is a complex of R-modules

$$\dots \xrightarrow{\partial} F_2 \xrightarrow{\partial} F_1 \xrightarrow{\partial} F_0$$

equipped with a product making $\bigoplus F_i$ into a ***graded commutative*** (i.e $ab = (-1)^{\deg(a)\deg(b)}ba$) R-algebra such that the Leibniz rule (product rule) holds for any homogeneous a, b (i.e. $a \in F_{\deg(a)}, b \in a \in F_{\deg(b)}$:

$$\partial(ab) = \partial(a)b + (-1)^{\deg(a)}a\partial(b)$$

Equivalently, one can verify the product rule by checking that $\mu : F_\bullet \otimes_R F_\bullet \to F_\bullet$ defined by taking $a \otimes b = ab$ is a chain map. As a consequence of this, $\mathrm{H}(F_\bullet)$ inherits a well defined product. Thus, $\mathrm{H}(F_\bullet) = \bigoplus_i H_i(F_\bullet)$ is also an R-algebra. For the Koszul Complex, you have an algebra structure on the Koszul homology ring, as you do with the singular cohomology groups on a topological space to form a cohomology ring. Let's see another example:

- Suppose k is a field, $R = k[x, y]$, $I = (x^2, xy)$, and let's put a DGA structure on the minimal resolution of R/I.

$$0 \to R \xrightarrow{\begin{bmatrix} -y \\ x \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} x^2 & xy \end{bmatrix}} R \to R/I \to 0$$

  Take the basis element $1 \in R$. We need a multiplication map, so define $R = F_0 \times R_i \xrightarrow{\mu_i} F_i$. We'd like to show that such a definition generalizes. Consider the map $F_1 \times F_1 \to F_2$ sending $(e_1, e_2) \mapsto e_1e_2$ as desired. We need this to satify the Liebniz rule.

$$\partial(e_1e_2) = \partial(e_1)e_2 - e_1\partial(e_2) = x^2e_2 - xye_1 = x(xe_2 - ye_1) = x\partial(y)$$

  Furthermore, $x\partial g = \partial xg$. We can thus define $ge_2 = xg$.

5.3.2 Commutative Algebra Motivation

There was a bit of study on resolutions if $R = k[x_1, \dots x_n]$ and M is a finitely generated module with finite length.

Lemma 5.3.1. ***(B,E, Dsottersag)*** *If $F_\bullet$ is a DGA, then the above criterion holds.*

Proof. We provide an outline below.

- Take a maximal regular sequence $f_1, \dots, f_n$ on $\mathrm{Ann}_R(M)$.

- Form a Koszul Complex $K = R\langle e_1, \dots, e_n\rangle$ where $\partial(e_i) = f_i$.

- If K is a "free" DGA then $\exists \varphi_\bullet : K \to F$ a map of DGAs such that $\varphi(e_{i_1}, \ldots, e_{i_p}) = \varphi(e_{i_1}) \ldots \varphi(e_{i_p})$, with a specification of the map $\varphi_1 : K_1 = R^n \to F_1$ such that $e_i \mapsto f_i$ where $f_i = \sum \lambda t_i$ where t_i generate I.
- We can check that $\varphi_\bullet$ is injective (check that $\mathrm{H}_n(\varphi)$ is injective, which implies φ_n is injective, which implies that φ_i is injective for all i). Thus, the rank of F_i is $\geq$ the rank of K_i, which is $\binom{n}{i}$.

□

Open Problem 20. This implies that odd resolutions capture this; in particular $\mathrm{rank}(f_i) \geq \binom{n}{i}$; a major conjecture.

here does exist a weaker conjecture to the unsolved one above, called the total rank conjecture.

Open Problem 21. (Total Rank Conjecture) $\sum \mathrm{rank} F_i \geq 2^n$.

Now the question is, when is there a DGA (and thus, when does the conjecture hold?)

- If $\mathrm{ProjDim}(R/I) \leq 3$ (Herzog, B,E)
- If $\mathrm{ProjDim}(R/I) \leq 4$ and R is Gorenstein (Kustin, Miller)

In both these cases, these DGAs are in fact minimal resolutions. also know when there does NOT exist a DGA nor minimal resolutions. Avramov found a counterexample when $\mathrm{ProjDim}(R/I) = 4$ and Srinivasan found a counterexample when $\mathrm{ProjDim}(R/I) = 5$ (and in this case, R is even Gorenstein!). If we can forgo the minimality requirements, we actually have a general criterion:

Theorem 5.3.2. ***(Tate, 1956)*** *There always exists a (possibly non-minimal) free resolution of R/I with a DGA structure.*

5.4 Computing Local Cohomology in Characteristic 0 - Walther

We're going to let $\mathbb{C}$ be a field of characteristic 0 (not necessarily algebraically closed, but in practice the case of the "real" $\mathbb{C}$ is all that matters). Let $R = \mathbb{C}[x_1, \ldots, x_n]$, and choose $I = (f_1, \ldots, f_k) \subseteq R$. We can then consider the Cech Complex

$$0 \to R \to \bigoplus_{i=1}^{k} R_{f_i} \to \bigoplus_{i<j}^{k} R_{f_i f_j} \to \cdots \to R_{f_1 \ldots f_k} \to 0$$

The cohomology of this Cech Complex is precisely $\mathrm{H}^i_I(R)$. In general, $\mathrm{H}^i_I(R)$ is either 0, R, or so big that they are not even finitely generated. Thus, we should instead consider computing these modules from a different perspective, at least if we have a hope to do so with a computer. We have seen that $\mathrm{H}^i_I(R)$ is finitely generated over the Weyl Algebra $\mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle =: D$. Conveniently, the Cech Complex above consists entirely of objects and morphisms that are compatible with the D-Module structure on local cohomology.

