Preliminary Info

These are notes taken from Math 553 (Algebraic Geometry II) at the University of Illinois Chicago in Spring 2023. The class was taught by Professor Kevin Tucker, and these notes are TeXed by Vignesh Jagathese (me!). I've added some supplements in the earlier sections from last year's course, taught by Lawrence Ein, as well as a bit of exposition from Hartshorne that I found helpful at times. I am still missing notes for 2/17, 2/20, and 3/03, so if you have a copy of these notes and are reading this, please send them to me so I can update this document. Otherwise, these notes roughly follow chapters 2 and 3 of Hartshorne in order, with a bit of a supplement at the end previewing some of the results in chapter 4. This is by no means a complete summary of the material in Hartshorne (in particular, very few exercises are presented here) so I would use these notes as a companion when reading the book, not a substitute for reading it.

There are still some minor errors in these notes, and by reading these and benefiting from them, all I ask in return is that you forward any errors to me. If there any suggestions you would recommend or additions that you would like to add, please email me at vjagat2 [at] (My graduate school's website).

Contents

Chapter 1

Schemes

1.1 An Introduction to Sheaves

Sheaves are a basic topic that is important not only to algebraic geometers, but everyone (basically). Suppose that X is a topological space.

1.1.1 Presheaves

We define a ***presheaf*** $\mathscr{F}$ on X of Abelian Groups as the following assignment:

(1) each $U \subseteq X$ open is assigned to an abelian group $\mathscr{F}(U)$ (where $\mathscr{F}(\varnothing) = 0$). We call the $\mathscr{F}(U)$ the group of ***sections*** of U.

(2) If $U \subseteq V \subseteq X$ are both open sets, then there is a group homomorphism $\rho_{VU} : \mathscr{F}(V) \to \mathscr{F}(U)$, which is the restriction morphism $s \mapsto s|_U$. It's clear that $\rho_{UU} = \text{Id}$ and if $U \subseteq V \subseteq W \subseteq X$, then $\rho_{VU} \circ \rho_{WV} = \rho_{WU}$, since for a section$s$ of W, it ought to be the case that $(s|_V)_U = s_U$.

To encode this information, we can define the category $\text{Top}(X)$ to be the category where open sets of X are the objects, and

$$\text{Hom}_{\text{Top}(X)}(U_1, U_2) = \begin{cases} \{i : U_1 \hookrightarrow U_2\} & U_1 \subseteq U_2 \\ \varnothing & U_1 \not\subseteq U_2 \end{cases}$$

With this nomenclature, we can simply define a presheaf of abelian groups to be a contravariant functor $\mathscr{F} : \text{Top}(X) \to \text{AbGp}$. It's worth noting that we can define presheaf that takes values in categories other than Abelian Groups, but this category is the natural setting for algebraic geometry.

1.1.2 Sheaves

A ***sheaf*** is a presheaf that is locally determined. More explicitly, if $U \subset X$ is an open set, we suppose that $U = \bigcup U_\alpha$ for smaller open sets U_α, where $U_{\alpha\beta} := U_\alpha \cap U_\beta$. If $s_\alpha \in U_\alpha$ (and similarly for U_β where $s_\alpha|_{U_{\alpha\beta}} = s_\beta|_{U_{\alpha\beta}}$ (i.e. the sections agree on the intersection, for any α and β) there must exist a unique section $s \in \mathscr{F}(U)$ so that $s|_{U_\alpha} = s_\alpha$. The "uniqueness" part of this definition is equivalent to saying that $s = 0 \iff s_\alpha = 0$ for all α, which is often used to define a sheaf in literature. You may often see this requirement split into two axioms as follows:

(a) (Separation) For $V \subset X$ open, suppose that V has an open cover $V = \bigcup_{i\in\mathcal{I}} U_i$, and choose $t, s \in \Gamma(V, \mathscr{F})$. Then $\exists s_i, t_i \in \mathscr{F}(U_i) = \Gamma(U_i, \mathscr{F})$ such that $s|_{U_i} = s_i$ and $t|_{U_i} = t_i$ for all $i \in \mathcal{I}$ (Note: $s|_{U_i}$ is defined to be the image of s under the (canonically defined) morphism $\mathscr{F}(V) \to \mathscr{F}(U_i)$.)

    The Separation Axiom states that if s and t agree on all components of the open cover (i.e. $s_i = t_i \ \forall i \in \mathcal{I}$) then they must agree globally (i.e. $s = t$). Intuitively, this means that behavior can be defined locally.

(b) (Gluing). For $V \subset X$ open, suppose that V has an open cover $V = \bigcup_{i\in\mathcal{I}} U_i$ as before, and choose $s_i \in \Gamma(U_i, \mathscr{F})$. If $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j} \ \forall i, j \in \mathcal{I}$, then there must exist $s \in \Gamma(V, \mathscr{F})$ such that $s|_{U_i} = s_i$. Intuitively, this means that local behavior must be defined compatibly, or in other words, sections are compatible with each other.

The object s constructed from the s_i via the Gluing Axiom is guaranteed to unique, via the Separation Axiom. Thus, one can intuitively define a sheaf as **a presheaf with compatible sections that can be uniquely glued together.** Less intuitively, one can define a sheaf as a presheaf with the following exact equalizer sequence for any open $V \subseteq X$ with open cover $V = \bigcup_{i\in\mathcal{I}} U_i$.

$$\begin{tikzcd} \mathscr{F}(V) \arrow[r, "\psi"] & \prod_{i\in\mathcal{I}} \mathscr{F}(U_i) \arrow[r, shift left, "\varphi_1"] \arrow[r, shift right, "\varphi_2"'] & \prod_{i,j\in\mathcal{I}} \mathscr{F}(U_i \cap U_j) \end{tikzcd}$$

Where $\psi(s) = (s|_{U_i})_{i\in\mathcal{I}}$, $\varphi_1((s_\ell)) = (s_i|_{U_i\cap U_j})_{i,j\in\mathcal{I}}$, and $\varphi_2((s_\ell)) = (s_j|_{U_i\cap U_j})_{i,j\in\mathcal{I}}$. For those of you complaining that the Sheaf definition requires that one chooses elements from $\mathscr{F}(V)$ (and thus makes assumptions about the category the sheaves take values in), the less intuitive definition above can sate your concerns. Morphisms of sheaves are defined identically to morphisms on presheaves (more concretely, as sheaves are presheaves, for sheaves $\mathscr{F}, \mathscr{G}$, $\mathcal{H}om(\mathscr{F}, \mathscr{G}) := \mathrm{Hom}_{\mathrm{Sh}}(\mathscr{F}, \mathscr{G}) := \mathrm{Hom}_{\mathrm{PreSh}}(\mathscr{F}, \mathscr{G})$.) We can also define a notion of a ***Subsheaf***, where $\mathscr{G} \subset \mathscr{F}$ as sheaves if $\mathscr{G}(U)$ is a substructure (subgroup, subring, etc.) of $\mathscr{F}(U)$ for any $U \subset X$.

Examples

- Let A be any abelian group. Let $\mathscr{F}(U) = A$ if $U \neq \varnothing$ and $\mathscr{F}(\varnothing) = 0$. This is the ***constant** A**-presheaf*** on X. The sections of this sheaf can be viewed as constant morphisms $U \to A$ where U maps to that specified element in A. This is a presheaf, but not necessarily a sheaf. If X were to have multiple components, for example, this would not satisfy the above sheaf axiom.

- Let $\underline{A}_X$ be the sheaf of locally constant functions from $U \to A$. This is indeed a sheaf.

- We could assign $\mathscr{F}(U) = \{\text{continuous functions } U \to \mathbb{R}\}$. This is the sheaf of continuous functions on a topological space (often referred to as $C^1(U)$). This is very clearly a sheaf, as continuity is a local property. You can replace $\mathbb{R}$ with any topological group and this will still be a sheaf.

- Suppose now that X is a smooth manifold. We can define an analogous sheaf to the previous example, but this time assigning $\mathscr{F}(U)$ to be the set of all smooth functions $U \to \mathbb{R}$. This is often denoted $C^\infty(U)$. You can replace $\mathbb{R}$ with $\mathbb{C}$ and instead consider sheaves of analytic functions.

- Suppose now that X is an algebraic variety over an algebraically closed field k. Letting $\mathcal{O}_X(U)$ be the set of regular functions $U \to k$, we can see that $\mathcal{O}_X$ is a sheaf.

1.1.3 Morphisms of (Pre)Sheaves

As presheaves are functors, morphisms of presheaves are just natural transformations between functors. Explicitly, if $\mathscr{F}$ and $\mathscr{G}$ are presheaves, then $\varphi : \mathscr{F} \to \mathscr{G}$ must satisfy the following diagram for any $U \subseteq V \subseteq X$:

$$\begin{tikzcd}
\mathscr{F}(V) \arrow[r, "\varphi_V"] \arrow[d, "\rho^{\mathscr{F}}_{VU}"] & \mathscr{G}(V) \arrow[d, "\rho^{\mathscr{G}}_{VU}"] \\
\mathscr{F}(U) \arrow[r, "\varphi_U"] & \mathscr{G}(U)
\end{tikzcd}$$

An analogous notion of morphisms for sheaves also holds. In practice, we write $\Gamma(U, \mathscr{F}) = \mathscr{F}(U)$ to denote the sections of U over the presheaf $\mathscr{F}$. We use this notation because $\Gamma(U, -)$ is naturally a covariant functor from the category of presheaves to AbGp. In fact, it is left exact! The good news is that the category of presheaves is an abelian category. For instance, we can define $\ker(\varphi)$ to be the presheaf $\ker(\varphi)(U) = \ker(\mathscr{F}(U) \to \mathscr{G}(U))$ and analogously define $\operatorname{coker}(\varphi)$ and $\operatorname{im}(\varphi)$.

The issue is that, in the category of sheaves, $\operatorname{im}(\varphi)$ and $\operatorname{coker}(\varphi)$ are not necessarily sheaves, just presheaves ($\ker(\varphi)$ is still a sheaf, though). Thus, to justify that the category of sheaves is abelian (it is) there is more work to be done.

1.1.4 Stalks

Suppose that $\mathscr{F}$ is a presheaf on X, and $p \in X$. Then we define $\mathscr{F}_p = \varinjlim_{p \in U} \mathscr{F}(U)$ to be the ***stalk*** of $\mathscr{F}$ at p. In practice, this is just the set of tuples (U, s) where $p \in U \subseteq X$ and $s \in \mathscr{F}(U)$, modulo the relation that $(U_1, s_1) \sim (U_2, s_2)$ when $\exists p \in V \subseteq U_1 \cap U_2$ such that $s_1|_V = s_2|_V$.

If $p \in U$, we get an induced map $\mathscr{F}(U) \to \mathscr{F}_p$ assigning a section s to the equivalence class $[(U, s)]$. This equivalence class is called the ***germ*** of s at p. If I have a morphism of (pre)sheaves $\varphi : \mathscr{F} \to \mathscr{G}$, this induces a map $\varphi_p : \mathscr{F}_p \to \mathscr{G}_p$, where $[(U, s)] \mapsto [(U, \varphi_U(s))]$.

If $\mathscr{F}$ takes values in AbGp, then

$$\mathscr{F}_p = \varinjlim \mathscr{F}(U_i) \cong \bigoplus_{i \in \mathcal{I}} \mathscr{F}(U_i) / \sim$$

With a similarly defined equivalence. Most generally, direct limits have a universal property where $\forall i, j \in \mathcal{I}$ such that $U_i \leq U_j$ (i.e. $U_j \subset U_i$), object A, and maps $\psi_i : \mathscr{F}(U_i) \to A$, $\psi_j : \mathscr{F}(U_j) \to A$, $\exists!$ morphism u_{ij} that makes the following diagram commute:

$$\begin{tikzcd}
\mathscr{F}(U_i) \arrow[rr] \arrow[dr, "\varphi_i"] \arrow[ddr, bend right, "\psi_i"'] & & \mathscr{F}(U_j) \arrow[dl, "\varphi_j"'] \arrow[ddl, bend left, "\psi_j"] \\
& \varinjlim \mathscr{F}(U_i) \arrow[d, dashed, "\exists! u_{ij}"] & \\
& A &
\end{tikzcd}$$

Here, φ_i, φ_j are canonically defined via the directed system. By construction of $\text{Top}(X)$, there is only one morphism $\mathscr{F}(U_i) \to \mathscr{F}(U_j)$, so this is canonically defined as well. Such a universal property suggests that objects in $\varinjlim \mathscr{F}(U_i)$ are of the form $(s_i)_{s_i \in \Gamma(U_i, \mathscr{F})}$ where $(s_i) \sim (s_j)$ if $\exists k$ such that $p \in U_k \subset U_i \cap U_j$ and $s_i|_{U_k} = s_j|_{U_k}$ (compare this to the gluing axiom, and let $U_k = U_i \cap U_j$).

1.1.5 Sheafification

At first glance, there seems to be a "gap" between a presheaf and a sheaf. A presheaf is merely a contravariant functor, defined purely algebraically/categorically. A sheaf on the other hand, defines associations that are intuitive and geometric, which a presheaf does not necessarily have to do. Given a presheaf $\mathscr{F}$, however, one can uniquely associate a sheaf to it, denoted $\mathscr{F}^+$, such that any morphism of presheaves $\mathscr{F} \to \mathscr{G}$ (for $\mathscr{G}$ a sheaf) factors uniquely through a morphism $\mathscr{F}^+ \to \mathscr{G}$. The process of converting $\mathscr{F}$ to $\mathscr{F}^+$ (or equivalently, constructing a sheaf $\mathscr{F}^+$ uniquely associated to $\mathscr{F}$) is called **Sheafification**.

Lemma 1.1.1. *(Existence of sheafified presheaves) For any presheaf $\mathscr{F}$, there exists a unique sheaf $\mathscr{F}^+$ and a morphism of presheaves $\theta : \mathscr{F} \to \mathscr{F}^+$ that extends any morphism to a sheaf of presheaves to a morphism of sheaves. Said in other words, if $\mathscr{G}$ is a sheaf and $\mathscr{F} \to \mathscr{G}$ is a morphism of presheaves, then we have a unqiue morphism of sheaves $\mathscr{F}^+ \to \mathscr{G}$ such that*

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\theta"] \arrow[dr] & \mathscr{F}^+ \arrow[d, dashed, "\exists!"] \\
& \mathscr{G}
\end{tikzcd}$$

We can construct $\mathscr{F}$ as follows: For $U \subset X$ open, set $\mathscr{F}^+(U)$ to be the set of all functions $U \to \sqcup_{p \in U} \mathscr{F}_p$ with the conditions that:

- $s(p) \in \mathscr{F}_p \forall p \in U$
- $\exists$ an open cover $U = \bigcup_\alpha U_\alpha$ of X such that $s_\alpha \in \mathscr{F}(U_\alpha)$ where $s(p) = (s_\alpha)_p \; \forall p \in U_\alpha$.

It's clear from this description that a presheaf $\mathscr{F}$ is a sheaf if and only if $\theta : \mathscr{F} \to \mathscr{F}^+$ is an isomorphism. This tells us that any sheaf can be viewed as a sheaf of functions where restriction maps in the target are precisely restrictions of those functions.

So what is being done here? Our presheaf does not necessarily track local data properly, so to remedy this, we "break up" the data into the "smallest" pieces possible (stalks at points). Then, we carefully glue the data at the stalk level together in a way that is (uniquely) compatible. If $\mathscr{F}$ was actually a sheaf to begin with, one easily checks that we are disassembling the sheaf and putting it back together again, suggesting that $\mathscr{F} \simeq \mathscr{F}^+$. Otherwise, we are disassembling $\mathscr{F}$ and putting it together in a compatible manner, and are doing so independent of choice of open sets and/or sections, suggesting that this construction is unique up to unique isomorphism.

Lemma 1.1.2. *In general,* $\mathscr{F}_p = (\mathscr{F}^+)_p \; \forall p \in X$.

Proof. The map θ is defined at an open set of U as a map $\mathscr{F}(U) \to \mathscr{F}^+(U)$ sending a section s to the map sending $p \mapsto s_p$ at each point $p \in U$. It follow that this defines a map on limits $\varinjlim \mathscr{F}(U) \to \varinjlim \mathscr{F}^+(U)$, as the morphisms are compatible with the directed system. It is easy to see that this is an isomorphism. $\square$

Sheafification allows us to define kernels, cokernels, and images in the category of sheaves. In particular, we just take those objects within the category of presheaves and sheafify them.

Lemma 1.1.3. *If* $\varphi : \mathscr{F} \to \mathscr{G}$ *is a morphism of sheaves, then* φ *has property* $\mathfrak{P}$ *if and only if* φ_p *has property* $\mathfrak{P}$ *at all stalks.*

Property $\mathfrak{P}$ could be being an injective, surjective, or an isomorphism. Do be warned though that this logic does not translate to sections of open sets, i.e. the statement "φ is a surjection $\iff$ φ_U is a surjection for all $U \subseteq X$" is **NOT** TRUE (though this hold for injections).

An important example of this is verifying exactness.

Lemma 1.1.4. *Let* $\mathscr{F}, \mathscr{G}, \mathscr{H}$ *be sheaves on X. Let* $\varphi : \mathscr{F} \to \mathscr{G}$ *and* $\psi : \mathscr{G} \to \mathscr{H}$ *be morphisms of sheaves.*

(a) $\mathscr{F} = 0 \iff \mathscr{F}_p = 0 \; \forall p \in X$.

(b) $\varphi = 0 \iff \varphi_p = 0 \; \forall p \in X$.

(c) $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ *is a short exact sequence of sheaves* $\iff 0 \to \mathscr{F}_p \to \mathscr{G}_p \to \mathscr{H}_p \to 0$ *is exact* $\forall p \in X$.

Proof. For (a), the forward case is immediate so we focus on the reverse case. If $\mathscr{F}_p = 0$ for all $p \in X$, then $\coprod_{p\in X} \mathscr{F}_p = 0$, suggesting that all morphisms $U \to \coprod_{p\in X} \mathscr{F}_p$ are trivial, for any choice of $U \subset X$ open. Therefore, we can conclude that $\mathscr{F}^+$ is trivial, but as $\mathscr{F}$ is a sheaf $\mathscr{F} \simeq \mathscr{F}^+ = 0$, and we can conclude.

The forward case for (b) is immediate as well, and for the reverse case we follow similar logic to part (a). $\varphi_p = 0 \; \forall p \in X$ suggests that $\coprod \varphi : \coprod \mathscr{F}_p \to \coprod \mathscr{G}_p$ is trivial, suggesting that $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ is trivial. $\mathscr{F}, \mathscr{G}$ are both sheaves, so $\varphi = \varphi^+ = 0$.

All that is left then is the reverse case of part (c). Recall that a complex is exact if and only if $\varphi : \mathscr{F} \to \mathscr{G}$ is injective (i.e. trivial kernel) and $\psi : \mathscr{G} \to \mathscr{H}$ is trivial (i.e. trivial cokernel), with exactness in the middle. Well, $0 \to \mathscr{F}_p \to \mathscr{G}_p \to \mathscr{H}_p \to 0$ being exact $\forall p \in X$ suggests that $\ker(\varphi_p) = 0$, $\operatorname{coker}(\psi_p) = 0$. and $\operatorname{im}(\varphi_p) = \ker(\psi_p)$. for all $p \in X$.

Part (a) coupled with lemma 1.1.4 tells us that $\ker(\varphi) = \operatorname{coker}(\psi) = 0$, so $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is a short exact sequence of sheaves. □

1.1.6 Abstract Algebra of Sheaves

We say that $\mathscr{F}' \subseteq \mathscr{F}$ is a ***subsheaf*** if $\mathscr{F}'(U) \subset \mathscr{F}(U)$ for all $U \subset X$ open. This is also something that we can check on stalks, i.e. $\mathscr{F}'$ is a subsheaf of $\mathscr{F}$ if there exists a morphism $\mathscr{F}' \to \mathscr{F}$ such that $\mathscr{F}'_p \to \mathscr{F}_p$ is an inclusion for any p.

A quotient sheaf is defined using this nomenclature. If $\mathscr{F}' \subseteq \mathscr{F}'$, the ***quotient sheaf*** $\mathscr{F}/\mathscr{F}'$ is the sheafification of the presheaf of morphisms $U \to \mathscr{F}(U)/\mathscr{F}'(U)$ satisfying the standard conditions above.

We can define notions of exactness of complexes of sheaves identically to the case of modules. Note that exactness is also a property that can be checked on stalks. One can also take direct sums, (inverse) limits, and Hom sets (often denoted $\mathcal{H}om$ to clarify that this is a sheaf Hom) of sheaves. The interesting thing about these is that all of these defined naively (i.e. pointwise) are actually already sheaves, and there is no need to sheafify.

1.1.7 Switching Base Spaces on Sheaves

Suppose that $f : Y \to X$ is a continuous function of topological spaces. We'd like to define a functor f_* that sends sheaves on Y to sheaves on X. If $\mathscr{G}$ is a sheaf on Y, let

$$(f_*\mathscr{G})(U) := \mathscr{G}(f^{-1}(U))$$

For $U \subset X$. This indeed defines a sheaf on X, allowing us to pull back sheaves. We'd also like to define an analogous functor f^{-1} that sends sheaves on X to sheaves on Y. Define

$$(f^{-1}\mathscr{F})(V) := \left(\varinjlim_{U \supseteq f(V)} \mathscr{F}(U) \right)^+$$

This allows us to pushforward sheaves. It turns out that f_* is left exact and f^{-1} right exact, as determined by the adjunction

$$\operatorname{Hom}_{\mathrm{Sh}(X)}(\mathscr{F}, f_*\mathscr{G}) = \operatorname{Hom}_{\mathscr{H}(Y)}(f^{-1}\mathscr{F}, \mathscr{G})$$

There is a nice duality between these two constructions. For pullbacks, it is easy to write down sections but hard to compute stalks. Conversely, it is hard to write down sections of pushforwards, but stalks are easy to compute. In particular,

$$(f^{-1}\mathscr{F})_y = \Big(V \mapsto \varinjlim \mathscr{F}(U)\Big)_y = \varinjlim_{y\in V}\left(\varinjlim_{U\supset f(V)\ni f(y)} \mathscr{F}(U)\right) = \varinjlim_{U\ni f(y)} \mathscr{F}(U) = \mathscr{F}_{f(y)}$$

Examples

- If $Z \subset X$ is a subspace, then we can restrict a sheaf $\mathscr{F}$ over X to a sheaf on Z (denoted $\mathscr{F}|_Z$) by taking the pushing forward with respect to the inclusion map. This we can define $\mathscr{F}|_Z := i^{-1}\mathscr{F}$
- If I take a point $p \in X$, I can take the map $i : \{p\} \to X$ and an abelian group A to define $\mathscr{F} := i_*A$ (A is representing the constant sheaf in this abuse of notation). This sheaf is precisely

  $$\mathscr{F}(U) = \begin{cases} A & p \in U \\ 0 & p \notin U \end{cases}$$

  where on stalks,

  $$\mathscr{F}_q = \begin{cases} A & q \in \overline{\{p\}} \\ 0 & q \notin \overline{\{p\}} \end{cases}$$

  Thus if p is a closed point, this sheaf assigns the stalk of A to a point p of X, and assigns zero elsewhere. This is called a ***skyscraper sheaf***.
- Suppose now that $Z \subset X$ is a closed subspace, and let $U = X/Z$. Let $i : Z \hookrightarrow X$ and $j : I \hookrightarrow X$. Now suppose that $\mathscr{F}$ is a sheaf on Z. Then $i_*\mathscr{F}$ is an extension by 0 on X
- If $\mathscr{G}$ is a sheaf on U, $j_!\mathscr{G}$ is the extension of U by zero on X, where

  $$j_!(\mathscr{G}) = \begin{cases} \mathscr{G}(V) & V \subseteq U \\ 0 & V \nsubseteq U \end{cases}$$

- Let X be a topological space. Define $\mathcal{O}_X^{\text{cont}}$ to be the **sheaf of continuous functions** on X, where

  $$\mathcal{O}_X^{\text{cont}}(U) = \{f : U \to \mathbb{R} \mid f \text{ continuous}\}$$

  As $\mathbb{R}$ has a commutative ring structure, we can define $(f+g)(x) := f(x)+g(x)$ and $(fg)(x) = f(x)g(x)$ to show that $\mathcal{O}_X^{\text{cont}}(U)$ is a commutative ring. $\mathcal{O}_X^{\text{cont}}$ functions similarly on morphisms, as $\mathcal{O}_X^{\text{cont}}(i : U \hookrightarrow V) : (f : V \to \mathbb{R}) \mapsto (i \circ f : U \to \mathbb{R})$. Thus, $\mathcal{O}_X^{\text{cont}}$ defines a contravariant functor from $\text{Top}(X) \to \text{CRing}$. It is easy to check that this is in fact a sheaf. If a notion of differentiability or smoothness exists on X (i.e. X is some C^r, C^∞ manifold), then $\mathcal{O}_X^{C^r}, \mathcal{O}_X^{C^\infty}$ can be similarly defined.

- Similarly, if X were a complex manifold, one can define sheaves $\mathcal{O}_X^{\text{hol}}$ and $\mathcal{O}_X^{*\text{hol}}$ as follows:

$$\mathcal{O}_X^{\text{hol}}(U) = \{f : U \to \mathbb{C} \mid f \text{ holomorphic}\}$$
$$\mathcal{O}_X^{*\text{hol}}(U) = \{f : U \to \mathbb{C}^* \mid f \text{ holomorphic}\}$$

  $\mathcal{O}_X^{\text{hol}}$ takes values in CRing as in the previous example, and $\mathcal{O}_X^{*\text{hol}}$ takes values in abelian groups, as $\mathbb{C}^*$ is an abelian group.

- Let X be a variety over k (algebraically closed). One can define a sheaf of regular functions $\mathcal{O}_X^{\text{reg}}$ similarly:

$$\mathcal{O}_X^{\text{reg}}(U) = \{f : U \to \mathbb{A}_k^1 \mid f \text{ regular}\}$$

- Suppose X and Y are topological spaces. Recall that $\text{Hom}(X, Y)$ is the set of continuous functions $X \to Y$, and that $\text{Hom}(-, Y)$ is contravariant. Thus, one can define a sheaf $\text{Hom}(-, Y)$ with values in Set as follows:

$$\text{Hom}(-, Y)(U) = \text{Hom}(U, Y)$$

- if $\mathscr{H}$ is a sheaf on X, we get a short exact sequence of sheaves

$$0 \longrightarrow j_! \mathscr{H}_U \longrightarrow \mathscr{H} \longrightarrow i_* \mathscr{H}|_Z \longrightarrow 0$$

1.2 An Introduction to Schemes

Schemes are essentially “glued together” rings, where the gluing occurs topologically. This gluing, despite seeming innocent, is generally hard to state, so we’ll do that over the course of this section.

1.2.1 The Zariski Topology

Suppose that R is a commutative ring. $X = \text{Spec}(R)$ is the set of prime ideals of R, equipped with the topology where we assign closed sets to be of the form $V(S)$ for a subset $S \subset R$, where $V(S) = \{P \in \text{Spec}(R) \mid S \subseteq P\}$. It’s easy to check that

$$V(S) = \bigcap_{f \in S} V(f) = \bigcap_{f \in S} V(R \cdot f) = V\left(\bigcap_{f \in S} R \cdot f\right)$$

A topology must be closed under finite unions and arbitrary intersections of closed sets, as well as include the empty set and itself. It is easily verified that

$$V(0) = X$$

$$V(1) = \varnothing$$
$$V(S_1) \cup V(S_2) = V(S_1 \cup S_2)$$
$$V(f) \cup V(g) = V(f \cdot g)$$
$$\bigcap V(S_\alpha) = V\left(\bigcup S_\alpha\right)$$

Applying $V(-)$ is order reversing on the radicals of ideals i.e.

$$V(I) \subset V(J) \iff \sqrt{I} \supseteq \sqrt{J}$$

We also have a Nullstellensatz-style correspondence, where $\{P\} \subset X$ is closed if and only if P is a maximal ideal. Furthermore, there is an opposite operation to $V(-)$, namely for a subset $Z \subseteq X$, we have that

$$I(Z) := \bigcap_{P \in Z} P$$

Where

$$V(I(Z)) = \overline{Z}$$
$$I(V(J)) = \sqrt{J}$$

Furthermore, $Z \subset X$ is a closed irreducible subset of X if and only if $I(Z)$ is a prime ideal. In this sense, $I(Z) \in X$ as a point, so we call $I(Z)$ the ***generic point*** of Z.

What about open sets, though? We let $D(f) = X \backslash V(f) = \{P \in \mathrm{Spec}(R) \mid\mid f \notin P\}$ be the set of ***distinguished open sets*** of X. These give a (topological) basis of all open sets. Observe that

$$D(f) \cap D(g) = D(fg)$$

We can also augment the ring and look at the new spectra.

$$\mathrm{Spec}(R/J) = V(J)$$
$$\mathrm{Spec}(R[1/f]) = D(f)$$

Lemma 1.2.1. *$D(f)$ is quasicompact.*

Proof. Suppose that $D(f)$ is contained in $\bigcup_\alpha D(g_\alpha)$ (we can assume all these open sets are of the form $D(g_\alpha)$ because we know it is a basis). It follows by taking complements that

$$V(f) \supseteq \bigcap_\alpha V(g_\alpha) = V((g_\alpha)_\alpha)$$

Thus, $\sqrt{(f)} \subset \sqrt{(g_\alpha)_\alpha}$, so $f^N = h_1 g_1 + \ldots h_r g_r$ for some $g_1, \ldots, g_r \in (g_\alpha)_\alpha$. It follows that

$$V(f) = V(f^N) \supseteq V(g_1, \ldots, g_r)$$

So

$$D(f) \subseteq \bigcup_{i=1}^{r} D(g_i)$$

□

Using this fact, we can conclude the following facts about the topology of $\mathrm{Spec}(R)$

- Quasicompact
- T_0
- Quasicompact open sets give a basis.
- The intersection of two quasicompact open sets is quasicompact
- nonempty irreducible closed subsets have generic points.

In general, topological spaces obeying these laws are called ***spectral spaces***, and it turns out these laws are sufficient to conclude that this topology arises from taking a ring spectra.

Theorem 1.2.2. ***(Hochster)*** *Any Spectral space is homeomorphic to* $\mathrm{Spec}(R)$ *for some ring R.*

1.2.2 Affine Schemes

Now that $\mathrm{Spec}(R) = X$ is a topological space, we'd like to associate it to a sheaf of rings $\mathcal{O}_X$. The idea is that, for a given open subset $D(g) \subset X$, we want to associate to it the ring $R[1/g]$, then sheafify it. Notice that under this presheaf assignment, the stalk at a point P is $\varinjlim_{g \notin P} R[1/g] = R_P$.

The construction hinted at above can be given explicitly as follows. choose an open set $U \subset X$. Assign $\Gamma(U, \mathcal{O}_X)$ to be the following set

$$\left\{ s : U \to \coprod_{P \in U} R_P \;\middle|\; s(P) \in R_P, \exists U = \bigcup_{\alpha} D(g_\alpha) \text{ and } \frac{f_\alpha}{g_\alpha^{m_\alpha}} \in R[1/g_\alpha] \text{ s.t. } s(P) = \frac{f_\alpha}{g_\alpha^{m_\alpha}} \text{ for } P \in D(g_\alpha) \right\}$$

Sheafifcation ought to preserve stalks, so we'd like the following lemma to hold (and it does):

Lemma 1.2.3. $\mathcal{O}_{X,P} = R_P$.

Proof. For $f/g \in R_P$ (where $g \notin P$) to the equivalence class $[(D(g), Q \mapsto \frac{f}{g} \in R_Q)]$ (since $\mathcal{O}_{X,P}$ is a stalk, it is a colimit, so we can construct the assignment to each given equivalence class). To check this is well defined, just construct an analogous map $R \to \mathcal{O}_{X,P}$ and use the universal property of localization to conclude that it factors through R_P, and gives the above map via this factorization.

To verify injectivity, suppose that $f/g \mapsto 0$. This means that the in the class $(D(g), f/g)$, we can shrink $D(g)$ to $D(h) \subseteq D(g)$ such that $\frac{f}{g} = 0$ in R_Q for all $Q \in D(h)$. Thus, $\mathrm{Ann}_R(f) \not\subseteq Q \; \forall Q \in D(h)$. Thus, $V(\mathrm{Ann}_R(f)) \subseteq V(h)$, so some large power of h lives in the annihilator of f, so $h^N \cdot f = 0$. Since $h \notin P$, we see that

$$\frac{f}{g} = \frac{f \cdot h^N}{g \cdot h^N} = 0$$

in R_P. To check surjectivity, choose $[(U, s)] \in \mathcal{O}_{X,P}$. We can choose our equivalence class wisely, such that it is of the form $[(D(g), s)]$. Our sections are defined in such a way that we know that there is this local patching property, i.e. if $D(g) = \bigcup_\alpha D(g_\alpha)$, for each α we have a $\frac{f_\alpha}{g_\alpha^{m_\alpha}} \in R[1/g_\alpha]$ such that $s(\alpha)$ agrees with this fraction on $D(g_\alpha)$, and there are finitely many such α. A priori, $D(g_i) = D(g_i^{m_i})$, so we can assume that each of these $m_\alpha = 1$, and all our fractions are of the form $\frac{f_\alpha}{g_\alpha}$. Because $\mathcal{O}_X$ is a sheaf, we know that $\frac{f_\alpha}{g_\alpha} = \frac{f_\beta}{g_\beta}$ on $D(g_\alpha) \cap D(g_\beta) = D(g_\alpha g_\beta)$. It follows then that this equality holds inside $R[1/g_\alpha g_\beta]$, implying that

$$(g_\alpha g_\beta)^{M_{\alpha\beta}}(f_\alpha g_\beta - f_\beta g_\alpha) = 0$$

For some $M_{\alpha\beta} \gg 0$. there are finitely many patches here by quasi compactness, so we can choose M sufficiently large where

$$(g_\alpha g_\beta)^M(f_\alpha g_\beta - f_\beta g_\alpha) = 0$$

for ALL $\alpha, beta$. Now replace f_α with $f_\alpha g_\alpha^M$ and g_α with g_α^{M+1}. This corresponds to multiplying numerators and denominators by g_α^M, which is OK. From how we chose $D(g_\alpha)$, we know that $V(g) = V(g_1, \dots, g_n)$ for some n. Thus, $g^N = h_1 g_1 + \cdots + h_n g_n$. Now, define $f = h_1 f_1 + \cdots + h_n f_n$. Then,

$$g_i f = h_1 g_i f_1 + \cdots + h_n g_i f_n = h_1 g_1 f_i + \cdots + h_n g_n + f_i = g^N f_i$$

It follows that

$$\frac{f_i}{g_i} = \frac{f}{g^N}$$

for all i. It follows that $\frac{f}{g^N}$ is the preimage we are after.

Thus we can conclude that $\frac{f_\alpha}{g_\alpha^{m_\alpha}} \in R[1/g_\alpha] \subset R_P$ is precisely the preimage we want. $\square$

We can prove a similar statement about localizing at a point.

Lemma 1.2.4. $R[1/g] \cong \Gamma(D(g), \mathcal{O}_X)$

Proof. We assign $r/g^m \in R[1/g]$ to the section mapping $Q \mapsto r/g^m \in R_Q$ in $\Gamma(D(g), \mathcal{O}_X)$. Call this association φ. We'd like to show that this association is bijective. If $r/g^m \in \ker(\varphi)$, $\exists h \in R \backslash Q$ such that $hr = 0$, i.e. $\mathrm{Ann}_R(r) \subseteq Q$. Thus, $V(\mathrm{Ann}_R(r) \cap D(g) = \varnothing$, so $V(\mathrm{Ann}_R(r)) \subset V(g)$, so $\sqrt{(g)} \subseteq \sqrt{\mathrm{Ann}_R(r)}$, so g^N, for some N, lives inside $\mathrm{Ann}_R(r)$. Thus, we conclude similarly to the previous case:

$$\frac{f}{g^m} = \frac{f \cdot g^N}{g^{m+N}} = \frac{0}{g^{m+N}} = 0$$

Surjectivity is checked analogously to the previous lemma. $\square$

A neat corollary of these results is that $\Gamma(X, \mathcal{O}_X) = R$, which is what we'd expect. We can get this by letting $g = 1$.

1.2.3 Locally Ringed Spaces

Let the category of ***locally ringed spaces***, denoted LRSp, be the category where the objects are $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X with local stalks. A morphism $(f, f^\#) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a tuple where $f : Y \to X$ is a morphism of topological spaces and $f^\# : \mathcal{O}_X \to f_* \mathcal{O}_Y$ is a map of sheaves of rings on X (equivalently, by adjointness $f^\#$ can be viewed as a map $f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ which is a map of sheaves of rings on Y).We also enforce that the induced stalk maps are local maps. More explicitly, when $f(Q) = P$, we get a map $f^{-1}(\mathcal{O}_X)_Q \to \mathcal{O}_{Y,Q}$, which must be a local map of local rings.

The category of Affine Schemes, denoted AffSch, is just the full category of LRSp where the objects $(X, \mathcal{O}_X)$ isomorphic to (Spec$(R), \mathcal{O}_{\text{Spec}(R)}$ for some ring. Luckily we can throw out all this formalism at a category level through the following theorem:

Theorem 1.2.5. AffSch *is contravariantly equivalent to* CRing.

Proof. We already know that $\Gamma(\text{Spec}(A)) = A$ and $\text{Spec}(\Gamma(X)) = X$. It is sufficient to verify that there exists an (order reversing)

If we have a ring homomorphism $\varphi : A \to B$, this yields a morphism $(f, f^\#) : (Y = \text{Spec}(B), \mathcal{O}_Y) \to (X = \text{Spec}(A), \mathcal{O}_Y)$ where $f(Q) = \varphi^{-1}(Q)(= Q \cap A)$. To see that f is continuous under the respective Zariski topologies, check that

$$f^{-1}(V(\Sigma)) = V(\varphi(\Sigma))$$

For any prime $Q \in \text{Spec}(B), Q \cap A \in \text{Spec}(A)$, we get a localization map $\varphi_Q : A_{Q\cap A} \to B_Q$. This gives us a map on structure sheaves $\mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}(U))$ where $(s : U \to \bigsqcup_{P \in U} A_P)$ maps to $(Q \mapsto \varphi_Q(s(Q \cap A)))$ where $\varphi_Q(s(Q \cap A))) \in \bigsqcup_{Q \in f^{-1}(U)} B_Q$. Conversely, all maps of locally ringed spaces gives a ring map on global sections. One can check that these two identifications are mutual inverses. □

Examples

- Let X be a topological space, and let $\mathcal{O}_X = \mathbb{Z}_X$, or the constant sheaf of rings. Choose $f : X \to Y$ continuous, and $U \subset X, V \subset Y$ open.

$$\Gamma(V, \mathbb{Z}_Y) = \{s : V \to \mathbb{Z} \mid s \text{ continuous}\}$$

$$\Gamma(f^{-1}(V), \mathbb{Z}_X) = \{t : f^{-1}(V) \to \mathbb{Z} \mid t \text{ continuous}\}$$

  There is a natural morphism $\Gamma(V, \mathbb{Z}_Y) \to \Gamma(f^{-1}(V), \mathbb{Z}_X)$ sending $s \mapsto s \circ f|_{f^{-1}(V)}$. One can easily check that this is a morphism of ringed spaces.

- Take X, Y as topological spaces, with $\mathcal{O}_X, \mathcal{O}_Y$ being sheaves of continuous functions on X, Y (note that a similar construction can be done for sheaves of smooth or holomorphic functions, if X, Y were equipped with such structures). For $p \in X$, Take $M_p \subset \mathcal{O}_{X,p}$ to be the ideal of functions vanishing at p. It is easily checked that this ideal contains all other ideals of $\mathcal{O}_{X,p}$, implying that it is the unique maximal ideal. Thus, $\mathcal{O}_{X,p}$ is local, and thus X is a locally ringed space (similarly, as is Y). For any continuous map $f : X \to Y$, let $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ be the morphism induced by composition, where $f^\#_V$ sends $(V \to \mathbb{R}) \mapsto (f^{-1}(V) \to V \to \mathbb{R})$. For $p \in X, f(p) = q \in Y$, let $M_q \subset \mathcal{O}_{Y,q}$ be the set $\{\varphi : V \to \mathbb{R} \mid \varphi(q) = 0\}$. Under $f^\#_q : \mathcal{O}_{Y,q} \to (f_*\mathcal{O}_X)_q$, this lifts to the set of morphism $\{f \circ \varphi : f^{-1}(V) \to \mathbb{R} \mid f \circ \varphi(q) = 0\}$, which is a maximal ideal in $(f_*\mathcal{O}_X)_q$. It follows that f is indeed a morphism of locally ringed spaces.

- We can use the language of sheaves to define an n-dimension C^∞ (smooth) Manifold. Let X be a topological space (Hausdorff/Second Countable) and let $\mathcal{O}_X$ be a sheaf of rings on X. X is a ***smooth manifold*** if $\forall p \in X$, $\exists U \ni p$ (an open neighborhood) that can be used to define an isomorphism of locally ringed spaces

$$(\varphi_U, \varphi_U^\#) : (U, \mathcal{O}_X) \to (\mathbb{R}^n, \mathcal{O}^{C^\infty}_{\mathbb{R}^n})$$

This gives us local homeomorphisms $U \cong \mathbb{R}^n$, as for any continuous map $f : U \to \mathbb{R}$, there exists a unique map $f : \mathbb{R}^n \to \mathbb{R}$ to make the corresponding diagram commute and vice versa, suggesting that the induced map $U \to \mathbb{R}^n$ is continuous with continuous inverse.

1.2.4 Schemes

A ***Scheme*** is a locally ringed space $(X, \mathcal{O}_X)$ so that every point has an open neighborhood U where $(U, \mathcal{O}_X|_U)$ is an affine scheme. Morphisms of schemes are just morphisms of locally ringed spaces.

Examples

- Consider $\mathrm{Spec}(k)$ for k a field. This is a point with structure sheaf k. One can then define schemes over k to be schemes X with an equipped morphism $X \to \mathrm{Spec}(k)$. This suggests that all local rings of X are k-algebras, and maps of local rings are k-algebra maps.

- If R is a DVR, then $\mathrm{Spec}(R)$ is a topological space consists of 2 points; the unique maximal ideal t_0 (a closed point) and a point t_1 corresponding to the zero ideal, which is a generic (open) point. Notice that the maps $R \to \mathrm{Frac}(R) = K$ and the map $R \to R/m = k$ both induce maps $\mathrm{Spec}(K) \to \mathrm{Spec}(R)$ and $\mathrm{Spec}(k) \to \mathrm{Spec}(R)$, so the residue field and fraction field both correspond to schemes over R.

- $\mathrm{Spec}(\mathbb{Z})$ has closed points corresponding to each prime and a generic point at the 0 ideal. This implies that the open sets of $\mathrm{Spec}(\mathbb{Z})$ are just $\mathrm{Spec}(\mathbb{Z})$ with finitely many primes (closed points) omitted.

- When k is algebraically closed, $\mathrm{Spec}(k[t])$ has a closed point for each irreducible polynomial, which are of the form $t - a$ for $a \in k$, and a generic point for 0. Thus, $\mathrm{Spec}(k[t])$ is just an affine line $\mathbb{A}^1_k$, and is in bijection with points in k. Similarly, $\mathrm{Spec}(k[t_1, \ldots, t_n])$ is affine n-space, denoted $\mathbb{A}^n_k$.

- So far all the above examples were affine schemes. Here is an example of a non-Affine scheme.

  Consider $U = \mathbb{A}_k^2 \backslash \{(x, y)\}$. This is the affine plane with the point at 0 removed. One can view $U = U_x \cup U_y$, where U_x and U_y (i.e. $\operatorname{Spec}(k[x,y]_x)$ and $\operatorname{Spec}(k[x,y]_y)$) are affine schemes. So U is a scheme, but why is U not affine? Notice that

  $$\Gamma(U_x, \mathcal{O}_U) = k[x^{\pm 1}, y]$$

  $$\Gamma(U_y, \mathcal{O}_U) = k[x, y^{\pm 1}]$$

  $$\Gamma(U_{xy}, \mathcal{O}_U) = k[x^{\pm 1}, y^{\pm 1}]$$

  Therefore,

  $$\Gamma(U, \mathcal{O}_U) = \ker\left(k[x^{\pm 1}, y] \oplus k[x, y^{\pm 1}] \to k[x^{\pm 1}, y^{\pm 1}]\right) = k[x^{\pm 1}, y] \cap k[x, y^{\pm 1}] = k[x, y]$$

  But this is an issue, since $\operatorname{Spec}(k[x,y]) = \mathbb{A}_k^2 \neq U$.