Do be warned, however! When $R = \mathbb{C}[x, y, z, w]$ and $f = xy + wz$, $R\left[\frac{1}{xy+zw}\right] \neq D \cdot \frac{1}{xy+zw}$, so we do not always generate in the way we'd expect. To resolve this, and continue to be able to interact with a computer to handle this computational problem, we need:

- A Gröbner Basis on D.
- To write R_f as a module over D.

5.4.1 Bernstein Sato Polynomials

Gelligo, Castor-Jiminez, and Kauri-Rudy/Saspfenning figured out how to deal with the first issue, provided that $x_i \cdot \partial_i \geq 1$. The second issue, is more interesting and has some inherent geometry baked into the question. Malgrange, Sato, and Bernstein have made some progress on this issue.

Theorem 5.4.1. *Suppose s is a new variable and we consider f^s. Then $\exists P(s) \in D[s]$ and $b(s) \in \mathbb{C}[s]$ such that*

$$P(s) \circ f \circ f^S = b(s) \circ f^s \neq 0$$

Let $B_f = \{b(s)\}$ be the set of such polynomials $b(s)$. B_f is an ideal of $\mathbb{C}[s]$, so it is principally generated as $\mathbb{C}[s]$ is a PID. Let $b_f(s)$ be this principal generator, denoted the ***Bernstein-Sato Polynomial*** of f. Such P may not be canonical or easier to find, but we can take $b_f(s)$ to be unique with respect to a given f.

Let r_f be the minimal $\mathbb{Z}$ root of $b_f(s)$, and take $A_f(s) := \mathrm{Ann}_{D(s)}(f^s)$. We can see from the construction above that $P(s) \cdot f - b(s) \in A_f(s)$. It follows that $b_f(s) \in D \cdot (A_f(s), f)$. Thus it is important to compute what $A_f(s)$ is. Before we do that, we state some facts about Bernstein Sato Polynomials:

- $b_f(s)$ factors completely over $\mathbb{Q}$, proved by Kashiwara.
- Furthermore, due to Saito, all roots are contained between $(-n, 0)$.
- -1 is also a guaranteed root.

5.4.2 Computing Annihilators

Due to Malgrange, we can consider $D\langle t, D_t \rangle$. Let $\mathcal{H} = \mathrm{H}_{t-f}(R[t]) = D\langle t, D_t \rangle \cdot \underbrace{\overline{\frac{1}{t-f}}}_{=:\delta}$.

$$\mathrm{Ann}_{D\langle t, D_t \rangle}(\delta) = D\langle t, D_t \rangle \cdot (t - f, \{\partial_i + f_i \partial_t\}_{1 \leq i \leq n})$$

We have a morphism $D[s] \to D\langle t, D_t \rangle$ where D is mapped to itself and $s \mapsto -\partial_t t$. Thus we can ask, what is the $D[s]$ module generated by δ within $D\langle t, D_t \rangle$. Well,

$$sf_i \cdot d = -\partial_t t \cdot f_i \cdot \delta = -\partial_t \overline{\frac{tf_i}{t-f}} = -\partial_t \overline{\frac{ff_i}{t-f}} = \overline{\frac{ff_i}{(t-f)^2}}$$

Also,

$$f \cdot \partial_i \cdot \delta = f \partial_i \overline{\frac{1}{t-f}} = \overline{\frac{ff_i}{(t-f)^2}}$$

Yielding the same thing. It follows that $sf_i - f\partial_i \in \mathrm{Ann}_{D(s)}(\delta)$. Thus, we've reduced to finding $\mathrm{Ann}_{D(s)}(\delta)$, and we know what $\mathrm{Ann}_{D\langle t, D_t \rangle}(\delta)$ looks like! This yields the general problem: Given some left ideal $J \subseteq D\langle t, D_t \rangle$, we'd like to compute $J \cap D[s]$. Oaku in 1995 said the following: Take new variables u, v with a weighting $u \mapsto 1, v \mapsto -1, t \mapsto 1, \partial_t \mapsto -1$. Using this, homogeneous I to get $\widetilde{I}$. Now consider the Gröbner basis of $\widetilde{I} + D\langle u, v, t, \partial_t \rangle + (uv - 1)$ that eliminates u and v. As we've fed the Gröbner basis algorithm a homogeneous input, it yields a homogeneous output. Bring this output to degree 0 using only t or ∂_t on the left. The ideal this generates is precisely $J \cdot D\langle -\partial_t t \rangle$. As $-\partial_t t$ lives in $D[s]$ which is principally generated, you are now done.