- Suppose X is a line with two origins. let $U_1 = \mathbb{A}_k^1, U_2 = \mathbb{A}_k^1$. We can glue $U_1 \backslash \{0\}$ and $U_2 \backslash \{0\}$ to get X, so

  $$X = U_1 \sqcup U_2 / \sim$$

  The two origins are precisely the origins of U_1 and U_2, which are still different and not identified. The structure sheaf, along this identification is of the form

  $$\mathcal{O}_X(V) = \{(s_1, s_2) \mid s_i \in \Gamma(V \cap U_i, \mathcal{O}_{U_i}) \text{ s.t. } s_1|_{V \cap U_1 \cap U_2} = s_2|_{V \cap U_1 \cap U_2} \text{ on } U_1 \cap U_2\}$$

  This is a "bad" gluing (and as we'll soon see, an example of a non-separated scheme). In most courses on manifolds, this is an example of a space that is locally Euclidean and second countable, but not Hausdorff, and is often used to show why we'd like manifolds to be Hausdorff. In Algebraic Geometry land, though, such requirements are unnecessary. In fact, such a space is locally affine but not globally so, and is our first example of a Scheme that is not an Affine Scheme.

- More generally, let X_1, X_2 be arbitrary schemes, and choose $U_1 \subset X_1, U_2 \subset X_2$ open subsets that are isomorphic to each other as subschemes. We can define a scheme X by gluing X_1 and X_2 together along the isomorphism $\varphi : U_1 \cong U_2$. In particular, we see that $X = X_1 \sqcup X_2 / \sim$ where $x_1 \sim \varphi(x_1)$. Let $i_j : X_j \to X$ denote the respective inclusion maps. The structure sheaf is defined as follows:

  $$\mathcal{O}_X(U) = \left\{(s_1, s_2) \mid s_1 \in \mathcal{O}_{X_1}(i_1^{-1}(U)), s_2 \in \mathcal{O}_{X_1}(i_2^{-1}(U)), \varphi(s_1|_{i_1^{-1}(U) \cap U_1}) = \varphi(s_2|_{i_2^{-1}(U) \cap U_2})\right\}$$

  Unpacking this, this agrees with the structure sheaves on X_1, X_2 outside of U_1, U_2 (i.e. where there is no gluing) but makes a very explicit notion of identity between structure sheaves on U_1, U_2, and we in a sense are declaring that the behavior of this structure sheaf "agrees" with the structure sheaves on the subsheaves U_1, U_2.

- $\mathbb{P}^1_k$ (first attempt):
$$U_1 = \mathbb{A}^1_k = \mathrm{Spec}(k[t])$$
$$U_2 = \mathbb{A}^1_k = \mathrm{Spec}(k[s])$$
Glue $U_1/\{0\} \to U_2\backslash\{0\}$ (i.e. the map $\mathrm{Spec}(k[t^{\pm 1}]) \to \mathrm{Spec}(k[s^{\pm 1}])$) induced by the ring map $k[s^{\pm 1}] \to k[t^{\pm 1}]$ sending $s \mapsto t^{-1}, s^{-1} \mapsto t$.

  The corresponding space, $\mathbb{P}^1_k$, is indeed not affine, as it has global sections $k[t] \cap k[t^{-1}] = k$. However, we expect $\mathbb{P}^1_k$ to have at least 2 points, namely 0 and ∞, whereas k ought to only have 1 point.

We can also construct projective space functorially. Suppose $S = \bigoplus S_d$ were a graded ring. $X = \mathrm{Proj}(S)$ is defined to be the scheme covered by open affine schemes $\mathrm{Spec}(S[1/f]_0)$ (often denoted Spec($S_{(f)}$ for short). This is precisely the distinguished open set $D_+(f) = X\backslash V(f)$. We can then glue them as follows:

Suppose that $\deg(f_1) = d_1, \deg(f_2) = d_2$. Then we have maps

$$S[1/f_1]_0 \to S[1/f_1 f_2]_0 \leftarrow S[1/f_2]_0$$

By inverting $\frac{f_2^{d_1}}{f_1^{d_2}}$ and $\frac{f_1^{d_2}}{f_2^{d_1}}$ respectively.

For example, if $S = k[x_0, \ldots, x_n]$, then Proj(S) is what we'd expect; $\mathbb{P}^n_k$. The affine patches in this case are of the form $D_+(x_i)$ for $0 \leq i \leq n$. One can also define Proj as follows:

$$\mathrm{Proj}(S) = \{\text{All homogeneous prime ideals that do not completely contain } S_+\}$$

If $X = \mathrm{Proj}(S)$¡ it has a natural sheaf $\mathcal{O}_X$ where $\mathcal{O}_X(U)$ is defined as follows:

$$\left\{ s : U \to \coprod_{p \in U} S_{(p)} \;\middle|\; s(p) \in S_{(p)}, \exists g_\alpha \text{ s.t. } U = \bigcup_\alpha D_+(g_\alpha), s(p) = \frac{f_\alpha}{g_\alpha^{m_\alpha}} \in S_{(p)}, S(g_\alpha)\ \forall p \in D_+(g_\alpha) \right\}$$

1.3 Fibered Products and Other Scheme Structures

1.3.1 Reduced Structure

Suppose $Z \subset X$ is a closed subscheme. The smallest scheme structure on Z has an induced reduced structure, denoted Z_{red}. This reduction comes along with the following universal property: If Z' is another scheme with the same topological structure as Z, then

$$\begin{tikzcd}
Z_{\mathrm{red}} \arrow[d, dashed] \arrow[dr, hook] & \\
Z' \arrow[r] & X
\end{tikzcd}$$

It is enough to verify this for affine schemes; you just assign every ideal in the reduced scheme to be the radical of a given ideal in the initial scheme. Alternatively, if Spec(R) is our affine scheme, then its canonical reduction is $\mathrm{Spec}(R/\mathrm{Nil}(R))$.

1.3.2 Fibered Products

Let S be a scheme. An S-***Scheme*** is a scheme X together with a **structure morphism** $q_X : X \to S$. Naturally, a morphism of S schemes X and Y is a morphism of schemes $\varphi : X \to Y$ satisfying the following commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, bend right, "q_X"'] & & Y \arrow[dl, bend left, "q_Y"] \\ & S & \end{tikzcd}$$

With canonical maps $X \to S, Y \to S$, it makes sense to consider pushouts of the diagram $X \longrightarrow S \longleftarrow Y$. We say the **fiber product** (or fibre product, depending on where you learned English) of X, Y is the universal object $X \times_S Y$ couples with projection maps $\pi_X : X \times_S Y \to X$ and $\pi_Y \to X \times_S Y \to Y$ such that the follow diagram commutes:

$$\begin{tikzcd} X \times_S Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "q_Y"] \\ X \arrow[r, "q_X"] & S \end{tikzcd}$$

In addition, the fiber product must satisfy the universal property associated to pushouts; i.e. for any Scheme W with morphisms $\varphi_X : W \to X, \varphi_Y : W \to Y$ such that $q_X \circ \varphi_X = q_Y \circ \varphi_Y$, then there exists a unique map $\widetilde{\varphi} : W \to X \times_S Y$ that makes the pushout diagram commute:

$$\begin{tikzcd} W \arrow[dr, dashed, "\exists! \widetilde{\varphi}"] \arrow[drr, bend left, "\varphi_Y"] \arrow[ddr, bend right, "\varphi_X"'] & & \\ & X \times_S Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "q_Y"] \\ & X \arrow[r, "q_X"] & S \end{tikzcd}$$

This sort of construction is similar to that of the Seifert-Van-Kampen Theorem, a very popular and known result in Algebraic Topology. As one would expect Fiber Products not only exist (Theorem 2.3.3 in Hartshorne) in Sch(S) (the category of S-Schemes), but are also uniquely defined up to unique isomorphism. In a slight abuse of notation, for any ring R we denote category of Schemes over Spec(R) as Sch(R).

Examples

- Suppose that X, Y, S are just sets. We already know the Cartesian product $X \times Y$ as tuples (x, y) where $x \in X, y \in Y$. If $X, Y \subset S$, and we take the natural inclusion maps, then the fiber product $X \times_S Y = \{(x, y) \mid x \in X, y \in Y, i_X(x) = i_Y(y)\}$ are precisely the elements that "include" into the same element with in S. This occurs only if $x = y$. Thus, $X \times_S Y$ consists of pairs (x, x) where $x \in X$ and $y \in Y$, implying that $X \times_S Y = X \cap Y$. (Note that such a conclusion can be made in any category; If $X, Y \subset Z$ then $X \times_Z Y = X \cap Y$ in the category of schemes as well)

- In the category $\mathrm{Sch}(S)$, $X \times_S Y$ is the natural product.

While fiber products certainly exist over the category of Schemes (we'll prove this later, but for now just believe me), it is not (necessarily) guaranteed that the fiber product of schemes yields an affine scheme. This, however, is in fact the case, and we have a unique characterization of the fiber product of affine schemes.

Lemma 1.3.1. *Suppose* $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B)$ *are affine* S*-schemes for* $S = \mathrm{Spec}(C)$*. Then we have*

$$\mathrm{Spec}(A) \times_{\mathrm{Spec}(C)} \mathrm{Spec}(B) = \mathrm{Spec}(A \otimes_C B)$$

Proof. First, observe that morphisms $\mathrm{Spec}(A) \to \mathrm{Spec}(C)$, $\mathrm{Spec}(B) \to \mathrm{Spec}(C)$ descend to ring morphisms $C \to A, C \to B$. This puts a natural C-module structure on A and B. Now consider the following diagram:

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & B \arrow[d, hook] \\
A \arrow[r, hook] & A \otimes_C B
\end{tikzcd}$$

Suppose for some ring aP there were ring homomorphisms $A \to P, B \to P$ (This, by the way, puts a C-module structure on P). We can apply the universal property of tensor products (via factoring through the product $A \times B$ to get the following diagram:

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & B \arrow[d, hook] \arrow[r, hook, dotted] \arrow[ddr, bend left] & A \times B \arrow[dl, dotted, "\pi"] \arrow[dd, dotted] \\
A \arrow[r, hook] \arrow[d, hook, dotted] \arrow[ddrr, bend right, phantom] \arrow[drr, bend right] & A \otimes_C B \arrow[dr, dashed, "\exists!"] & \\
A \times B \arrow[ur, dotted, "\pi"] \arrow[rr, dotted] & & P
\end{tikzcd}$$

As we can induced the dashed map uniquely by applying the universal property on either the bottom dotted diagram or the top; it's uniqueness guarantees the same map. Applying the contravariant functor $\mathrm{Spec}(-)$ to the non-dotted portions of this diagram, we have

$$\begin{tikzcd}
\mathrm{Spec}(P) \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathrm{Spec}(A \otimes_C B) \arrow[r] \arrow[d] & \mathrm{Spec}(B) \arrow[d] \\
& \mathrm{Spec}(A) \arrow[r] & \mathrm{Spec}(C)
\end{tikzcd}$$

As this holds for any affine scheme $\mathrm{Spec}(P)$, $\mathrm{Spec}(A \otimes_C B)$ satisfies the universal property of fiber products. $\square$

As a corollary to this, we see that the fiber product of affine schemes over an affine scheme is again an affine scheme (we even gave an explicit description of it!). In fact, if you aren't convinced fiber products exist, one can define the fiber product for affine schemes this way. Now we'd like to show that taking fiber products of subschemes of affine schemes (or even better, elements of an open affine cover of schemes) respects the global structure.

Lemma 1.3.2. *Given the fiber product and maps defined below:*

$$\begin{tikzcd}
X \times_S Y \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & S
\end{tikzcd}$$

Where $U \subset S, V \subset X, W \subset Y$ *are all open subschemes such that* $f(V), g(W) \subset U$*, the morphism* $V \times_U W \to X \times_S Y$ *induced by the universal property of fiber products is an open map which identifies* $V \times_U W$ *with* $p^{-1}(V) \cap p^{-1}(W)$*.*

$$\begin{tikzcd}
V \times_U W \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & W \arrow[dr, hook] \arrow[dd] & \\
& X \times_S Y \arrow[rr, "q"] \arrow[dd, "p"] & & Y \arrow[dd, "g"] \\
V \arrow[rr] \arrow[dr, hook] & & U \arrow[dr, hook] & \\
& X \arrow[rr, "f"] & & S
\end{tikzcd}$$

Proof. Suppose we had morphisms $a : T \to V, b : T \to W$ such that $f \circ a = g \circ b$. This induces a morphism $T \to V \times_U W$, which we can compose with the dashed morphism above to get a morphism $T \to X \times_S Y$. This has image that lies in $p^{-1}(V) \cap p^{-1}(W)$ by commutativity of the above diagram, implying that the following diagram commutes:

$$\begin{tikzcd}
T \arrow[dr, dashed] \arrow[rrrd, bend left, "b"] \arrow[dddr, bend right, "a"'] & & & & \\
& p^{-1}(V) \cap p^{-1}(W) \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & W \arrow[dr, hook] \arrow[dd] & \\
& & X \times_S Y \arrow[rr, "q"] \arrow[dd, "p"] & & Y \arrow[dd, "g"] \\
& V \arrow[rr] \arrow[dr, hook] & & U \arrow[dr, hook] & \\
& & X \arrow[rr, "f"] & & S
\end{tikzcd}$$

By uniqueness of fiber products, $V \times UW \cong p^{-1}(V) \cap p^{-1}(W)$. From the diagram above, it is clear that $T \to p^{-1}(V) \cap p^{-1}(W)$ is an open map. □

As we can just take $U = S$, applying similar logic we see that $V \times_U W = V \times_S W$, and we can view $V \times_U W$ as sitting inside $X \times_S Y$. We can extend this principle to open affine covers of schemes below.

Lemma 1.3.3. *Take f, g, p, q, X, S, Y as in the previous lemma. Let $S = \bigcup_{i \in \mathcal{I}} U_i$ be an open covering of S by affine schemes, and for each $i \in \mathcal{I}$, let $f^{-1}(U_i) = \bigcup_{j \in \mathcal{J}_i} V_j$ be an open covering of $f^{-1}(U_i)$ by affine schemes, and let $g^{-1}(U_i) = \bigcup_{k \in \mathcal{K}_i} W_k$ be an open covering of $g^{-1}(U_i)$ by affine schemes. Then,*

$$X \times_S Y = \bigcup_{i \in \mathcal{I}} \bigcup_{j \in \mathcal{J}_i, k \in \mathcal{K}_i} V_j \times_{U_i} W_k$$

Proof. Lemma 2.2.2 tells us that each $V_j \times_{U_i} W_k$ is affine, and from our construction in Lemma 2.2.3, we can clearly cover $X \times_S Y$ by sets of these form. □

Tying all these together, we can conclude that fiber products exist and are unique.

Theorem 1.3.4. *Fiber products exist in the category of schemes, and they are unique.*

Proof. Over affine schemes, define $\mathrm{Spec}(A) \times_{\mathrm{Spec}(C)} \mathrm{Spec}(B)$ to be $\mathrm{Spec}(A \otimes_C B)$. $\mathrm{Spec}(A \times_C B)$ is known to satisfy the universal property of fiber products. For general schemes X, S, Y, just define $X \times_S Y$ to be $\bigcup_{i \in \mathcal{I}} \bigcup_{j \in \mathcal{J}_i, k \in \mathcal{K}_i} V_j \times_{U_i} W_k$ as constructed above. Uniqueness, as with all other things defined by universal properties, follows from staring at these diagrams:

$$\begin{tikzcd}
X \times_S Y \arrow[dr, dashed] \arrow[ddrr, dotted, "\mathrm{Id}"'] \arrow[ddrrr, bend left] \arrow[dddrr, bend right] & & & \\
& (X \times_S Y)' \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& & X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& & X \arrow[r] & S
\end{tikzcd}$$

$$\begin{tikzcd}
(X \times_S Y)' \arrow[dr, dashed] \arrow[ddrr, dotted, "\mathrm{Id}"'] \arrow[ddrrr, bend left] \arrow[dddrr, bend right] & & & \\
& X \times_S Y \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& & (X \times_S Y)' \arrow[r] \arrow[d] & Y \arrow[d] \\
& & X \arrow[r] & S
\end{tikzcd}$$

Dashed morphisms are unique, and thus inverses of each other, defining an isomorphism of schemes. □

Now this is all fine and good, but how do we compute fiber products? First, we need to define something. Let X be a scheme, and choose $x \in X$. As $X = \bigcup U_i$ for $U_i = \text{Spec}(A_i)$ affine, we know that x corresponds uniquely to a prime ideal $P \in \text{Spec}(A_i)$. We define the ***Residue Field*** of $x \in X$ to be the field $(A_i)_P / P(A_i)_P$. We often denote this $\mathcal{K}(x)$.

Lemma 1.3.5. *Take f, g, p, q, X, S, Y as in the previous lemmas. Points $z \in X \times_S Y$ are in bijective correspondence to tuples (x, y, s, P), for $x \in X, y \in Y, s \in S$ such that $f(x) = g(y) = s$, and $P \in \text{Spec}(\mathcal{K}(x) \otimes_{\mathcal{K}(s)} \mathcal{K}(y))$.*

Proof. See Tag 01JT in the Stacks Project. □

The notion of a Scheme "over" something rigorizes the notion of taking a variety over a field. In some sense, a variety X over a field k is a scheme with an associated morphism $X \to \text{Spec}(k)$ ($\text{Spec}(k)$ is just a point, so this map is trivial). More generally, we can say that X is a scheme "over R, when it is really a scheme over $\text{Spec}(R)$.

When we consider X to be a scheme over S, we're less concerned with the geometry of X specifically, and more-so concerned with the geometry of X relative to S. For example, one can define a notion of base change for schemes. If X is an S-scheme and $S' \to S$ is a morphism, the ***Base Change*** of X over S' is the scheme $X_{S'} := S' \times_S X$, which is naturally a scheme over S' (Hartshorne refers to this as a ***Base Extension***). As suspected, a morphisms of S-schemes $f : X \to Y$ induces a morphism of S' schemes $f' : X_{S'} \to Y_{S'}$.

For $f : X \to S$ a morphism of schemes, for any $s \in S$ the ***fiber of** f **over*** s is the scheme $X_s := \text{Spec}(\mathcal{K}(s)) \times_S X$, or the base change of X to $\mathcal{K}(s)$. It's easy to check that X_s embeds into X (in a topological sense). An interesting kind of family of such schemes arises when we have a scheme X over $\text{Spec}(\mathbb{Z})$. Recall that $\mathbb{Z}$ is initial in CommRing, so it is final in the category of affine schemes. Taking the fiber over the generic point gives a scheme $X_{\mathbb{Q}}$ over $\mathbb{Q}$, while taking the fiber over a closed point corresponding to $p\mathbb{Z}$ gives a scheme X_p over the finite field $\mathbb{F}_p$. We say that X_p arises by ***reduction*** mod p of the scheme X.

The idea of this is to study schemes in a "relative" context. Just as we study manifolds over a base field (usually $\mathbb{R}$ or $\mathbb{C}$) or a variety over some algebraically closed field, schemes should also be "based" in something. Rather than studying schemes in their own right, we should consider morphisms of schemes $X \to S$ and study how that morphism behaves. If a property is preserved under base extension (i.e. we can change the base of the scheme and still have the property hold) then we say the property is ***stable under base extension.*** (Conversely, it is not stable if the property fails to hold under some base change.)

1.3.3 Dimension

One useful invariant of a scheme is its dimension. In the affine case, we can just take the dimension of $X = \text{Spec}(A)$ to be the krull dimension of A. In the case of general schemes, we let the dimension of X as a scheme be the dimension of X as a topological space. If $Z \subset X$ is an irreducible closed subset, then the codimension of Z (denoted $\operatorname{codim}(Z, X)$ in Hartshorne) is the length of the maximal saturated chain of irreducible schemes from Z to X. For a general closed subset $Y \subset X$, we define $\operatorname{codim}(Y, X) = \inf_{Z \subset Y}(Z, X)$.

Do be warned, though; while the notion of dimension "makes sense" to our minds in the affine case, the general case may seem unintuitive. On a general scheme dimension may shift locally in ways one might not expect. For example, one can cover a scheme X with an open affine cover $\operatorname{Spec}(A_i)$, where each A_i can have wildly different krull dimension.

1.3.4 Properties of Schemes

Schemes can have certain properties. Here are some of them:

- Purely Topological: Connected, irreducible, (quasi)-compact, dimension
- Being Reduced: (TFAE)
  - $\mathcal{O}_X(U)$ is reduced for any open set $U \subset X$
  - $\mathcal{O}_X(U)$ is reduced for any affine open set $U \subset X$
  - $\mathcal{O}_{X,x}$ is reduced for any $x \in X$.
- Being Integral:
  - $\mathcal{O}_X(U)$ is a domain for any open $U \subset X$
  - NOT equivalent definition: $\mathcal{O}_{X,x}$ is a domain for every $x \in X$ (this is called being locally integral and is not equivalent to the previous definition. Consider for instance $\operatorname{Spec}(R \times S) = \operatorname{Spec}(R) \coprod \operatorname{Spec}(S)$ where R and S are domains.

  Lemma 1.3.6. *Being integral is equivalent to being reduced and irreducible.*

- Locally Noetherian: $\exists$ an open affine cover $X = \bigcup \operatorname{Spec}(R_\alpha)$ where R_α is Noetherian.
- Noetherian: X is locally Noetherian and quasi-compact.

  Lemma 1.3.7. *If X is locally Noetherian, then $\mathcal{O}_{X,x}$ is a Noetherian ring for all $x \in X$. The converse is false.*

  Lemma 1.3.8. *If X is a Noetherian scheme, it is a Noetherian topological space. Following the trend, the converse is false.*

  Lemma 1.3.9. *X is locally Noetherian $\iff$ all open affine patches are Noetherian.*

  Proof. The special case follows from the main result, and the reverse case is the definition of a locally Noetherian scheme. Thus, it just remains to prove the forward direction. Notice that for any affine subset $U = \operatorname{Spec}(B)$, it is covered by open subset of the form $D(f) = \mathrm{Spec}(B_f)$, where B_f is Noetherian (since B is Noetherian). Thus, any affine open set can be covered by spectra of Noetherian rings, allowing us to conclude once we've checked the affine case. In other words, if $X = \mathrm{Spec}(A)$ is an affine scheme that is covered by spectra of Noetherian rings, then A is Noetherian.

Let $U = \mathrm{Spec}(B)$ be an affine open subset of X, with B Noetherian. Choose $f \in A$ such that $D(f) \subset U$, and let $\overline{f}$ be the image of f under the projection morphism $A \twoheadrightarrow B$. It follows that $A_f \cong B_{\overline{f}}$, and as B is Noetherian, $B_{\overline{f}}$ is Noetherian, so A_f is Noetherian. Thus X can be covered by open sets of the form $\mathrm{Spec}(A_f)$ for A_f Noetherian. As X is quasi-compact, we can restrict to a finite subcover of these.

Algebraically, A is generated by some $f_1, \dots, f_r$ where each localization A_{f_i} is Noetherian. Let φ_i denote each respective localization map. For any ideal $I \subset A$, I claim that

$$I = \bigcap_{i=1}^{r} \varphi_i^{-1}(\varphi(I) \cdot A_{f_i})$$

A domain/range checking argument shows $\subset$. Conversely, choose $b \in A$ contained in the intersection. This allows us to write $\varphi_i(b) = a_i / f_i^{n_i}$ in each A_{f_i} for some $n_i > 0$ and $a_i \in I$. Let $n = \max_i n_i$. It follows that $f_i^{m_i}(f_i^n b - a_i) = 0$ for some m_i. As before, let $m = \max_i m_i$. It follows that $f_i^{m+n} b \in I$ for any i, and as $A = (f_1, \dots, f_r)$, it follows that $\exists c_i$ such that $\sum c_i f_i^{n+m} = 1$. Writing b as $b \sum c_i f_i^{n+m}$, we see that it lies in I.

Now we can verify that A is Noetherian. Let $I_1 \subset I_2 \subset \dots$ be an ascending chain in A. Then for each i,

$$\varphi_i(I_1) A_{f_i} \subset \varphi_i(I_2) A_{f_i} \subset \varphi_i(I_3) A_{f_i} \subset \dots$$

As A_{f_i} is Noetherian, this chain terminates at a certain point. As there are finitely many such A_{f_i}, take the maximal index of halting, and the initial chain $I_1 \subset I_2 \subset \dots$ will halt there as well, from the above result. This means that A satisfies the ascending chain condition, and is thus Noetherian. $\square$

It's worth noting that, topologically schemes are gross and (generally) hard to parse. These properties, if they hold, are often excruciatingly difficult to verify for a general scheme in practice.

1.3.5 Properties of Morphisms

Let $f : Y \to X$ be a morphism of schemes.

- Quasicompact: $\exists$ an open affine cover $X = \bigcup U_\alpha$ such that $f^{-1}(U_\alpha)$ is quasicompact for all α.

  Lemma 1.3.10. *f is Quasicompact $\iff$ $\forall U \subset X$ open affine, $f^{-1}(U)$ is quasicompact.*

Proof. Let U be an open affine subset of X. Then $U = \bigcup(U \cap U_\alpha)$. By quasicompactness of U (recall all affine schemes are quasicompact), we know need finitely many U_α, denoted $U_{\alpha_1}, \dots, U_{\alpha_n}$. It follows that $U = \bigcup_{i=1}^n (U \cap U_{\alpha_i})$. This is the same as

$$\bigcup_{\substack{i=1,\dots,n \\ j=1,\dots,m_i}} W_{\alpha_i,j}$$

Where $W_{\alpha_i,j} = (U_{\alpha_i})_{\varphi_j}$ where $\varphi_j = \Gamma(X, U_{\alpha_j}$. Looking at $f^{-1}(U_{\alpha_j})$, by our hypothesis we know it is covered by a finite number of open affines $V_{\alpha_i,1}, V_{\alpha_i,\ell_i}$. Thus,

$$f^{-1}(W_{\alpha_i,j} = (f^{-1}(U_{\alpha_i})_{f^\#(\varphi_j)} = (V_{\alpha_i,1})_{f^\#(\varphi_j)} \cup \dots \cup (V_{\alpha_i,\ell_i})_{f^\#(\varphi_j)}$$

As the $V_{\alpha_i,j}$'s are all open affines as well. □

- Locally of Finite Type: Suppose that $X = \bigcup U_\alpha$ is an affine oven cover, and let $f^{-1}U_\alpha = \bigcup V_{\alpha\beta}$ be an affine open cover of each fiber. Then f is of locally finite type if $f|_{V_{\alpha\beta}} : V_{\alpha\beta} \to U_\alpha$ is Spec of a map of finite type (i.e. if $V_{\alpha\beta} = \operatorname{Spec}(B_{\alpha\beta})$ and $U_\alpha = \operatorname{Spec}(A_\alpha)$, then $B_{\alpha\beta}$ is a finitely generated A_α-algebra.
- Finite Type: f is of finite type if it is locally of finite type and quasicompact.

Lemma 1.3.11. *Suppose that $f : Y \to X$ is of finite type, and* $\operatorname{Spec}(A) = U \subset X$ *is an affine open, and* $\operatorname{Spec}(A) = V \subset Y$ *is an affine open such that $f(V) \subset U$. Then, the ring map $A \to B$ is of finite type.*

Proof. Pick open covers $\{U_\alpha\}$ and $\{V_{\alpha\beta}\}$ respectively, and shrink them such that $\forall \beta$ and a fixed α, $U_\alpha \subseteq U$, $V_{\alpha\beta} \subseteq V$. Assuming these are distinguished affine opens, we have the diagram

$$\begin{tikzcd}
B \arrow[ddd] & & & A \arrow[lll] \arrow[ddd] \\
& V \arrow[r] & U & \\
& V_{\alpha\beta} \arrow[r] \arrow[u, hook] & U_\alpha \arrow[u, hook] & \\
B[1/y_{\alpha\beta}] & & & A[1/x_\alpha] \arrow[lll]
\end{tikzcd}$$

$A \to A[1/x_\alpha]$ is of finite type, as is $A[1/x_\alpha] \to B[1/y_{\alpha\beta}]$, so the composition is finite. Let g_{ij} denote the set of generators of $B[1/y_{\alpha\beta}]$ as an A-algebra. We see that all $(y_{\alpha\beta})$ (of which there are finitely many by quasicompactness) generate the unit ideal of B, i.e. $\exists h_{\alpha\beta}$ such that $\sum h_{\alpha\beta} y_{\alpha\beta} = 1$. Letting φ denote the $A \to B$ map, we claim that

$$\varphi(A)[h_{\alpha\beta}, y_{\alpha\beta}, g_{ij}] = B$$

This is already a subring, so it is sufficient to check that I can find any element in B inside $\varphi(A)[h_{\alpha\beta}, y_{\alpha\beta}, g_{ij}]$. We can check this by localizing at every $y_{\alpha\beta}$; geometrically this corresponds to matching the open covers, and algebraically this means that you can match generators. This allows us to coclude that B is a finite A-algebra. $\square$

- Locally Finitely Presented: Suppose that $X = \bigcup U_\alpha$ is an affine oven cover, and let $f^{-1}U_\alpha = \bigcup V_{\alpha\beta}$ be an affine open cover of each fiber. Then f is locally finite presented if $f|_{V_{\alpha\beta}} : V_{\alpha\beta} \to U_\alpha$ is Spec of a map that is finite presented (i.e. if $V_{\alpha\beta} = \operatorname{Spec}(B_{\alpha\beta})$ and $U_\alpha = \operatorname{Spec}(A_\alpha)$, then $B_{\alpha\beta}$ is a finitely presented A_α-algebra.
- Finitely Presented: f is finitely presented if it is locally finitely presented, quasicompact, and quasi-separated.
- $f : Y \to X$ is affine if $\exists$ an affine open cover $X = \bigcup U_\alpha$ such that $f^{-1}(U_\alpha)$ is affine for each α. It immediately follows that affine morphisms are quasicompact.

  This definition leads us to a question: how do we determine whether a scheme is affine?

  Theorem 1.3.12. *Let X be a scheme. $f_1, \dots, f_n \in \Gamma(X, \mathcal{O}_X)$ must satisfy the following criterion:*

  (1) $X_{f_i} = X \setminus V(f_i)$ are all open affine
  (2) $f_1, \dots, f_n$ generate the unit ideal of $\Gamma(X, \mathcal{O}_X)$.

  These criteria are satisfied if and only if X is an affine scheme.

  Proof. Take the exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X) \longrightarrow \bigoplus_i \Gamma(X_{f_i}, \mathcal{O}_X) \longrightarrow \bigoplus_{i<j} \Gamma(X_{f_i f_j}, \mathcal{O}_X)$$

  These are all $\Gamma(X, \mathcal{O}_X)$-modules. We claim that this sequence remains exact after localizing at $g \in \Gamma(X, \mathcal{O}_X)$. This follows from the fact that localizing is exact. We then get the sequence

$$0 \longrightarrow \Gamma(X_g, \mathcal{O}_X) \longrightarrow \bigoplus_i \Gamma(X_{g, f_i}, \mathcal{O}_X) \longrightarrow \bigoplus_{i<j} \Gamma(X_{g, f_i f_j}, \mathcal{O}_X)$$

  We could take g to be any one of the f_i terms. If we do this, we get the identification

$$\Gamma(X_{f_i}, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)[1/f_i]$$

  Where $\Gamma(X, \mathcal{O}_X)[1/f_i]$ cover $\Gamma(X, \mathcal{O}_X)$. It follows that the natural map

$$\operatorname{Spec}(\Gamma(X, \mathcal{O}_X)) \to X$$

  has to be an isomorphism, as it is an isomorphism after inverting each f_i, and we now know that $\Gamma(X_{f_i}, \mathcal{O}_X)$ covers the space. $\square$

We can use this to prove the following lemma:

Lemma 1.3.13. *If $f : Y \to X$ is an affine map and $U \subset X$ is affine, then $f^{-1}(U)$ is affine.*

Proof. We can reduce to the case where $X = U = \operatorname{Spec}(A)$ and $g_1, \ldots, g_r \in A$ so that $X = \bigcup X_{g_i}$, i.e. $(g_1, \ldots, g_r) = 1 \in A$, and $f^{-1}(X_{g_i}) = Y_{f^\#_{g_i}}$ From here we use the affine criterion, where the finitely many global sections needed are $f^\#_{g_1}, \ldots, f^\#_{g_r} \in \Gamma(Y, f^\# \mathcal{O}_X)$. □

- $f : Y \to X$ is an integral map (resp. finite) if $\exists$ an open affine cover $X = \bigcup U_\alpha = \bigcup \operatorname{Spec}(A_\alpha)$ so that $f^{-1}(U_\alpha) = \operatorname{Spec}(B_\alpha)$ are all affine, where the induced map $A_\alpha \to B_\alpha$ is integral (resp. finite). Do note that all integral and finite maps are immediately affine. We also have the following result directly pulled from commutative algebra:

  Theorem 1.3.14. *A map f is finite if and only if it is of finite type and integral.*

  Theorem 1.3.15. *This follows from the fact that a ring map $\varphi : A \to B$ is module finite if and only if it of finite type and it is integral.*

  Lemma 1.3.16. *f is integral (resp. finite), $\operatorname{Spec}(A) = U \subset X$ is an affine open subset. Then $f^{-1}(U) = \operatorname{Spec}(B)$ and the induced map $A \to B$ is integral (resp. finite).*

  Lemma 1.3.17. *Integral maps are always closed.*

  Proof. Follows from the lying over theorem. □

- f is quasi-finite if all fibers are finite (some may say that quasi-finite means it is of locally finite type, quasicompact, and has finite fibers).

  Lemma 1.3.18. *finite maps are quasi-finite.*

  Proof. Follows from the incomparability theorem. □

1.3.6 Immersions

Let X be a scheme, and $U \subset X$ open. An ***open subscheme*** of X is an open set U equipped with a structure sheaf $\mathcal{O}_U := \mathcal{O}_X|_U$. An ***open immersion*** is a map of schemes $i : U \to X$ such that i determines a topological homeomorphism between U and its image in X¡ and induces an isomorphism on structure sheaves $i^\# \mathcal{O}_X|_{i(U)} \to \mathcal{O}_U$.

Lemma 1.3.19. *$i : U \to X$ is an open immersion if i is injective and $i(U)$ is an open subscheme of X.*

For example, consider $X = \mathrm{Spec}(R)$, and $f \in R$. Then $D(f) = \mathrm{Spec}(R[1/f])$ is an open subscheme of X, with $\mathrm{Spec}(R[1/f]) \hookrightarrow \mathrm{Spec}(R)$ being an open immersion. To prove this, $D(f)$ is already an open subscheme (clearly; it in fact forms a basis of $\mathrm{Spec}(R)$) and the map is definitively injective.

As one can define open objects and morphisms, one can also defined analogous closed objects and morphisms. A ***closed subscheme*** $Z \subseteq X$ is a pair $(Z, \mathcal{O}_Z)$ where $Z \subset X$ is closed in X topologically, and $\mathcal{I}_Z \subset \mathcal{O}_X$ is a sheaf of $\mathcal{O}_X$-ideals such that $\mathcal{O}_{Z,p} \to \mathcal{O}_{X,p}$ is an isomorphism $\iff p \notin Z$, denoted the ***ideal sheaf*** of Z in X. Define $\mathcal{O}_Z$ as $\mathcal{O}_X/\mathcal{I}_Z|_Z$.

For example, if $X = \mathrm{Spec}(k[t])$ and $Z = V(t)$, with $\mathcal{I}_Z = (t^p)$ (over affine schemes, ideal sheaves can be associated to an ideal. Thus they sometimes can just be written as I_Z instead of $\mathcal{I}_Z$). Then $\Gamma(Z, \mathcal{O}_Z) = k[t]/(t^p)$. This motivates or definition of a closed immersion; A map of schemes $f : Y \to X$ is a ***closed immersion*** if f is a topological homeomorphism from Y to a closed subset o X and the corresponding map on structure sheaves $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.

Examples of closed immersions, like the examples above, arise naturally over affine schemes. If $X = \mathrm{Spec}(R)$ and $I \subset R$ is an ideal, then the canonical projection $R \twoheadrightarrow R/I$ determines a closed immersion on spectra $\mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$. This can be seen by the fact that $\mathrm{Spec}(R/I) = V(I) \subseteq \mathrm{Spec}(R)$, and on structure sheaves the map just composes sections with the aforementioned projection map $R \twoheadrightarrow R/I$, which is clearly surjective.

On charts, we'd hope that all closed immersions look like this.

Theorem 1.3.20. *Let $f : Y \to X$ be a closed immersion.*

(1) If $U \subset X$ is an affine open set, then $f^{-1}(U)$ is an affine open subset of Y.

(2) f induces an isomorphism of Y onto a closed subscheme of X.

Here we define $\mathcal{O}_{f(Y)}|_X \subset \mathcal{O}_X$ as $\ker(\mathcal{O}_X \to f_*\mathcal{O}_Y)$.

Proof. (Sketch) You may as well assume that X itself is affine for part (1), then use the affine criterion. by looking at $f^{-1}(X)$ and covering it by finitely many open sets $D(f_i)$ who's defining functions generate the global section ring $\Gamma(Y, \mathcal{O}_Y)$. Here you use surjectivity of the the map on structure sheaves. □

Observe that there is a natural bijection between closed subschemes of X and sheaves of ideals in $\mathcal{O}_X$, where $(Z, \mathcal{O}_Z)$ is identified with $\mathcal{I}_Z$.

Examples

- As we saw before, for $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(A/I)$ for some ideal I, the ring homomorphism $A \to A/I$ induces a map of affine schemes $Y \hookrightarrow X$ which is a closed immersion. $Y \cong V(I)$ induces a homeomorphism onto its image. The map $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective since at stalks, $\mathcal{O}_{X,P} \to (f_*\mathcal{O}_Y)_P$ is just the map $A_P \to (A/I)_P = A_P/IA_P$, which is just the natural projection. As the map is surjective at stalks, we can conclude that it is a surjective morphism of sheaves. This gives us a correspondence between ideals of A and closed subschemes of X. We check that such a correspondence is injective in the discussion above, but it also turns out that every closed subscheme structure on a closed subscheme $Y \subset X$ arises this way in the affine case (i.e. writing $Y = \operatorname{Spec}(A/I)$ for some ideal I), with a similar notion holding in the general case.

- Let $A = k[x,y]$, and $I = (xy)$. I induces a closed subscheme of the 2 axis on $\mathbb{A}^2_k$, which is naturally reducible into the x and y axes. The ideal $J = (x^2)$ gives a subscheme structure with nilpotents on the y-axis (i.e. where $x^2 = 0 = x$). The ideal (x^2, xy) yields a different subscheme structure on the y-axis, but this time the nilpotents are concentrated at the origin.

- Let V be some variety over k, and suppose $W \subset V$ is a closed subvariety. Then W corresponds to some $P \in \operatorname{Spec}(k[V])$.

Most morphisms in the realm of topology also arise from preservation of topological invariants, such as open maps, closed maps, etc. The issue (as you've seen above) is that the topology of a scheme does not do a great job tracking the wealth of geometric data. Thus, morphisms of schemes need to be constructed carefully. When dealing with varieties over $\mathbb{C}$ (i.e. an integral scheme of finite type over $\mathbb{C}$) a lot of these definitions agree with the definitions we know of already with respect to complex analytic topological spaces.

1.4 Separatedness and Properness

1.4.1 Separated Morphisms

$f : Y \to X$ is a ***separated morphism*** if the canonically defined map $\Delta : Y \to Y \times_X Y$ is a closed immersion. For example, when $Y = \operatorname{Spec}(S), X = \operatorname{Spec}(R)$, $\Delta : \operatorname{Spec}(S) \to \operatorname{Spec}(S) \times_{\operatorname{Spec}(R)} \operatorname{Spec}(S)$ just the map $\operatorname{Spec}(S) \to \operatorname{Spec}(S \otimes_R S)$ defined by the multiplication map $S \otimes_R S \to S$ on rings.

Lemma 1.4.1. *Suppose that* $X \to \operatorname{Spec}(R)$ *is a separated morphism. Then if* $U_1, U_2 \subseteq X$ *is affine, then* $U_1 \cap U_2$ *is affine.*

Proof. By our hypothesis, $\Delta : X \to X \times_{\operatorname{Spec}(R)} X$ is a closed immersion, so it is an affine morphism. It is easy to see that $U_1 \times_{\operatorname{Spec}(R)} U_2 \subseteq X \times_{\operatorname{Spec}(R)} X$ is an affine patch. The inverse image of this under the diagonal morphism (which, since it is affine, will also be affine) is precisely $U_1 \cap U_2$. □

Lemma 1.4.2. *Suppose* X, Y *are separated schemes over* $\operatorname{Spec}(R)$, *and* $f_1, f_2 : Y \to X$ *are two morphisms of* $\operatorname{Spec}(R)$*-schemes. The locus where they agree, i.e.* $\{y \in Y \mid f_1(y) = f_2(y)\} \subseteq Y$ *is closed.*

Proof. We have the following diagram

$$\begin{tikzcd}
Y \arrow[drr, bend left, "f_2"] \arrow[dr, "\exists!(f_1{,}f_2)"] \arrow[ddr, bend right, "f_1"'] & & \\
& X \times_{\mathrm{Spec}(R)} X \arrow[r] \arrow[d] & X \arrow[d] \\
& X \arrow[r] & \mathrm{Spec}(R)
\end{tikzcd}$$

via the universal property of fiber products. Now consider the inverse image of the diagonal $\Delta \subseteq X \times_{\mathrm{Spec}(R)} X$ under the morphism (f_1, f_2) in Y. This is precisely $\{y \in Y \mid f_1(y) = f_2(y)\} \subseteq Y$. Separatedness tells us that (f_1, f_2) is a closed map, so we can conclude that the inverse image of Δ (a closed set) must also be closed. $\square$

Lemma 1.4.3. *Any morphism of affine schemes is separated.*

Proof. Let $f : X \to Y$ be a morphism of affine schemes, where $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B)$. f descends to a morphism of rings $B \to A$, giving A a B-algebra structure. $X \times_Y X = \mathrm{Spec}(A \otimes_B A)$ (see section 2.2), and the diagonal morphism $\Delta : X \to X \times_Y X$ is the image under $\mathrm{Spec}(-)$ of the ring homomorphism $A \otimes_B A \to A$ defined by $a \otimes a' = aa'$. This morphism is clearly surjective ($a \otimes 1 \mapsto a \ \forall a \in A$) so Δ is a closed immersion. $\square$

Lemma 1.4.4. *Assume all morphisms in the following statements are morphisms of noetherian schemes.*

(a) Open and closed immersions are separated.

(b) A composition of two separated morphisms is separated.

(c) Separated morphisms are stable under base extension.

(d) If $f : X \to Y, f' : X' \to Y'$ are separated morphisms of schemes over a base scheme S, then the product morphism $f \times f' : X \times_S X' \to Y \times_S Y'$ is also separated.

(e) If $f : X \to Y, g : Y \to Z$ are two morphisms and $g \circ f$ is separated, then f is separated.

(f) A morphism $f : X \to Y$ is separated $\iff$ Y can be covered by open subsets V_i such that $f^{-1}(V_i) \to V_i$ is separated for each i.

Proof. These results follow cleanly from the valuative criterion, and staring at the requisite diagram.

(a) Clear.

(b) Let $f : X \to Y, g : Y \to Z$ be two separated morphisms. Taking U, T as above, we have at most 1 of each dashed morphism:

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "i"] \arrow[dr] & X \arrow[d, "f"] \\
T \arrow[ur, dashed] \arrow[r] \arrow[r, dashed, bend right] \arrow[dr] & Y \arrow[d, "g"] \\
 & Z
\end{tikzcd}$$

Removing some arrow from this diagram, we immediately see that $X \to Y \to Z$ satisfies the valuative criterion:

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\
T \arrow[ur, dashed] \arrow[dr] & Y \arrow[d, "g"] \\
 & Z
\end{tikzcd}$$

(c) Let $f : X \to Y$ be a separated morphism. Suppose we have a morphism $Y' \to Y$ and base change X to a scheme over Y'. This gives us the diagram

$$\begin{tikzcd}
X \times_Y Y' \arrow[r] \arrow[d, "f_{Y'}"] & X \arrow[d, "f"] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

We check the valuative criterion. Suppose have two maps $T \to X \times_Y Y'$ making the following diagram commute:

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & X \times_Y Y' \arrow[r] \arrow[d, "f_{Y'}"] & X \arrow[d, "f"] \\
T \arrow[ur, dashed, shift left] \arrow[ur, dashed, shift right] \arrow[r] & Y' \arrow[r] & Y
\end{tikzcd}$$

Well, both of these maps extend to maps $T \to X \times_Y Y' \to X$, and as f is separated, they must be the same maps. It follows from the universal property of fiber products that $T \to X \times_Y Y'$ is uniquely determined, so $f_{Y'}$ satisfies the valuative criterion, and is thus separated.

$$\begin{tikzcd}
T \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "T \to X \times_Y Y' \to X"] \arrow[ddr, bend right] & & \\
 & X \times_Y Y' \arrow[r] \arrow[d, "f_{Y'}"] & X \arrow[d, "f"] \\
 & Y' \arrow[r] & Y
\end{tikzcd}$$

(d) Here's a diagram describing the situation.

$$\begin{tikzcd}
 & & X\times_S X' \arrow[dl] \arrow[dr] \arrow[d, dashed, "f\times f'"] & & \\
U \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] & Y\times_S Y' \arrow[dl] \arrow[dr] & X' \arrow[d, "f'"] & U' \arrow[l] \arrow[d, "i'"] \\
T \arrow[r] \arrow[ur, dashed] & Y & & Y' & T' \arrow[l] \arrow[ul, dashed]
\end{tikzcd}$$

If we have maps $U, U' \to X\times_S X'$ and maps $T, T' \to Y\times_S Y'$, We'd like to show uniquness of the dotted morphisms below:

$$\begin{tikzcd}
 & & X\times_S X' \arrow[dl] \arrow[dr] \arrow[d, dashed, "f\times f'"] & & \\
U \arrow[r] \arrow[d, "i"'] \arrow[urr, bend left] & X \arrow[d, "f"] & Y\times_S Y' \arrow[dl] \arrow[dr] & X' \arrow[d, "f'"] & U' \arrow[l] \arrow[d, "i'"] \arrow[ull, bend right] \\
T \arrow[r] \arrow[ur, dashed] \arrow[uurr, dotted, bend left] \arrow[urr, bend right=60] & Y & & Y' & T' \arrow[l] \arrow[ul, dashed] \arrow[uull, dotted, bend right] \arrow[ull, bend left=60]
\end{tikzcd}$$

Any map $T, T' \to X\times_S X'$ can be composed with the maps to X, X' to get maps $T \to X', T' \to X$. If there were two maps $T \to X\times_S X'$, then there would be two maps $T \to X'$, but since f' is separated, checking the valuative criterion on the diagram with the map $T \to Y\times_S Y' \to Y'$ implies that they have to be the same map, via logic similar to the previous part. A similar argument can be made for T', using the fact that f is separated.

$$\begin{tikzcd}
U \arrow[r, "U\to X\times_S X'\to X'"] \arrow[d, "i"] & X' \arrow[d, "f'"] \\
T \arrow[r, "T\to Y\times_S Y'\to Y'"'] \arrow[ur, dashed, shift left] \arrow[ur, dashed, shift right] & Y'
\end{tikzcd}
\qquad
\begin{tikzcd}
U' \arrow[r, "U'\to X\times_S X'\to X"] \arrow[d, "i'"] & X \arrow[d, "f"] \\
T' \arrow[r, "T'\to Y\times_S Y'\to Y"'] \arrow[ur, dashed, shift left] \arrow[ur, dashed, shift right] & Y
\end{tikzcd}$$

Thus, we can apply the universal property of fiber products (as below) to induce a unique map $T \to X\times_S X'$

$$\begin{tikzcd}
& T \arrow[d, dotted, "\exists!"] \arrow[dr, bend left] \arrow[dd, dashed, bend right] & \\
T' \arrow[r, dotted, "\exists!"] \arrow[rr, dashed, bend left] \arrow[dr, bend right] & X \times_S X' \arrow[r] \arrow[d] & X' \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

As there are at most 1 of each dashed morphism, there is at most 1 of each dotted morphism, implying that $f \times f'$ satisfies the valuative criterion, and is separated.

(e) Consider the diagram from part (b), which is satisfied when $g \circ f$ is separated:

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\
T \arrow[ur, dashed] \arrow[dr] & Y \arrow[d, "g"] \\
& Z
\end{tikzcd}$$

If there were a map $T \to Y$ making the diagram commute, then surely there could only be one map $T \to X$ making the diagram commute, as if there were multiple, then it would also let the above diagram commute, which would be a contradiction, as $g \circ f$ satisfies the valuative criterion.

(f) The forward case is immediate. For the reverse case, take the diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\
T \arrow[ur, dashed] \arrow[r] & Y
\end{tikzcd}$$

and apply the valuative criterion to the fibers of each element of the open cover on Y in T. There are finitely many of these things (as Y is Noetherian) so a simple gluing argument works to construct (at most) 1 map $T \to X$, as being able to construct two would mean that the maps differ on the fiber of some element of the open cover, which would mean there were two separate maps from the fiber of the open cover, which would be a contradiction to the separatedness in the local case.

It is also worth noting that $(a), (b), (c)$ imply $(d), (e)$ for any property of a morphism of schemes. See Hartshorne exercise 2.4.8. $\square$

Examples

- Let k be a field and let X be the affine line with doubled origin (see previous example in section 1.2.4). X is **not** separated over k. $X \times_k X$ is the affine plane with doubled axes and 4 origins, and the image of Δ is the diagonal line in $X \times_k X$, but with the origin simply doubled, not quadrupled. Taking the closure of $\Delta(X)$ would contain all 4 origins, suggesting that Δ does not have a closed image and is thus not a closed immersion.
- Let V be a variety over an algebraically closed field k. Then (as a scheme) V is separated over k. (This will be checked later, but indicated that the notion of being separated is a trivial notion outside of schemes)

1.4.2 Proper Morphisms

A ***proper morphism*** is a morphism that is

- Separated
- of finite type
- universally closed (i.e. all base changes are closed maps)

There is a method to check that morphisms are proper or separated:

Theorem 1.4.5. ***(Valuative Criterion)*** *Suppose $f : Y \to X$ is a morphism of schemes where Y is Noetherian. f is separated if for all valuation rings R, we have the diagram*

$$\begin{tikzcd}
\operatorname{Spec}(Frac(R)) \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
\operatorname{Spec}(R) \arrow[r] \arrow[ur, dashed, "\exists at\ most\ one"] & X
\end{tikzcd}$$

This is because $\operatorname{Spec}(R)$ *is just a closed point, and there should only be one way to lift it to Y (but no more) that maintains commutativity of the diagram if f is separated. For f to be proper, f must be of finite type and there exists EXACTLY one map* $\operatorname{Spec}(R) \to Y$.

As we are assuming R is a DVR, another way to think of the diagram above is like this:

$$\begin{tikzcd}
- - \circ - - \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
- - \bullet - - \arrow[r] \arrow[ur, dashed, "\exists \text{at most one}"] & X
\end{tikzcd}$$

As DVRs are local PIDs, they correspond to the local ring of a point on a curve, so you should get a shape like this.

Lemma 1.4.6. *Proper morphisms are:*

- *Closed immersions*
- *finite morphisms*
- *stable under base change*
- *stable under taking products (over a fixed based scheme)*

We say that a map $f : Y \to X$ is a ***projective morphism*** if the following diagram commutes:

$$\begin{tikzcd}
Y \arrow[r, "\text{closed immersion}"] \arrow[dr, "f"] & \mathbb{P}^n_X(:= \mathbb{P}^n \times_{\mathrm{Spec}(\mathbb{Z})} X) \arrow[d, "\pi"] \\
& X
\end{tikzcd}$$

Theorem 1.4.7. *Projective morphisms are proper.*

1.5 $\mathcal{O}_X$-Modules

Let X be a scheme, with associated structure sheaf $\mathcal{O}_X$. A ***Sheaf of $\mathcal{O}_X$-Modules*** $\mathscr{F}$ is a sheaf of abelian groups on X where $\mathscr{F}(U)$ is an $\mathcal{O}_X(U)$-module for all open sets $U \subset X$¡ and where the corresponding restriction maps are compatible with the module structures. $\mathcal{O}_X$-Mod is the category of $\mathcal{O}_X$-modules (where morphisms are $\mathcal{O}_X$-linear maps) and is an abelian category. Choose $\mathscr{F}, \mathscr{G}$ $\mathcal{O}_X$-modules. $\mathcal{H}om_{\mathcal{O}_x}(\mathscr{F}, \mathscr{G})$ is itself a sheaf of $\mathcal{O}_X$ modules assigning

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathscr{F}(U), \mathscr{G}(U))$$

(It's worth noting that you don't need to sheafify this, it is a sheaf by default). This is also different than $\mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$, which is just the set of $\mathcal{O}_X$-linear homomorphisms from $\mathscr{F} \to \mathscr{G}$. As the tensor product of modules is a module, we'd like $\mathcal{O}_X$-Mod to have tensor products. Thus, we can define $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$ to be the sheafification of the assignment

$$U \mapsto \mathscr{F}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{G}(U)$$

1.5.1 Pushforwards and Pullbacks

We'd also like to check compatibility over morphisms between two different schemes. Suppose that $f : Y \to X$ is a map of schemes, and $\mathscr{G}$ is an $\mathcal{O}_Y$-module. $f_*\mathscr{G}$ is a sheaf of $f_*\mathcal{O}_Y$ modules, but is not necessarily an $\mathcal{O}_X$ modules.You can, however, give $f_*\mathscr{G}$ the structure of an $\mathcal{O}_X$ module by applying restriction of scalars via the map $\mathcal{O}_X \to f_*\mathcal{O}_Y$. The pushforward functor $f_* : \mathcal{O}_Y$- Mod $\to \mathcal{O}_X$- Mod is indeed left exact (but not always right exact!) so it has a right adjoint, f^*, which is just the f^{-1} functor plus an extension of scalars over $f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$. In other words, if $\mathscr{F}$ is an $\mathcal{O}_X$ module, then $f^{-1}\mathscr{F}$ is an $f^{-1}\,\mathcal{O}_X$-modules, and $f^*\mathscr{F} := \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathscr{F}$ is an $\mathcal{O}_Y$-module, and the functor $f^* : \mathcal{O}_X$- Mod $\to \mathcal{O}_Y$- Mod is right exact (as it is the right adjoint of f_*, which is left exact). This gives us the isomorphism

$$\mathrm{Hom}_{\mathcal{O}_Y}(f^*\mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}, f_*\mathscr{G})$$

1.5.2 Quasi-Coherent Sheaves

Let $X = \operatorname{Spec}(R)$, and let M be an R-module. We'd like to construct a sheaf $\widetilde{M}$ with M in a way that is similar to the structure sheaf construction on $\operatorname{Spec}(R)$.

$$\widetilde{M}(U) = \left\{ s : U \to \prod_{p \in U} M_p \;\middle|\; s(p) \in M_p, \exists U = \bigcup D(g_\alpha), \frac{m_\alpha}{g_\alpha^{n_\alpha}} \in M\left[\frac{1}{g_\alpha}\right] \text{ s.t. } s(p) = \frac{m_\alpha}{g_\alpha^{n_\alpha}} \forall p \in D(g_\alpha) \right\}$$

Any sheaf that arises is clearly an $\mathcal{O}_X$-module, and is called ***Quasi-Coherent*** (i.e. if $\mathscr{F} = \widetilde{M}$ for some R-module M, it is quasi-coherent). If M is finitely generated, then we say that $\mathscr{F}$ is ***Coherent***. These definitions (for now) only hold for affine schemes.

As one would expect,

Lemma 1.5.1. *$\widetilde{M}(D(g)) = M\left[\frac{1}{g}\right]$. In particular, $\Gamma(X, \widetilde{M}) = M$.*

This immediately implies that $\widetilde{M}_P = M_P$, so stalks behave how we'd expect, similar to the case over $\mathcal{O}_X$. Furthermore, the assignment $M \to \widetilde{M}$ defines a fully faithful exact functor from R-Mod to $\mathcal{O}_X$-Mod. This, however, is not essentially surjective, so it does NOT define an equivalence of categories. This assignment does, however define an equivalence of categories R-Mod $\longleftrightarrow$ $\operatorname{QCoh}(\operatorname{Spec}(R))$. Similarly, the category of finitely generated R-modules (over a Noetherian ring R) is equivalent to the category of coherent sheaves on $\operatorname{Spec}(R)$.

Lemma 1.5.2. $\widetilde{R} = \mathcal{O}_X$.

Proof. Clear. ☐

Lemma 1.5.3. $X = \operatorname{Spec}(R)$ *is an affine scheme. $\mathscr{F}$ is a quasicoherent sheaf of $\mathcal{O}_X$ modules $\iff$ there is an exact sequence*

$$\mathcal{O}_X^{\oplus\beta} \to \mathcal{O}_X^{\oplus\alpha} \to \mathscr{F} \to 0$$

Proof. Suppose $\mathscr{F} = \widetilde{M}$. Consider the exact sequence

$$R^{\oplus\beta} \to R^{\oplus\alpha} \to M \to 0$$

Applying $\widetilde{(-)}$ yields the desired result. ☐

Lemma 1.5.4. *Let $X = \operatorname{Spec}(R)$ and M, N, M_α be R-modules for $\alpha \in \mathcal{A}$ for some indexing set $\mathcal{A}$.*

$$\widetilde{M \otimes_R N} = \widetilde{M} \otimes_R \widetilde{N}$$
$$\widetilde{\prod_{\alpha \in \mathcal{A}} M_\alpha} = \bigoplus_{\alpha \in \mathcal{A}} \widetilde{M_\alpha}$$

However,

$$\widetilde{\operatorname{Hom}_R(M, N)} = \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$$

only when M is finitely presented OR M and N are finitely generated.

Now suppose that $f : Y \to X$ is a morphism of schemes where $Y = \operatorname{Spec}(B), X = \operatorname{Spec}(A)$, and $f^\# : A \to B$ is the corresponding map on rings. Let N be a B-module and M an A module. Let ${}_A N$ be N viewed as an A-module via restriction of scalars.

Lemma 1.5.5. *$f_*(\widetilde{N}) = \widetilde{{}_A N}$ and $f^*(\widetilde{M}) = \widetilde{B \otimes_A N}$.*

Thus the tilde operation determines a compatibility between pushforward/pullback and extension/restriction of scalars.

We're now ready to define (quasi) coherence in the general case. Let X be ANY scheme and $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say $\mathscr{F}$ is ***Quasi-Coherent*** if $\exists X = \bigcup U_\alpha$ an affine open cover (i.e. $U_\alpha = \operatorname{Spec}(A_\alpha)$) and A_α-modules M_α such that $\mathscr{F}|_{U_\alpha} = \widetilde{M}_\alpha$. If all the M_α are finitely generated, then $\mathscr{F}$ is ***coherent***.

Equivalently, a sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ is quasicoherent if and only if $\exists$ a presentation

$$\mathcal{O}_{U_\alpha}^{\oplus\gamma} \to \mathcal{O}_{U_\alpha}^{\oplus\beta} \to \mathscr{F}|_{U_\alpha} \to 0$$

on each U_α. If X is locally noetherian, $\mathscr{F}$ is coherent if and only if this is a finite presentation.

Lemma 1.5.6. *$\mathscr{F}$ is a quasicoherent $\mathcal{O}_X$-module if and only if for every $\operatorname{Spec}(A) = U \subseteq X$ that's affine and open, $\mathscr{F}|_U = \widetilde{\mathscr{F}(U)}$. Moreover, if X is locally noetherian, $\mathscr{F}$ is coherent if and only if M is finitely generated.*

Lemma 1.5.7. *Let $X = \operatorname{Spec}(R)$ and let $\mathscr{F}, \mathscr{F}', \mathscr{F}''$ be sheaves of $\mathcal{O}_X$-modules such that*

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

is a short exact sequence. If $\mathscr{F}'$ is quasicoherent, then

$$0 \to \Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'') \to 0$$

is still a short exact sequence.

Proof. If all these sheaves were quasicoherent, then exactness of the tilde functor $\widetilde{(-)}$ would allow us to conclude immediately. We won't do the general case here, but you want to cover X by some finite number of basic open sets $D(g_i)$. Then pull sections of each basic open set back one by one from $\Gamma(X, \mathscr{F}'')$ to $\Gamma(X, \mathscr{F})$, using various lifting properties, to conclude surjectivity, and thus exactness. $\square$

We'll be able to prove this later using sheaf cohomology. In particular, this result follow immediately from the fact that, when X is affine and $\mathscr{F}'$ is quasicoherent, $\mathrm{H}^i(X, \mathscr{F}') = 0$ for $i > 0$.

Lemma 1.5.8. *$\operatorname{QCoh}(X)$ has kernels, cokernels, and images. It is also closed under extensions, i.e. if we have a short exact sequence of sheaves*

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

and $\mathscr{F}', \mathscr{F}''$ are quasicoherent, then $\mathscr{F}$ is quasicoherent. If X is Noetherian, these statements are also true about $\operatorname{Coh}(X)$.

Lemma 1.5.9. *Suppose that $f : Y \to X$ is a morphism of schemes and $\mathscr{G}$ is an $\mathcal{O}_Y$-module, and $\mathscr{F}$ is an $\mathcal{O}_X$-module.*

1. *(1) If $\mathscr{F}$ is (quasi) coherent, then $f^*\mathscr{F}$ is (quasi) coherent, with the implication following for coherence when X and Y are Noetherian.*
2. *(2) If Y is Noetherian OR f is quasicompact and quasi-separated, then $\mathscr{G}$ being quasicoherent implies that $f_*\mathscr{G}$ is quasicoherent.*

Do note that the second case tells us nothing about coherence, just quasicoherence. It is a major result of Grothendieck that, if f is proper, then pushforwards preserve coherence (and not only pushforwards, but derived pushforwards!).

Proof. We prove (1) first. Without loss of generality, let $X = \text{Spec}(A), Y = \text{Spec}(B)$ both be affine. Then $f : Y \to X$ induces a ring map $A \to B$. Thus, $f^*\widetilde{M} = \widetilde{B \otimes_A M}$. The quasicoherence of the pullback follows, and if X and Y are Noetherian, finite generation is preserved so coherence follows.

For (2), we can only assume that X is affine. As f is quasicompact, Y is covered by finitely many open affine sets. As Y is Noetherian OR quasiseparated, then the intersection of any two open affines can be covered by finitely many open affines. Thus, we get the short exact sequence

$$0 \to f_*\mathscr{G} \to \bigoplus_i f_*\mathscr{G}|_{U_i} \to \bigoplus_i , j f_*\mathscr{G}|_{U_i \cap U_j}$$

Where the last two terms are exact as they are merely restriction of scalars along a quasicoherent sheaf. It then follows that $f_*\mathscr{G}$ is quasicoherent. ☐

1.5.3 Sheaves on Projective Schemes

Let $S = \bigoplus_{n \geq 0} S_n$ be a graded ring, and let $X = \text{Proj}(S)$. Let $M = \bigoplus_{m \in \mathbb{Z}} M_n$ be a graded S module. One can use this to construct $\widetilde{M}$, a sheaf of $\mathcal{O}_X$-modules, in a way nearly identical to the construction of $\mathcal{O}_X$ from S. This is very similar to the normal tilde functor, but is constructed in a way that respects the graded structure.

Lemma 1.5.10. *Let $X = \text{Proj}(S)$ and M be a graded S-module.*

1. *(1) $\forall P \in X, \widetilde{M}_P = M_{(P)}$*
2. *(2) $\forall f \in S_n, n > 0$, $\widetilde{M}|_{D_+(f)} = \widetilde{M}_{(f)} = \widetilde{M\left[\frac{1}{f}\right]}$*
3. *(3) If $\widetilde{M}$ is a quasicoherent $\mathcal{O}_X$-module, S is Noetherian, and M is finitely generated, then $\widetilde{M}$ is a coherent $\mathcal{O}_X$-module.*

Recall that we can "twist" a graded S-module M to make $M(\ell)$, where $M(\ell)_n = M_{\ell+n}$. Similarly for $\mathcal{O}_X$-modules, if $X = \text{Proj}(S)$, then $\mathcal{O}_X(\ell) := \widetilde{S(\ell)}$ is the twisted sheaf of $\mathcal{O}_X$

by ℓ. Similarly, if $\mathscr{F} = \widetilde{M}$ is quasicoherent, then $\mathscr{F}(\ell) := \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\ell)$ is the ℓth twist of $\mathscr{F}$.

It's worth noting that twists behave well when S has a "standardized" grading, i.e. $S = S_0[S_1]$. If S is Noetherian, then $\exists S_0[x_0, \dots, x_n] \twoheadrightarrow S$ where x_i are all of degree 1. Taking Proj of this surjection induces a morphism $\operatorname{Proj}(S) \hookrightarrow \mathbb{P}^n_{S_0}$, recovering the standard definition of a projective scheme. For a quasicoherent sheaf $\mathscr{F} = \widetilde{M}$ over $X = \operatorname{Proj}(S)$, we define $\Gamma_*(\mathscr{F}) := \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathscr{F}(n)))$. This is akin to taking sections, but returns a graded module that behaves in the expected way.

1.5.4 Sheafy Constructions

Let X be any scheme and $\mathscr{A}$ a quasicoherent sheaf of $\mathcal{O}_X$-algebras. Then define $Y = \mathcal{S}pec(\mathscr{A})$ to be the scheme projecting down via an affine morphism f to X such that for every open affine $U \subset X$, $f^{-1}(U) = \operatorname{Spec}(\mathscr{A}(U))$. This is a "sheafy" Spec. It is a nontrivial fact that all affine morphisms into X arise this way, i.e. if $Y \to X$ is an affine morphism, then $Y = \mathcal{S}pec(\mathscr{A})$ for some quasicoherent sheaf $\mathscr{A}$.

Similarly, we can define a "sheafy" Proj. Suppose X is any scheme, $\mathscr{S}$ is a quasicoherent sheaf of **graded** $\mathcal{O}_X$ algebras. We can then define $\mathcal{P}roj(\mathscr{S}) \to X$. Using this construction, we can see that

$$\mathbb{P}^n_Y = \mathcal{P}roj\, \mathcal{O}_Y[x_0, \dots, x_n]$$

Which is equivalent to the formulation

$$\mathbb{P}^n_Y = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}(\mathbb{Z})} Y$$

For another example, take $\mathscr{E}$ to be a locally free sheaf of rank $r+1$ on X. Then $\mathbb{P}(\mathscr{E}) = \mathcal{P}roj\, \operatorname{Sym}(\mathscr{E})$. and the total space $\mathbb{E}$ is of the form $\mathcal{S}pec\, \operatorname{Sym}(\mathcal{E}^*)$, where $\mathcal{E}^*$ denotes the dual of $\mathcal{E}$.

Suppose X is a Γ-Scheme. $\mathcal{L}$ is an invertible sheaf on X. Then $\mathcal{L}$ is ***very ample*** provided that there exists a closed immersion of Γ-Schemes $i : X \hookrightarrow \mathbb{P}^n_\Gamma$ where $i^*\mathcal{O}_{\mathbb{P}^n_\Gamma}(1) \cong \mathcal{L}$.

We now give a definition. Suppose that A is a ring and $I \subset A[x_0, \dots, x_n]$ is a homogeneous ideal. It's clear that $I \subseteq (I : (x_0, \dots, x_n)) \subseteq (I : (x_0, \dots, x_n)^2) \subseteq \dots$. We define

$$I^{\text{sat}} := \bigcup_{\ell \in \mathbb{N}} (I : (x_0, \dots, x_n)^\ell) = \{f \in A[x_0, \dots, x_n] \mid \exists \ell \gg 0 \text{ s.t. } x_i^\ell f \in I\}$$

This is the ***saturation*** of I. If ideals I and J have the same saturation, we say they ***agree up to saturation***.

Lemma 1.5.11. *Suppose that A is any ring. Then,*

(a) Closed subschemes of $\mathbb{P}^n_A$ are in one to one correspondence with homogeneous ideals of $A[x_0, \dots, x_n]/$ where $I_1 \sim I_2$ if I_1 and I_2 agree in large degrees, i.e. up to saturation.

(b) *If I have a map from* $Y \to \mathrm{Spec}(A)$*, it is projective if and only if* $Y = \mathrm{Proj}(S)$ *for some graded* $S = S_0[S_1]$ *where* S_1 *is finitely generated over* S_0*, and* $S_0 = A$*.*

Suppose X is a scheme and $\mathscr{F}$ is a sheaf of $\mathcal{O}_X$-Modules. Then $\mathscr{F}$ is ***finitely globally generated*** if $\exists s_0, \dots, s_r \in \Gamma(X, \mathscr{F})$ such that $(s_0)_x, \dots, (s_r)_x$ generate $\mathscr{F}_x$ for every $x \in X$. Equivalently, we have a surjection $\mathcal{O}_X^{\oplus r+1} \twoheadrightarrow \mathscr{F}$, where the images of the generators are precisely $s_0, \dots, s_r$.

Theorem 1.5.12. *$X = \mathrm{Proj}(S)$, where $S = S_0[S_1]$ and S is Noetherian, then if $\mathscr{F}$ is a coherent $\mathcal{O}_X$-module, then $\mathscr{F}(n)$ is globally generated by finitely many sections for $n \gg 0$.*

1.6 Divisors

Currently our focus is on Weil Divisors $\eta \in X$, where X is a Noetherian normal integral separated scheme. Let $K = \mathcal{O}_{X,\eta}$. Let $Y \subseteq X$ be a ***prime divisor***, which is a irreducible closed integral subscheme of codimension 1. It follows that $\mathcal{O}_{X,Y} := \mathcal{O}_{X,\eta_Y}$ is a DVR contained inside K. This gives us a discrete valuation $\nu_Y : K^* \to \mathbb{Z}$. For $P = \eta_Y \in U = \mathrm{Spec}(R) \subseteq X$, where U is an open affine subset, we see that $\mathcal{O}_{X,Y} = R_P$. It follows that the set $\{x \in K \mid \nu_Y(x) \geq 0\}$ is just R_P. Furthermore, $P = \{x \in R \mid \nu_Y(x) > 0\}$.

Lemma 1.6.1. *Choose $f \in K^*$ nonzero. Then $\nu_Y(f) \neq 0$ for only finitely many choices of $Y \subseteq X$.*

This is in line with what we'd expect about divisors.

Proof. X is Noetherian, so it is covered by finitely many affine patches. Thus, it is sufficient to check that there are only finitely many Y as above on each patch. In particular, by the Noetherian condition, there are only finitely many prime divisors in $X \backslash U$, where $U = \mathrm{Spec}(R)$ is any open affine patch, i.e. there are finitely many Y such that $Y \cap U \neq \varnothing$. Taking $K = \mathrm{Frac}(R)$ and $f = \frac{a}{b}$ for $a, b \in R$ and nonzero, then $\nu_Y(f) = \nu_Y(a) - \nu_Y(b)$. It suffices to check then, for $a \in R$, that $\nu_Y(a) \neq 0$ for finitely many Y. This valuation being nonzero holds precisely when a is contained in the prime associated to Y, and as all Y are codimemsion 1, we just need to show that any $0 \neq a \in R$ is contained in finitely many height 1 primes. Well, $V(a)$ has finitely many irreducible components by Krull's principal ideal theorem, so the result follows. $\square$

1.6.1 The Class Group

Let $\mathrm{Div}(X)$ be the free abelian group generated on all prime divisors on X. Within this, we consider the set of ***principal divisors***, which are the divisors of the form

$$\mathrm{div}_X(f) := \sum_{Y \subset X \text{ prime divisor}} \nu_Y(f) \cdot y$$

For some $f \in K^*$. It's easy to check that this is an abelian subgroup of $\mathrm{Div}(X)$. Thus we can define ***class group*** $\mathrm{Cl}(X)$ to be the quotient $\mathrm{Div}(X)/\{\text{principal divisors}\}$. Alternatively, you can define the class group to be the group of divisors modulo the linear equivalence that $D_1 \sim D_2$ if $D_1 - D_2$ is a principal divisor.

For example, if $X = \mathrm{Spec}(R)$ where R is a Noetherian normal domain, then it is a classical number theory fact that $\mathrm{Cl}(X) = 0$ if and only if R is a UFD. equivalently, $\mathrm{Cl}(X) = 0$ if and only if all height 1 primes are principal, which holds if and only if all irreducible elements are prime. It follows from this discussion that $\mathrm{Cl}(\mathbb{A}^n_R) = 0$ precisely when R is a UFD.

For an explicit example, consider $R = \mathrm{Spec}\left(\frac{k[x,y,u,v]}{(xy-uv)}\right)$. We claim that the class group of $\mathrm{Spec}(R)$ is nonzero. To see why, we just need to exhibit a non-principal height 1 prime. Consider $P = (u, x)$. R/P is just $k[y, v]$, which is a domain, so P is prime. Furthermore, $k[y, v]$ is dimension 2 but R is dimension 3, so P has height 1, and it clearly is not principal. It follows that R is not a UFD, so it's spectrum has nontrivial class group.

1.6.2 Effective Divisors

We say a divisor $D = \sum a_i Y_i$ is ***effective*** provided that $a_i \geq 0$ for all i. Do be warned; this is a property about a divisor D, NOT a property about the equivalence class of divisors $[D] \in \mathrm{Cl}(X)$. An equivalence class can contain both effective and non-effective divisors. We say that $D \geq 0$ if it is effective, and that $D_1 \supseteq D_2 \iff D_1 - D_2 \geq 0$, i.e. is effective.

Lemma 1.6.2. *If $U \subset X$ is any open (not necessarily affine) subset of X (where X has all the hypotheses for divisors to be define), then if $f \in K^*$, then $f \in \Gamma(X, \mathcal{O}_X) \iff \mathrm{div}_U(f) \geq 0$.*

Proof. Let $U = U_1 \cup \cdots \cup U_r$ be a finite open affine cover. $\Gamma(U, \mathcal{O}_X) = \bigcap_i \Gamma(U_i, \mathcal{O}_X)$. It suffices to then check this is true when U is affine, in which case it follows from the fact that, for R a normal noetherian integral domain, $R = \bigcap R_P$, where we are intersecting over all the primes of height 1. □

As a corollary, we have

Lemma 1.6.3. *If $0 \neq f, g \in \Gamma(X, \mathcal{O}_X)$. Then $\mathrm{div}_X(f) = \mathrm{div}_X(g) \iff f = ug$ where $u \in \Gamma(X, \mathcal{O}_X)$ is unit.*

For example, for k a field, consider $\mathrm{Cl}(\mathbb{P}^n_k)$. This has a natural map into $\mathbb{Z}$ by taking degree, sending $D = \sum a_i Y_i$ to $\sum a_i \deg(Y_i)$. A prime divisor of $\mathbb{P}^n_k$ bijectively corresponds to a prime homogeneous ideal $(f) = P \subset k[x_0, \ldots, x_d]$ of height 1 where $d = \deg(f) = \deg(Y)$ (We know P to be principal because it is a prime homogeneous ideal of height 1, and that f is irreducible). It follows that the aforementioned map is bijection, and indeed $\mathrm{Cl}(\mathbb{P}^n_k) = \mathbb{Z}$.

Now consider the hyperplane $H = V(x_0)$ in $\mathbb{P}^n_k$. It follows that $\frac{f}{x_0^d} \in K(\mathbb{P}^n) = K\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$ where $\mathrm{div}\left(\frac{f}{x_0^d}\right) = Y - dH$.

Lemma 1.6.4. *Suppose X is a Noetherian normal integral separated scheme. Suppose $Z \subsetneq X$ is closed and $U = X \backslash Z$ is open.*

(a) The map $Cl(X) \twoheadrightarrow Cl(U)$ is surjective.

(b) If $\operatorname{codim}_X(Z) \geq 2$, then the above map is an isomorphism.

(c) If there is a prime divisor $Z \subsetneq X$ of codimension 1, then we get a right exact sequence

$$\mathbb{Z} \to Cl(X) \twoheadrightarrow Cl(U) \to 0$$

Where $\mathbb{Z} \to Cl(X)$ is defined by the map $\ell \mapsto \ell \cdot [Z]$.

Let's compute some more examples:

1.6.3 Example Computations

- Suppose that f is an irreducible polynomial of degree d. Then $V(f) \subseteq \mathbb{P}^n_k$ is a closed subvariety, and in particular,

  $$\mathrm{Cl}(\mathbb{P}^n_k \backslash V(f)) = \mathbb{Z}/d\mathbb{Z}$$

  by similar logic to the class group of projective space computation.

- Let $X = \text{Spec}\left(R = \frac{k[x,y,z]}{(xy-z)^2}\right)$. This is clearly not a UFD so X has nontrivial class group. Let Y be the prime divisor associated to $P = (y,z)$. In R_P, $\frac{1}{x} \in R_P$ so $y = \frac{1}{x}z^2$, and therefore PR_P is generated by z. It follows that $\mathrm{div}_X(y) = 2Y$. Thus $U = X \backslash Y = X \backslash V(y) = D(y) = \text{Spec}\,(R[1/y])$, where

  $$R[1/y] = \frac{k[x,y,z,1/y]}{(xy-z^2)} \cong k[y^{\pm 1}, z]$$

  Which is a UFD, and in particular, has trivial class group. It follows that $\mathbb{Z}$ surjects on to $\mathrm{Cl}(X)$. In particular, it follows from the fact that $\mathrm{div}_X(y) = 2Y$ that $\mathrm{Cl}(X) = \mathbb{Z}/2\mathbb{Z}$.

- The map $\pi : \mathbb{A}^1_X \to X$ induces a pullback map $\pi^* : \mathrm{Cl}(X) \to \mathrm{Cl}(\mathbb{A}^1_X)$ (this is not true in general! in general only flat morphisms pull back to class groups.) Explicitly this map sends $[Y]$ to $[pi^{-1}(Y)]$, mapping a prime ideal $P \subset R$ of height 1 to $P[t] \subsetneq R[t]$. This map is actually an isomorphism!

- Similarly, we can check that $\mathrm{Cl}(\mathbb{P}^1_X) = \mathrm{Cl}(X) \oplus \mathbb{Z}$.

- In general, $\mathrm{Cl}(X \times Y)$ is NOT isomorphic to $\mathrm{Cl}(X) \oplus \mathrm{Cl}(Y)$, so be careful!

1.6.4 Divisors on Curves

Recall that X is a curve (over an algebraically closed field k) if it is integral, separated of finite type and dimension 1. We'll assume that X is complete (i.e. proper over k) and nonsingular (i.e. all stalks are regular rings). On a nonsingular curve, prime divisors are just (closed) points! Thus $D \in \mathrm{Div}(X)$ is of the form $\sum n_i P_i$ where P_i is a closed point, and

$\deg(D) = \sum n_i$. If $f : Y \to X$ is a map of nonsingular complete curves, then either $f(Y) \subset X$ is a closed point, or $f(Y) = X$. In the latter case, one can check that f is a finite map of degree $[K(Y) : K(X)]$. We have a pullback map $f^* : \text{Div}(X) \to \text{Div}(Y)$ that can be defined as follows. If $P \in X$, then because finite maps are quasi-finite, $f^{-1}(P) = \{Q_1, \dots, Q_S\}$. Thus, we have a map $\mathcal{O}_{X,P} \to \mathcal{O}_{Y,Q_i}$ mapping some uniformizer t into the DVR $\mathcal{O}_{Y,Q_i}$. Thus we can define

$$f^*(P) := \sum_{i=1}^{s} \nu_{Q_i}(t) \cdot Q_i$$

One can check that $\deg(f^*D) = \deg(f)\deg(D)$. In particular, if I choose $\varphi \in K(X)$, then $f^*(\text{div}_X(\varphi)) = \text{div}_Y(f^\# \varphi)$. We even have the following lemma:

Lemma 1.6.5. *If X is a complete nonsingular curve and $\varphi \in K(X)$, then* $\deg(\mathit{div}_X(\varphi)) = 0$.

Proof. On an open set $U \subset X$, φ induces a map $U \to \mathbb{A}^1_k$ which extends to a map $\pi : X \to \mathbb{P}^1_k$. It's easy to check that $\text{div}_X(\varphi) = \pi^*(\{0\} - \{\infty\})$, so

$$\deg(\text{div}_X(\varphi)) = \deg\left(\pi^*(\{0\} - \{\infty\})\right) = \deg(\pi) \cdot \deg(\{0\} - \{\infty\}) = 0$$

$\square$

This lemma implies that taking the degree of a divisor induces a natural map $\text{Cl}(X) \to \mathbb{Z}$, just as in the case of $\mathbb{P}^n_k$. This phenomenon does not generalize to arbitrary projective varieties, not even arbitrary curves.

Lemma 1.6.6. *Suppose that X is a nonsingular complete curve over an algebraically closed field. X is rational if and only if $\exists P, Q \in X$ distinct points such that $P \sim Q$ as divisors.*

In the case where X is elliptic, you get a really nice group law on the class group that is reminiscent of the construction from number theory. This is certainly worth looking into, and will be revisited in chapter 4 of Hartshorne. Now, however, we'll move our attention to Cartier Divisors.

1.6.5 Cartier Divisors

We'll take the same setup as before: X is a Noetherian normal integral separated scheme. $D \in \text{Div}(X)$ is ***locally principal, or Cartier*** if $\exists X = \bigcup_i U_i$ an open affine cover and $\varphi_i \in K(X)^*$ where $\text{div}_{U_i}\varphi_i = D|_{U_i}$ for all i. We have the natural inclusions

$$\{\text{principal divisors}\} \subseteq \text{CaDiv}(X) := \{\text{Cartier Divisors}\} \subseteq \text{Div}(X)$$

This gives rise to the ***Cartier Class Group***, denoted $\text{CaCl}(X)$, which is just the group of Cartier divisors modulo the subgroup of principal divisors. This is naturally a subgroup of the class group of X.

Lemma 1.6.7. *If X is regular, then every divisor is locally principal. In particular, $CaDiv(X) = Div(X)$ if and only if $\mathcal{O}_{X,x}$ is a UFD for all $x \in X$.*

Proof. Recall that class groups are trivial in the affine setting if and only if X is Spec of a UFD. Thus if all divisors are principal if and only if we are in the spectrum of a UFD, then a scheme being locally the spectrum of a UFD is equivalent to all divisors being locally principal. This is precisely how one would interpret the condition that $\mathrm{CaDiv}(X) = \mathrm{Div}(X)$.

It's worth noting that this is more of a moral proof than an actual one, at least in the reverse case. $\square$

A natural consequence of this is that the class group and Cartier class group of X are equal provided that X is locally factorial (in particular, in the case that X is regular).

Lemma 1.6.8.

$$\mathrm{CaDiv}(X) = \Gamma\left(X, \frac{\underline{k(x)}^*}{\mathcal{O}_X^*}\right)$$

Where $\underline{k(x)}^*$ is the constant sheaf of $k(x)^*$ on X, and $\mathcal{O}_X^* \subseteq \underline{k(x)}^*$ is the sheaf of units of $\mathcal{O}_X$, i.e. the unique sheaf such that for any open set $U \subseteq X$, $\Gamma(U, \mathcal{O}_X^*) = \Gamma(U, \mathcal{O}_X)^*$.

This gives us the following construction. Consider the short exact sequence

$$0 \to \mathcal{O}_X^* \to \underline{k(x)}^* \to \underline{k(x)}^*/\mathcal{O}_X^* \to 0$$

taking global section yields the sequence

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X)^* \longrightarrow k(x) \longrightarrow \mathrm{CaDiv}(X) \longrightarrow \mathrm{H}^1(X, \mathcal{O}_X^*) \longrightarrow \underbrace{\mathrm{H}^1(X, \underline{k(x)}^*)}_{=0} \longrightarrow \ldots$$

Thus by exactness, one can get the isomorphism

$$\mathrm{H}^1(X, \mathcal{O}_X^*) \cong \mathrm{CaCl}(X)$$

1.6.6 The Picard Group

We define the Picard Group of X (Noetherian integral separated and normal), denoted $\mathrm{Pic}(X)$ as the group of isomorphism classes of invertible sheaves on X. By taking the sheaf of sections of a line bundle, this produces an invertible sheaf. Also, one can take an invertible sheaf, then use it construct a line bundle such that the sheaf of sections of that line bundle is the invertible sheaf you started with. This correspondence immediately tells us that $\mathrm{Pic}(X)$ is the group generated by the isomorphism classes of line bundles on X. With regards to a line bundle $\mathcal{L}$, one can prove the following easy facts:

$$\mathcal{L} \otimes \mathcal{O}_X \cong \mathcal{O}_X \otimes \mathcal{L} \cong \mathcal{L}$$

$$\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$$

$$\mathcal{L}^{-1} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) = \mathcal{L}^\vee$$

Theorem 1.6.9.

$$\mathrm{CaCl}(X) \cong \mathrm{Pic}(X)$$

Proof. We prove this using the correspondence sending $D \in \mathrm{Div}(X)$ to $\mathcal{O}_X(D)$, which is a sheaf on $\mathcal{O}_X$-modules. Recall that can associate

$$\Gamma(U, \mathcal{O}_X) = \left\{ f \in K(X) \mid \mathrm{div}_U(f) = \sum \nu_Y(u) \cdot Y \geq 0 \right\}$$

Where we are summing over prime divisors $Y \subseteq U$. Thus,

$$\Gamma(U, \mathcal{O}_X(D)) = \left\{ f \in K(X) \mid \mathrm{div}_U(f) + D = \sum \nu_y(u) \cdot y + D \geq 0 \right\}$$

Writing $D = \sum a_y \cdot Y$, It follows that this is the sheaf of all functions where, for all prime divisors $Y \subset U$, $\nu_Y(f) \geq -a_y$. It's easy to see that $\mathcal{O}_X(D) \subseteq \underline{K(X)}$ and is of rank 1. Thus, $\mathcal{O}_X(D) \leq \mathcal{O}_X \iff D \leq 0$.

$\mathcal{O}_X(D)$ is actually a ***reflexive sheaf***, i.e. under the functor $(-)^* = \mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)$, $\mathcal{O}_X(D)^{**} = \mathcal{O}_X(D)$. It is also easy to see that $\mathcal{O}_X(D)^* = \mathcal{O}_X(-D)$. The result follows from the fact that $\mathcal{O}_X(D)$ is invertible $\iff$ D is Cartier, which we will now prove.

For the reverse case, let $X = \bigcup U_i$ and choose $f_i \in K(X)^*$ such that $\mathrm{div}_{U_i}(f_i) = D|_{U_i}$ (we can do this because D is Cartier, i.e. locally principal). On U_i,

$$\Gamma(U_i, \mathcal{O}_X(D)) = \left\{ \varphi \in K(X) \mid \mathrm{div}_{U_i}(\varphi) + D|_{U_i} \geq 0 \right\}$$

Where $\mathrm{div}_{U_i}(f_i) = D|_{U_i}$. It follows that $f_i \varphi \in \Gamma(U_i, \mathcal{O}_X)$, so $\varphi \in \frac{1}{f_i}\Gamma(U_i, \mathcal{O}_X)$. It follows that $\mathcal{O}_X(D)|_{U_i} = \mathcal{O}_{U_i} \cdot \frac{1}{f_i} \cong \mathcal{O}_{U_i}$. It follows that $\mathcal{O}_X(D)$ is invertible.

Before moving on, we must issue a warning. In general, $\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$ is NOT isomorphic to $\mathcal{O}_X(D_1 + D_2)$. It is when D_1 (or equivalently, D_2) is Cartier. In general, we have that $(\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2))^{**} \cong \mathcal{O}_X(D_1 + D_2)$.

From this, we see that the map $D \mapsto \mathcal{O}_X(D)$ is a map of abelian groups between $\mathrm{CaDiv}(X) \to \mathrm{Pic}(X)$. We'd like to show that this map is surjective, and that it's kernel is precisely all the principal divisors. If $D = \mathrm{Div}_X(f)$ for some f, then $\mathcal{O}_X(D) = \frac{1}{f}\mathcal{O}_X$, which is isomorphic to $\mathcal{O}_X$, and thus will be sent to 0. Similarly if $\mathcal{O}_X(D) \cong \mathcal{O}_X$, if we look at the generic point $\eta \in X$, we see that $\mathcal{O}_X(D), \mathcal{O}_X \subseteq \underline{K(X)} = \mathcal{O}_X(D)_\eta$. The isomorphism $\mathcal{O}_X(D) \cong \mathcal{O}_X$ induces an endomorphism on $\underline{K(X)}$ which is a linear map between dimension 1 vector spaces, i.e. multiplication by an element. Say this map is multiplication by g. It's easy to check that $D = \mathrm{div}_X\left(\frac{1}{g}\right)$.

It is now sufficient to check that this map is surjective. Suppose that $\mathcal{L}$ is any invertible sheaf. I can consider $\mathcal{L} \subseteq \mathcal{L}_\eta \cong K(X)$ (to see why the latter isomorphism holds, we can see that $\mathcal{L}_\eta \cong \mathcal{O}_U$ because $\mathcal{L}$ is invertible, and η corresponds to the zero ideal, so $\mathcal{O}_{U,\eta}$ is just $K(X)$, implying that $\mathcal{L}_\eta \cong K(X)$, which yields the isomorphism). Thus, $\mathcal{L} \subset \underline{K(X)}$. Choose a distinguished section $1 \in \underline{K(X)}$. $1 \in K(X)$ corresponds to a rational section $s \in \Gamma(U, \mathcal{L})$ for a nonempty $U \subseteq X$ open. Now define

$$\mathrm{CaDiv}(X) \ni \mathrm{div}_X(s)$$

by working on patches. Take an open affine trivializing cover of $\mathcal{L}_U \cong \mathcal{O}_U$. f corresponds to a fraction on the affine open cover, and check that these things glue together nicely. This determines a divisor D such that $\mathcal{O}_X(D) \cong \mathcal{L}$, and we get surjectivity. □

What are some natural generalizations we can make? Well for starters, $\operatorname{Pic}(X)$ makes sense for any scheme X, not just the Noetherian integral normal separated ones we've been thinking about. We saw that $\operatorname{CaDiv}(X) \cong \Gamma\left(X, \underline{K(X)}/\mathcal{O}_X^*\right)$ in our standard setting, so we can define $\operatorname{CaDiv}(X)$ on an arbitrary scheme as

$$\operatorname{CaDiv}(X) \cong \Gamma\left(X, \mathcal{K}^*/\mathcal{O}_X^*\right)$$

Where $\mathcal{K}^*$ is the ***total ring of quotients***, i.e. for a given ring, localize at the set of all non-zero divisors S. This is the largest localization such that the map $R \to R_S$ is injective. When R is a domain, $S = R\backslash\{0\}$ and we just get the fraction field as before. We can use this to construct an analogous notion of $\operatorname{CaCl}(X)$. We have the short exact sequence

$$0 \to \mathcal{O}_X{}^* \to \mathcal{K}^* \to \mathcal{K}^*/\,\mathcal{O}_X{}^* \to 0$$

Passing this to global sections, we define the principal divisors to be

$$\operatorname{im}\left(\Gamma(X, \mathcal{K}^*) \to \Gamma(X, \mathcal{K}^*/\,\mathcal{O}_X{}^*)\right)$$

Thus, we can associated $\operatorname{CaCl}(X)$ with

$$\operatorname{coker}\left(\Gamma(X, \mathcal{K}^*) \to \Gamma(X, \mathcal{K}^*/\,\mathcal{O}_X{}^*)\right)$$

One can then ask, are the Cartier class group and the Picard group the same in this general setting, the answer is no in general, but the isomorphism does hold when X is integral.

1.7 Projective Morphisms

X is a scheme over $\operatorname{Spec}(A)$.

Theorem 1.7.1. *Morphisms $X \to \mathbb{P}^n_A$ are in a one to one bijection with specifying an invertible sheaf $\mathcal{L}$ on X with a collection $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ that globally generate X.*

Proof. Given a morphism $\pi : X \to \mathbb{P}^n_A$, let $\mathcal{L} := \mathcal{O}_{\mathbb{P}^n_A}(1)$ and take $s_i := \pi^* x_i$. For the reverse direction, Define the morphism $X \to \mathbb{P}^n_A$ by sending $x \mapsto [s_0(x) : \cdots : s_n(x)]$. This definition is a bit murky since each s_i has nothing to do with projective space. On an open set, however, the ratios of the sections make sense in some way. Thus on an open set we can map into $\mathbb{A}^n_S$ by mapping to the images $[1 : s_1/s_0 : \cdots : s_n/s_0]$. □

Lemma 1.7.2.

$$\operatorname{Aut}(\mathbb{P}^n_k) \cong \operatorname{PGL}_n(k) := \operatorname{GL}_n(k)/k^*$$

Proof. for any $\pi \in \operatorname{Aut}(\mathbb{P}^n_k)$, $\pi^*\mathcal{O}_{\mathbb{P}^n_k}(m) \cong \mathcal{O}_{\mathbb{P}^n_k}(m)$, where $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1))$ is the vector space of degree 1 polynomials in n variables. The isomorphism is thus a linear map of these two vector spaces on sections, and this argument can be generalized to considering maps on $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(m))$ for positive m (note that $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(-1)) = 0$ so we don't consider these). □

Lemma 1.7.3. *Let $\pi : X \to \mathbb{P}^n_A$ be a morphism of schemes over A. This is a closed immersion $\iff$*

- *$X_{s_i} = \pi^{-1}(D_+(x_i))$ is affine for each i.*
- *$A\left[\frac{x_j}{x_i}\right] \to \Gamma(X_s, \mathcal{O}_X)$ sending $\frac{x_j}{x_i} \to \frac{s_j}{s_i}$ must be surjective.*

We can specialize this lemma when we are working an algebraically closed field k.

Lemma 1.7.4. *Let $\pi : X \to \mathbb{P}^n_k$ be a morphism of schemes over k, and X is a projective scheme over k, and π is determined by $\mathcal{L}$ and $s_0, \dots, s_n$. Let $V = \langle s_0, \dots, s_n \rangle \subset \Gamma(X, \mathcal{L})$. π is a closed immersion $\iff$*

- *V separates points, i.e. for any two closed points $x_1, x_2 \in X$, $\exists s \in V$ such that $s_{x_1} \in m_{x_1}\mathcal{L}_{x_1}$, but $s_{x_2} \notin m_{x_2}\mathcal{L}_{x_2}$*
- *V separates tangent vectors, i.e. $\forall x \in X$ closed points, images of $s \in V$ with $s_x \in m_x\mathcal{L}_x$ generate $m_x\mathcal{L}_x / m_x^2\mathcal{L}_x$.*

1.7.1 Ample Line Bundles

Suppose that $\mathcal{L}$ is an invertible sheaf on X, which is a scheme over A. We say that $\mathcal{L}$ is ***very ample*** if $\exists$ an immersion $\pi : X \to \mathbb{P}^n_A$ such that $\pi^*\mathcal{O}_{\mathbb{P}^n_A}(1) = \mathcal{L}$.

Over a general Noetherian scheme X, an invertible sheaf $\mathcal{L}$ on X is ***ample*** if $\forall$ coherent sheaves $\mathscr{F}$ on X, we have $\mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ is globally generated for $n \gg 0$, where the size of n can depend on each choice of $\mathscr{F}$. It is fairly clear that very ample line bundles are ample line bundles.

Theorem 1.7.5. *Suppose X is finite type over a noetherian ring A. An invertible sheaf $\mathcal{L}$ on X is ample $\iff$ $\mathcal{L}^{\otimes n}$ is very ample for $n \gg 0$.*

Proof. Let's do the forward case (the reverse case is left as an easy exercise). Suppose that $\mathcal{L}$ is ample. Pick $x \in X$ and choose $U \subset X$ to be an affine open set containing x such that $\mathcal{L}|_U \cong \mathcal{O}_U$. Now take $Z = X \backslash U \subseteq X$ closed, and assign it its canonical reduced structure. This determines a coherent sheaf of ideals $\mathcal{I}_Z$. As $\mathcal{L}$ is ample, it follows that $\mathcal{I}_Z \otimes \mathcal{L}^{\otimes m}$ is globally generated for $m \gg 0$. Given an inclusion map $i : Z \hookrightarrow X$, We have a natural short exact sequence of sheaves

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0$$

Locally, tensoring by $\mathcal{L}$ (or really, any line bundle) is just tensoring by a copy of $\mathcal{O}_X$, so it must be flat. It follows that we have a short exact sequence

$$0 \to \mathcal{I}_Z \otimes \mathcal{L}^{\otimes m} \to \mathcal{L}^{\otimes m} \to i_*\mathcal{O}_Z \otimes \mathcal{L}^{\otimes m} \to 0$$

Where $i_*\mathcal{O}_Z \otimes \mathcal{L}^{\otimes m} \cong \mathcal{L}^{\otimes m}|_Z = i_*i^*\mathcal{L}^{\otimes m}$. Taking global sections yields a left exact sequence

$$0 \to \Gamma(X, \mathcal{I}_Z \otimes \mathcal{L}^{\otimes m}) \to \Gamma(X, \mathcal{L}^{\otimes m}) \to \Gamma(X, i_*\mathcal{O}_Z \otimes \mathcal{L}^{\otimes m})$$

It follows that sections $\Gamma(X, \mathcal{I}_Z \otimes \mathcal{L}^{\otimes m})$ are precisely the sections $\Gamma(X, \mathcal{L}^{\otimes m})$ that map to zero under the map $\Gamma(X, \mathcal{L}^{\otimes m}) \to \Gamma(X, i_* i^* \mathcal{L}^{\otimes m})$ viewed as $s \mapsto s|_Z$, i.e. we are considering the sections s such that $s|_Z = 0$. Thus, sections $\Gamma(X, \mathcal{I}_Z \otimes \mathcal{L}^{\otimes m})$ are precisely the sections in $\Gamma(X, \mathcal{L}^{\otimes m})$ that vanish on Z. Thus, $\exists s \in \Gamma(\mathcal{I}_Z \otimes \mathcal{L}^{\otimes m})$ so that $s_x \notin m_x \mathcal{I}_{Z_{m_x}} \otimes \mathcal{L}_x^{\otimes n}$. Locally $\mathcal{L}_x^{\otimes n}$ looks like $\mathcal{O}_X$, so we can ignore it for the most part. We can see that that $X_s \subseteq U$, where X_s is the locus of sections that are nonvanishing in the above sense. In particular, s does not vanish on x, so

$$x \in X_s \subseteq U \subseteq X$$

for U affine. Thus X_s is also affine, and in particular an affine open set containing x. As X is Noetherian, we can get a uniform m that is sufficiently large such that, for all points in X, X_s is an affine neighborhood of x. For such an m, we see that $\mathcal{L}^{\otimes m}$ is globally generated and determines a map $X \to \mathbb{P}^n_A$. Since X is finite type over A, we can pick enough sections to make this map an immersion, so $\mathcal{L}^{\otimes m}$ is very ample. □

Lemma 1.7.6. *Suppose that $\mathcal{L}$ is an invertible sheaf on a Noetherian scheme. Then, the following are equivalent:*

(1) $\mathcal{L}$ is ample.

(2) $\mathcal{L}^{\otimes m}$ is ample for any $m > 0$.

(3) $\mathcal{L}^{\otimes m}$ is ample for some $m > 0$.

Proof. Clearly (2) and (1) imply (3). Now suppose (3) holds. Say $\mathscr{F}$ is a coherent sheaf. Clearly $\mathscr{F} \otimes (\mathcal{L}^{\otimes m})^{\otimes n} = \mathscr{F} \otimes \mathcal{L}^{\otimes mn}$. Thus, you can tensor any of $\mathscr{F}, \mathscr{F} \otimes \mathcal{L}, \ldots, \mathscr{F} \otimes \mathcal{L}^{\otimes mn}$, when tensoring by $\mathcal{L}^{\otimes mn}$, are all globally generated for n large. Thus, $\mathscr{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated for any large n. Thus, $\mathcal{L}^{\otimes m}$ is ample for any $m > 0$, so in particular, (3) implies (2) and (1). □

1.7.2 Linear Systems

Suppose now that we're working over an algebraically closed field k and that X is an integral nonsingular projective variety over k. We know now that ll invertible sheaves have the form $\mathcal{O}_X(D)$ for some $D \in \operatorname{Div}(X)$. In particular,

$$\Gamma(X, \mathcal{O}_X(D)) = \{\varphi \in K(X) \mid \operatorname{div}_X(\varphi) + D \geq 0\}$$

Let $|D| := \mathbb{P}\Gamma(X, \mathcal{O}_X(D))$. One can see from the discussion above that this is precisely the set of all effective divisors linearly equivalent to D (by varying over $\varphi \in K(X)$ as above). In particular, if $\varphi_1, \varphi_2 \in \Gamma(X, \mathcal{O}_X(D))$, then we see that $\operatorname{div}_X \frac{\varphi_1}{\varphi_2} = 0$. Thus,

$$\frac{\varphi_1}{\varphi_2} \in \Gamma(X, \mathcal{O}_X) = k$$

As k is algebraically closed. Thus, $\varphi_1 = \lambda \varphi_2$ for some $\lambda \in k^*$. Given a complete linear system, you can map $X \to \mathbb{P}(\Gamma(X, \mathcal{O}_X(D))^*)$ *(take note that we are taking the dual, so instead* *of mapping to lines through the origin you map to hyperplane sections)* defined by mapping x to the set of hyperplanes of $\Gamma(X, \mathcal{O}_X(D))$ containing x. This map is well defined provided that all hyperplanes don't contain X, so in practice we want to restrict to the open locus where this will not happen.

Suppose that $\mathcal{E}$ is a locally free sheaf of finite rank $n+1$ on X. We can define $\mathbb{P}(\mathcal{E}) := \mathcal{P}roj\operatorname{Sym}^* \mathcal{E}$. Local freeness tells us that $\mathcal{E}|_U = \mathcal{O}_X^{\oplus n+1} = \widetilde{A^{\oplus n+1}}$, so $\operatorname{Sym}^* \mathcal{E}|_U \cong A[x_0, \ldots, x_n]$. In particular, $\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{E}$, so $\mathcal{E}$ can be recovered in the projectivization. In addition, if we take the direct sum over all twists, we recover the symmetric algebra:

$$\bigoplus_{\ell} \pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\ell) = \operatorname{Sym}^* \mathcal{E}$$

More generally,

$$\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\ell) = \begin{cases} 0 & \ell < 0 \\ \mathcal{O}_X & \ell = 0 \\ \operatorname{Sym}^\ell \mathcal{E} & \ell > 0 \end{cases}$$

We also have a natural surjection $\pi^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. This is because $\pi^*\mathcal{E}|_U = \pi^*\mathcal{O}_U^{\oplus n+1} = \mathcal{O}_{\mathbb{P}^n_A}^{\oplus n+1}$, we have an induced surjection $\mathcal{O}_{\mathbb{P}^n_A}^{\oplus n+1} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^n_A}(1)$ by assignment of global generators.

and in general surjections $f^*\mathcal{E} \to \mathcal{L}$ induce a morphism of schemes $Y \to X$ that factors through $\mathbb{P}(\mathcal{E})$.

Lemma 1.7.7. *For a fixed morphism $f : Y \to X$, There is a one to one bijection between*

$$\left\{ \text{Maps } Y \to \mathbb{P}(\mathcal{E}) \text{ such that } \begin{tikzcd} Y \arrow[r] \arrow[dr, "f"] & \mathbb{P}(\mathcal{E}) \arrow[d, "\pi"] \\ & X \end{tikzcd} \right\} \longleftrightarrow \{\text{surjections } f^*\mathcal{E} \twoheadrightarrow \mathcal{L}\}$$

Where $\mathcal{L}$ is a line bundle over Y.

As an example of this, just take $Y = \mathbb{P}(\mathbb{E}) \to X$ where $f = \pi$. This induces the surjection $Y = \pi^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{L}$. Such a surjection is identified to the identity map

$$\begin{tikzcd} \mathbb{P}(\mathcal{E}) \arrow[r, "\mathrm{Id}"] \arrow[dr, "f"] & \mathbb{P}(\mathcal{E}) \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

As another example, let's just let Y be a closed point. Suppose in this cae we're working with finite type varieties over a field. An element $x \in X$ can be viewed as a map $\operatorname{Spec}(k) \to X$. We have a diagram of morphisms

$$\begin{tikzcd}
y : \mathrm{Spec}(K) \arrow[r, "i_y"] \arrow[d] & Y \arrow[d, "f"] \\
f(y) = x : \mathrm{Spec}(k) \arrow[r, "i_x"] & X
\end{tikzcd}$$

Observe that $\mathcal{E}|_x = i_x^*(\mathcal{E}) = k^{n+1} = f^*\mathcal{E}|_y = i_y^* f^*\mathcal{E}$. Thus, $\mathbb{P}(\mathcal{E})|_x \cong \mathbb{P}(k^{n+1}) \cong \mathbb{P}^n$. Since $\mathbb{P}^n \subset \mathbb{P}(\mathcal{E})$, we can pull Y to $\mathbb{P}^n \subset \mathbb{P}(\mathcal{E})$ in a way that determines a surjection $k^{n+1} \twoheadrightarrow k = \mathcal{L}|_y$.

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, hook] & \mathbb{P}(\mathcal{E}) \arrow[d] & & \\
& & \Longrightarrow & k^{n+1} \twoheadrightarrow k = \mathcal{L}_y \\
y \in Y \arrow[uu] \arrow[uur, dashed] \arrow[r, "f"] & X \ni x \arrow[from=1-2, to=3-2] & &
\end{tikzcd}$$

Let's now move on to proving the result.

Proof. For the reverse case, given a morphism

$$\begin{tikzcd}
Y \arrow[r, "\varphi"] \arrow[dr, "f"'] & \mathbb{P}(\mathcal{E}) \arrow[d, "\pi"] \\
& X
\end{tikzcd}$$

As $\pi^*\mathcal{E} \twoheadrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, this induces a morphism $f^*\mathcal{E} = \varphi^*\pi^*\mathcal{E} \twoheadrightarrow \mathcal{L} := \varphi^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ by passing through φ^* functorially.

For the forward case, given a surjection $f^*\mathcal{E} \twoheadrightarrow \mathcal{L}$, on an open affine $U = \mathrm{Spec}(A) \subseteq X$ this looks like $\mathcal{O}_{f^{-1}(U)}^{n+1} \twoheadrightarrow \mathcal{L}|_{f^{-1}(U)}$. this surjection is uniquely determined by a choice of $s_0, \ldots, s_{n+1}$ that globally generates $\mathcal{L}|_{f^{-1}(U)}$, which in turn determine a morphism $g : f^{-1}(U) \to \mathbb{P}_A^n$ where $g^*\mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{L}_{f^{-1}(U)}$. □

1.7.3 Blowups

A coherent ideal sheaf $\mathcal{I} \subset X$ determines a Rees algebra $\mathscr{S} = \oplus_{n\geq 0}\mathcal{I}^n$, In other words, letting $X = \mathrm{Spec}(R)$ and $\mathcal{I} = \widetilde{I}$ for $I \subset R$, we see that $\mathscr{S} = \widetilde{\bigoplus_{n\geq 0} I^n} = R[It] \subset R[t]$. The ***Blow-up along*** $\mathcal{I}$ ***on*** X is the object $\widetilde{X} = \mathrm{Bl}_{\mathcal{I}}X := \mathcal{P}roj(\mathscr{S})$ with an associated projection $\widetilde{X} \to X$.

Examples

- Let $X = \mathbb{A}_k^n$, for k some field, $R = k[x_1, \ldots, x_n]$, and $\mathcal{I} = \widetilde{I}$, where $I = (x_1, \ldots, x_r)$ and $r \leq n$. In this case, letting $S = k[x_1, \ldots, x_n, x_1 t, \ldots, x_r t] = R \oplus It \oplus I^2t^2 \oplus \ldots,$

we see that $\widetilde{X} = \text{Proj}(S)$. $S^+ = (x_1, \dots, x_r)$, and we have a covering by patches $D_+(x;t) = \text{Spec}(S_{(x;t)} = \left[S\left[\frac{1}{x_i t}\right]\right]_{\deg=0}$ where

$$S_{(x;t)} = k\left[x_1, \dots, x_n, x_1 t, \dots, x_r t, \frac{1}{x_i t}\right]_0 = k\left[x_1, \dots, x_n, \frac{x_1}{x_i}, \dots, \frac{x_r}{x_i}\right]$$

Which is just $k\left[\frac{x_1}{x_i}, \dots, \frac{x_r}{x_i}, x_i, x_{i+1}, \dots, x_n\right]$. This is still a polynomial ring in n variables over k.

1.8 Differentials

1.8.1 Derivations

Suppose $A \to B$ is a ring map. A ***derivation*** (over A) is a morphism $d : B \to M$ such that d is additive, $d(A) = 0$, and d satisfies the Leibniz rule:

$$d(bb') = bd(b') + b'd(b)$$

Notice that from this we glean that

$$d(ab) = ad(b) + \underbrace{bd(a)}_{=0} = ad(b)$$

Thus d is naturally an A-module homomorphism. The set of all derivations is denoted $\text{Der}_A(B, M)$. This is a B-module, and in particular it is representable. Let $\Omega_{B/A}$ be this unique (universally canonically defined) object:

$$\text{Der}_A(B, M) = \text{Hom}_B(\Omega_{B/A}, M)$$

$\Omega_{B/A}$, equipped with a universal derivation $d : B \to \Omega_{B/A}$, has the following universal property: If $B \to M$ is a derivation, there exists a unique morphism such that the following diagram commutes:

$$\begin{tikzcd}
B \arrow[r, "d"] \arrow[dr] & \Omega_{B/A} \arrow[d, dashed, "\exists!"] \\
& M
\end{tikzcd}$$

So let's check that this is object exists. Take a free module on symbols db for $b \in B$, which is of the form $B^{\oplus B}$. Now quotient $B^{\oplus B}$ by all the relations that make a map $b \mapsto db$ a derivation. Thus,

$$\Omega_{B/A} := \frac{B^{\oplus B}}{\langle d(b_1 + b_2) - d(b_1) - d(b_2), d(b_1 b_2) - b_1 d(b_2) - b_2 d(b_1), da \mid b_1, b_2 \in B, a \in A\rangle}$$

As an example computation, if k is a field we have that

$$\Omega_{k[x_1,\dots,x_n]/k} = k[x]^{\oplus n} = \langle dx_1, \dots, dx_n\rangle$$

Here are some properties of the derivation:

- take the map $\Delta : B \otimes_A B \to B$ that sends $b \otimes b' \to bb'$. This corresponds to the diagonal map on schemes. Let $I = \ker(\Delta)$. It's clear that $1 \otimes b - b \otimes 1$ is a nonzero element of I when $b \neq 0$. Then the map $B \to I/I^2$ defined by sending $b \mapsto 1 \otimes b - b \otimes 1$. This is naturally a derivation, and one can see that $I/I^2 = \Omega_{B/A}$.
- Suppose we have another morphism $A \to A'$ and $B' = B \otimes_A A'$ is the base change of B to A'. It follows that
$$\Omega_{B'/A'} = \Omega_{B/A} \otimes_B B'$$
- if $S \subset B$ is a multiplicative system, $S^{-1}\Omega_{B/A} = \Omega_{S^{-1}B/A}$. This remarkable property allows us to sheafify everything.

1.8.2 Sheaves of Differentials

Suppose $\pi : Y \to X$ is a map of schemes. We can construct a ***sheaf of differentials*** $\Omega_{Y/X}$, which is a quasicoherent sheaf of $\mathcal{O}_Y$-Modules by gluing together at the patches. One way to construct this explicitly is via the diagonal again; let $\Delta_{Y/X} : Y \to Y \times_X Y$ be the diagonal map. Then,

$$\Omega_{Y/X} = \Delta^*_{Y/X} d_{\Delta(Y)} / d^2_{\Delta(Y)}$$

It's worth noting that this picture is complete when Y is separated, when Y is not, then $\Delta_{Y/X}$ is not closed, so instead we must map into a closed subset of $Y \times_X Y$ instead of all of $Y \times_X Y$. We also have a natural morphism $d : \mathcal{O}_Y \to \Delta^*_{Y/X} d_{\Delta(Y)} / d^2_{\Delta(Y)}$ via this construction.

Lemma 1.8.1. *Suppose $f : X \to Y$ and $g : Y \to Z$ are morphisms of schemes, and $h = g \circ f = X \to Z$. It follows that*
$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \twoheadrightarrow \Omega_{X/Y} \to 0$$
is a right exact sequence of $\mathcal{O}_X$-Modules.

At the level of rings, this says that if you have morphisms of rings $A \to A' \to B$, you have the right exact sequence

$$B \otimes_{A'} \Omega_{A'/A} \to \Omega_{B/A} \twoheadrightarrow \Omega_{B'/A} \to 0$$

These results tend to specialize nicely.

Lemma 1.8.2. *Suppose $f : X \to Y$ is a morphism of schemes and $Z \subset X$ is a closed subscheme with associated ideal sheaf $\mathcal{I}$. Then we have a right exact sequence*

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/Y} \otimes \mathcal{O}_Z \to \Omega_{Z/Y} \to 0$$

For example, let $Z = V(f) \hookrightarrow \mathbb{A}^n_k$, and $\mathbb{A}^n_k \to \operatorname{Spec}(k)$. It follows that we have the sequence
$$(f)/(f)^2 \to S/(f) \otimes \Omega_{k[x]/k} \to \Omega_{Z/k} \to 0$$

Chapter 2

Cohomology

We now move in to the second part of the course; cohomology. Most of the earlier sections of chapter 3 in Hartshorne are not great. This is because they tend to introduce terminology that is not used much again in the book, and in a broader context, largely outdated. Thus we'll do a quick recap of derived functors (not in the style of Hartshorne) and use that framework to understand cohomology and move on to section 3.2 from there.

2.1 Crash Course on Derived Functors

Let $\mathcal{A}$ denote an ***Abelian Category***. This is, broadly speaking, a category where we can do homological algebra, i.e. take direct sums/products, (co)kernels, and measure exactness.

Theorem 2.1.1. ***(Freyd, Mitchell)*** *Any Abelian Category embeds into the category $R - \mathrm{Mod}$ for some commutative ring R.*

If $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ is a functor of abelian categories, it is ***left/right exact*** if the image of a left/right exact sequence in $\mathcal{A}$ is a left/right exact sequence in $\mathcal{B}$. We say that an object $I \in \mathcal{A}$ is ***injective*** if the functor $\mathrm{Hom}_{\mathcal{A}}(-, I) : \mathcal{A} \to \mathrm{AbGp}$ is exact (this functor is always left exact, so equivalently we only need to check right exactness). Similarly, $P \in \mathcal{A}$ is ***projective*** if the functor $\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathrm{AbGp}$ is (right) exact.

An abelian category $\mathcal{A}$ has ***enough projectives*** if we are able to construct projective resolutions, i.e. for any object $M \in A$ we can construct a projective object P_M such that $P_M \to M$. Free objects are projective, so it is clear that any abelian category has enough projectives. Checking that any abelian category has enough injectives (defined analogously), however, is quite hard without additional machinery. If a category has enough injectives/projectives, then we can take injective/projective resolutions of any given object.

If $\mathcal{A}$ has enough injectives and $\mathscr{F} : \mathcal{A} \to \mathcal{B}$ is a left exact functor, we get ***right derived functors*** $\mathcal{R}^i\mathscr{F}$. To define $\mathcal{R}^i\mathscr{F}(M)$ for some $M \in \mathcal{A}$, first take an injective resolution $0 \to M \to I^\bullet$ of M, then let $\mathcal{R}^i\mathscr{F}(M) = \mathrm{H}^i(0 \to \mathscr{F}(M) \to \mathscr{F}(I^\bullet))$. It is clear that $\mathcal{R}^0\mathscr{F} \simeq \mathscr{F}$, and that $\mathcal{R}^i\mathscr{F}(I) = 0$ for $i > 0$ if I is injective (this is because I is its own injective resolution). Furthermore, derived functors are functorial over short exact sequences, in the sense that long exact sequences in cohomology encode over any short exact sequence.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[d, Rightarrow] & C \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{F}(A) \arrow[r] & \mathscr{F}(B) \arrow[r] & \mathscr{F}(C) \arrow[dll] & \\
& \mathcal{R}^1\mathscr{F}(A) \arrow[r] & \mathcal{R}^1\mathscr{F}(B) \arrow[r] & \mathcal{R}^1\mathscr{F}(C) \arrow[dll] & \\
& \mathcal{R}^2\mathscr{F}(A) \arrow[r] & \dots & &
\end{tikzcd}$$

This is functorial in the sense that, if we have a morphism of short exact sequences (φ, ψ, χ),

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\varphi"] & B \arrow[r] \arrow[d, "\psi"] & C \arrow[r] \arrow[d, "\chi"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

The larger diagram above yields a morphism of complexes

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}(A) \arrow[r] \arrow[d, "\mathscr{F}\varphi"] & \mathscr{F}(B) \arrow[r] \arrow[d, "\mathscr{F}\psi"] & \mathscr{F}(C) \arrow[r] \arrow[d, "\mathscr{F}\chi"] & \mathcal{R}^1\mathscr{F}(A) \arrow[r] \arrow[d, "\mathcal{R}^1\mathscr{F}\varphi"] & \mathcal{R}^1\mathscr{F}(B) \arrow[r] \arrow[d, "\mathcal{R}^1\mathscr{F}\psi"] & \dots \\
0 \arrow[r] & \mathscr{F}(A') \arrow[r] & \mathscr{F}(B') \arrow[r] & \mathscr{F}(C') \arrow[r] & \mathcal{R}^1\mathscr{F}(A') \arrow[r] & \mathcal{R}^1\mathscr{F}(B') \arrow[r] & \dots
\end{tikzcd}$$

One can also define a similar notion for projective modules; if $\mathcal{A}$ has enough projectives and $\mathscr{F}$ is right exact, we can analogously define the ***left derived functors*** $\mathcal{L}_i\mathscr{F}$ such that for any object M with associated projective resolution $P_\bullet \to M \to 0$, $\mathcal{L}_i\mathscr{F}(M) := \mathrm{H}_i(\mathscr{F}(P_\bullet) \to \mathscr{F}(M) \to 0)$. As before, $\mathcal{L}^0\mathscr{F} \simeq \mathscr{F}$ and $\mathcal{L}_i\mathscr{F}(P) = 0$ when $i > 0$ and P is projective. We also get an analogous encoding of long exact sequences in homology over short exact sequences.

It is by no means trivial that left or right derived functors are well defined; who's to say that we can't change the choice of injective or projective resolution and have it affect the (co)homology? well, one can show that homotopic complexes yield the same (co)homology, and that the image of any injective/projective resolution of a given object M are always going to be homotopic to the image of any other injective/projective resolution under a left/right exact functor $\mathscr{F}$.

2.2 Cohomology of Sheaves

Suppose X is a topological space and $\mathrm{Sh}(X)$ is the category of sheaves over X taking values in any abelian category $\mathcal{A}$ (in most cases, we can let $\mathcal{A}$ be AbGp). We define $\mathrm{H}^i(X,-) : \mathrm{Sh}(X) \to \mathrm{AbGp}$ to be $\mathcal{R}^i\Gamma(X,-)$. In other words, sheaf cohomology is precisely the derived functors of taking global sections. This definition belies some information that must be justified (or at least mentioned), however. For instance, we know that $\Gamma(X,-)$ is an additive left exact functor and that $\mathrm{Sh}(X)$ is an abelian category (to check this, just check properties on stalks), and one can check that $\mathrm{Sh}(X)$ has enough injectives. With issues with well-defined assuaged, we will now move on to discussing some properties of sheaf cohomology. Let X be a topological space and $\mathscr{F},\mathscr{G}$ both sheaves on X, with $\eta : \mathscr{F} \to \mathscr{G}$ being a natural transformation between them.

- $\mathrm{H}^0(X,\mathscr{F}) = \Gamma(X,\mathscr{F})$.

- $\mathrm{H}^i(X,-)$ is indeed a functor, i.e. $\eta : \mathscr{F} \to \mathscr{G}$ induces a map on cohomology. If $\mathscr{F}$ and an injective resolution $\mathscr{I}^\bullet$ and $\mathscr{G}$ has an injective resolution $\mathscr{J}^\bullet$, we see that η ascends to a map $\eta^\bullet : \mathscr{I}^\bullet \to \mathscr{J}^\bullet$, which then induces a map on cohomology groups via writing down a sufficiently nice homotopy. It's worth noting that this map being well defined is an immediate consequence of the abstract nonsense in the final chapter of Aluffi's Chapter 0, which you should be familiar with if you are reading this.

- $\mathrm{H}^i(X,\mathscr{I}) = 0$ for $i > 0$ provided that $\mathscr{I}$ is an injective sheaf. This is because $\mathcal{I}$ is an injective resolution of itself.

- $\mathrm{H}^\bullet(X,-)$ is a universal δ-functor, i.e. given a short exact sequence $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$, we get a long exact sequence in cohomology:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{G} \arrow[r] \arrow[d, Rightarrow] & \mathscr{H} \arrow[r] & 0 \\
0 \arrow[r] & \Gamma(X,\mathscr{F}) \arrow[r] & \Gamma(X,\mathscr{G}) \arrow[r] & \Gamma(X,\mathscr{H}) \arrow[dll] & \\
 & \mathrm{H}^1(X,\mathscr{F}) \arrow[r] & \mathrm{H}^1(X,\mathscr{G}) \arrow[r] & \mathrm{H}^1(X,\mathscr{H}) \arrow[dll] & \\
 & \mathrm{H}^2(X,\mathscr{F}) \arrow[r] & \ldots & &
\end{tikzcd}$$

  This is also functorial over short exact sequences, in the same manner as in the previous section.

- In the affine case, we can associate quasicoherent sheaves to modules. Very importantly, just because M may be an injective R-module, its associated quasicoherent sheaf $\widetilde{M}$ may not be injective. As we'll see soon, not all hope is lost, but one must be careful.

2.2.1 Flabby/Flasque Sheaves

While the method above works for computing sheaf cohomology in the abstract, it's important to note that computing an injective resolution of sheaves is nigh impossible. There are a couple of other ways to compute sheaf cohomology that are more reasonable. First, we introduce a new kind of sheaf that can take the place of injective sheaves. Over a scheme X, we say that a sheaf of $\mathcal{O}_X$-Modules $\mathscr{F}$ is ***flasque***, or sometimes referred to as ***flabby***, if For every $U \subset V \subset X$, the corresponding map $\mathscr{F}(V) \to \mathscr{F}(U)$ is a surjection.

Lemma 2.2.1. *Injective sheaves of $\mathcal{O}_X$-modules are flasque.*

We'll prove the above result in the following section. In addition, we can now resolve the problem of taking $\widetilde{(-)}$ in the affine setting.

Lemma 2.2.2. *Suppose that M is an injective R-module. Then, $\widetilde{M}$ is a flasque sheaf over* $\operatorname{Spec}(R)$.

This implies that we can nicely move between the ring setting and the sheaf setting. In particular, if we have an injective resolution $0 \to M \to I^\bullet$ of an R-module M, it follows that this passes to a flasque resolution of $\mathcal{O}_X$-Modules $0 \to \widetilde{M} \to \widetilde{I}^\bullet$. From this, we can see the importance of the following theorem; that flasque sheaves preserve the cohomological stability of injective sheaves.

Lemma 2.2.3. *Suppose X is a scheme and $\mathscr{F}$ is a flasque sheaf of $\mathcal{O}_X$-modules. Then,* $\mathrm{H}^i(X, \mathscr{F}) = 0$ *for $i > 0$. In particular, if $X =$* $\operatorname{Spec}(R)$ *is affine and $\mathscr{F} = \widetilde{M}$ is quasicoherent, then* $\mathrm{H}^0(X, \mathscr{F}) = M$.

Thus, we can compute sheaf cohomology by taking a flasque resolution. Using this construction, the following theorem is easily justifiable.

Theorem 2.2.4. *Suppose that X is a Noetherian topological space of dimension d. Then for any sheaf of abelian groups $\mathscr{A}$,* $\mathrm{H}^i(X, \mathscr{A}) = 0$ *for $i > d$.*

2.2.2 Serre's Affine Criterion

One of the first major utilizations of sheaf cohomology is a criterion for proving a scheme is affine, due to Serre.

Theorem 2.2.5. ***(Serre's Affine Criterion)*** *Suppose X is a scheme. Then, the following are equivalent:*

(1) X is affine.

(2) $\forall \mathscr{F} \in \operatorname{QCoh}(X)$, $\mathrm{H}^i(X, \mathscr{F}) = 0$ *for $i > 0$.*

(3) $\forall \mathscr{I} \in \operatorname{QCoh}(X)$ *which are ideal sheaves, i.e.* $\mathcal{I} \subseteq \mathcal{O}_X$, *we have that* $\mathrm{H}^1(X, \mathcal{I}) = 0$.

Proof. We need to check that (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1). The former is easy to check; suppose that $X = \mathrm{Spec}(R)$ is affine. It follows that any $\mathscr{F} \in \mathrm{QCoh}(X)$ is of the form $\mathscr{F} = \widetilde{M}$, where M is an R-module. We've seen that injective resolutions pass to flasque resolutions in the affine setting, so we can conclude by the theorem in the previous section that

$$\mathrm{H}^i(X, \widetilde{M}) = \begin{cases} M & i = 0 \\ 0 & i > 0 \end{cases}$$

So the result follows. We do need to justify the fact that, in the affine setting, I being injective implies that $\widetilde{I}$ is flasque, which we'll now do below.

Lemma 2.2.6. *Suppose I is an injective R-module, and $\mathfrak{a} \subseteq R$ is an ideal. This implies that $\Gamma_{\mathfrak{a}} I = \{x \in I \mid \exists N \gg 0 \text{ s.t. } \mathfrak{a}^N x = 0\}$ is also an injective R-module.*

We can justify this relatively easily in the case where R is Noetherian, so we'll prove it in that setting (though the result also holds for non-Noetherian rings). Suppose that $\mathfrak{a} = (f_1, \ldots, f_s)$ by Noetherianity; it follows that if we let $U = \bigcup D(f_i)$ and $\mathscr{F} = \widetilde{M}$, by left exactness of Γ we see that

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_{\mathfrak{a}} M \arrow[r] & M \arrow[r] & \bigoplus M\left[\frac{1}{f_i}\right] \\
0 \arrow[r] & \ker\varphi \arrow[r] \arrow[u, hook, two heads] & \Gamma(X, \mathscr{F}) \arrow[r, "\varphi"] \arrow[u, hook, two heads] & \Gamma(U, \mathscr{F}) \arrow[u, hook]
\end{tikzcd}$$

Observe that $\Gamma_{\mathfrak{a}}(-)$ is itself left exact (and clearly additive) on the category of R-modules, so it too has derived functors. We define $\mathrm{H}^i_{\mathfrak{a}}(M) := \mathcal{R}^i\Gamma_{\mathfrak{a}}(M)$ to be the ***local cohomology*** of M supported in $\mathfrak{a}$. Furthermore, as I being an injective R-module implies that $\widetilde{I}$ is a flasque $\mathrm{Spec}(R)$ valued sheaf, it follows that the morphism $I \twoheadrightarrow I\left[\frac{1}{f}\right]$ is surjective for any $f \in R$. To make life a bit easier, let's just check this over a domain. Choose $0 \neq f \in R$. Since I is injective, $\mathrm{Hom}_R(-, I)$ is exact. Thus,

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\cdot f"] & R \arrow[r] \arrow[d, Rightarrow, "\mathrm{Hom}_R(-,I)"] & R/f \arrow[r] & 0 \\
0 \arrow[r] & \substack{\mathrm{Hom}_R(R/f,I) \\ =\{x \in I \,\mid\, f^n x = 0\}} \arrow[r] & I \arrow[r, "\cdot f^n"] & I \arrow[r] & 0
\end{tikzcd}$$

Taking a direct limit of short exact sequences $\varinjlim \mathscr{S}_n$ where

$$\mathscr{S}_n = \qquad 0 \longrightarrow \{x \in I \mid f^n x = 0\} \longrightarrow I \xrightarrow{\cdot f^n} I \longrightarrow 0$$

We see that

$$\varinjlim \mathscr{S}_n = \qquad 0 \longrightarrow \Gamma_{(f)} I \longrightarrow I \longrightarrow I\left[\frac{1}{f}\right] \longrightarrow 0$$

So the surjectivity follows. This can then be used to show that $\widetilde{I}$ is flasque when I is injective via Noetherian induction on the support of $\widetilde{I}$. We'll leave this last part as an exercise.

We now move on to showing that (3) $\Rightarrow$ (1). We need to check that X is affine, via the affine criterion. That is, we'd like to find sections $f_1, \ldots, f_s$ that generate $\Gamma(X, \mathcal{O}_X)$, and where each X_{f_i} are all affine.

Choose $U \subset X$ to be an open affine patch. Let $Y = X \backslash U$. and $x \in U$ a closed point. Observe that $\mathcal{I}_{Y \cup \{x\}} \hookrightarrow \mathcal{O}_Y$, where the cokernel is precisely the skyscraper sheaf $\underline{k(x)}$. This induces a short exact sequence, and when we take a long exact sequence in cohomology, we see that $\mathrm{H}^1(X, \mathcal{I}_{Y \cup \{x\}}) = 0$. Thus, $\mathrm{H}^0(X, \mathcal{O}_Y) \twoheadrightarrow k(x)$. Thus $\exists f_x \in \mathrm{H}^0(X, \mathcal{O}_Y)$ such that $f_x \mapsto 1$. It follows that $Y \subseteq V(f_x)$, so $D(f) \subseteq U$. From here, we can conclude that X_{f_x} is affine, and we can repeat this across affine patches to find a set of f_i that generates $\Gamma(X, \mathcal{O}_X)$. □

2.3 Cech Cohomology

Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$ be an open cover, where $\mathcal{I}$ is a well ordered indexing set. Suppose $\mathscr{F}$ is a sheaf of abelian groups on X. Naively, we can understand section on X by looking at $\prod_{i \in \mathcal{I}} \Gamma(U_i, \mathscr{F})$, but we need to make sure that sections agree on the intersections. Thus, we should consider the map into the sections of intersections as follows:

$$\prod_{i \in \mathcal{I}} \Gamma(U_i, \mathscr{F}) \to \prod_{j<k} \Gamma(U_j \cap U_k, \mathscr{F})$$

Where one sends the section $s_i \mapsto (t_{jk})$ where $t_{jk} = s_i|_{/} U_j \cap U_k$ if $j = i$, and $t_{jk} = -s_i|_{/} U_j \cap U_k$ if $k = i$, with the other sections being zero sections. In a sense, the sign records what kind of overlap we are seeing. We can then additively extend this to maps on tuples of sections $(s_i) \in \prod_{i \in \mathcal{I}} \Gamma(U_i, \mathscr{F})$.

That is not all, however. What about triple intersections $U_i \cap U_j \cap U_k$? Or quadruple intersection? what about n-tuple intersections? We can cover our bases here by extending the sequence onwards

$$\prod_{i \in \mathcal{I}} \Gamma(U_i, \mathscr{F}) \to \prod_{i<j} \Gamma(U_i \cap U_j, \mathscr{F}) \to \prod_{i_1<i_2<i_3} \Gamma\left(\bigcap_{t=1}^{3} U_{i_t}, \mathscr{F}\right) \to \cdots \to \prod_{i_1<\cdots<i_\ell} \Gamma\left(\bigcap_{t=1}^{\ell} U_{i_t}, \mathscr{F}\right) \to \ldots$$

Define $C^\ell(\mathcal{U}, \mathscr{F}) := \prod_{i_1<\cdots<i_\ell} \Gamma(U_i \cap \cdots \cap U_{i_\ell}, \mathscr{F})$, and let $\widehat{\mathrm{H}}^i(\mathcal{U}, \mathscr{F}) = \mathrm{H}^i(C^\bullet(\mathcal{U}, \mathscr{F}))$. This is the ***Cech Cohomology*** of X with respect to an open cover $\mathcal{U}$. It follows from this construction that $\widehat{\mathrm{H}}^0(\mathcal{U}, \mathscr{F}) = \Gamma(X, \mathscr{F})$, and we can extend this even further. First, though, we need to make sure that we are independent of choice of cover. For a general topological space, Cech cohomology is only independent of choice of open cover when

you avoid taking coarse open covers. This is encapsulated by the direct limit:

$$\widehat{\mathrm{H}}^i(X,\mathscr{F}) := \varinjlim_{\mathcal{U}} \widehat{\mathrm{H}}^i(\mathcal{U},\mathscr{F})$$

Using this notation, we can now extend and use Cech complexes to compute sheaf cohomology:

Theorem 2.3.1. *Suppose X is a Noetherian separated scheme, $\mathscr{F} \in \mathrm{QCoh}(X)$, and $\mathcal{U} = \{U_i\}$ is an open affine cover. Then $\mathrm{H}^i(X,\mathscr{F}) = \widehat{\mathrm{H}}^i(\mathcal{U},\mathscr{F})$, i.e. Cech Cohomology agrees with Sheaf Cohomology.*

Philosophically, this works because, on each affine patch, $\mathscr{F}$ has trivial cohomology (via Serre's Affine criterion). Furthermore, because X is separated, the intersection of two affine patches are affine, so the higher cohomology of $\mathscr{F}$ on $U_1 \cap \cdots \cap U_\ell$ also vanishes.

We can also consider Cech Cohomology in a more sheafy setting. Suppose again that $\mathcal{U}$ is an open cover. Notice that for any open set U, $U \cap \mathcal{U} = \{U \cap U_i\}_{i\in\mathcal{I}}$ is an open cover of U. Furthermore, if $V \subseteq U$, we have a natural restriction map

$$C^\bullet(U \cap \mathcal{U},\mathscr{F}) \to C^\bullet(V \cap \mathcal{U},\mathscr{F})$$

Thus we can compute Cech Cohomology on sheaves by restricting the sheaf to each open set, then to each intersection, then to each triple intersection, and so on:

$$\prod_{i\in\mathcal{I}} \mathscr{F}|_{U_i} \to \prod_{i<j} \mathscr{F}|_{U_i\cap U_j} \to \cdots \to \prod_{i_1<\cdots<i_\ell} \mathscr{F}|_{U_1\cap\cdots\cap U_\ell} \to \cdots$$

Using much the same logic that we used when constructing Cech Cohomology in the topological setting. Be weary that this is a bit of an abuse of notation, as $\mathscr{F}|_{U_1\cap\cdots\cap U_\ell}$ is actually a sheaf on $U_1 \cap \cdots \cap U_\ell$, not X. Thus we are secretly pushing through $i : U_1 \cap \cdots \cap U_\ell \to X$ and considering the sheaf $i_*\mathscr{F}|_{U_1\cap\cdots\cap U_\ell}$, or equivalently, the sheaf $i_*i^{-1}\mathscr{F}$, so we are still working over sheaves on X at each term in the complex. Defining $\mathcal{C}^\ell(\mathcal{U},\mathscr{F}) := \prod_{i_1<\cdots<i_\ell} \mathscr{F}_{U_1\cap\cdots\cap U_\ell}$ we can see that $\mathcal{C}^\bullet(\mathcal{U},\mathscr{F})$, with the corresponding differential maps, is a resolution of $\mathscr{F}$. Interestingly enough, for an injective resolution $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ of $\mathscr{F}$, we can get a chain map between the two resolutions $\mathcal{I}^\bullet \to \mathcal{C}^\bullet(\mathcal{U},\mathscr{F})$ that is compatible with taking refinements of $\mathcal{U}$, and in particular, indeuces a homotopy. Thus, these complexes have the same homology. Therefore, we can compute sheaf cohomology using the complex $\mathcal{C}^\bullet(\mathcal{U},\mathscr{F})$, which is much easier to write down and much more concrete.

Theorem 2.3.2. *Let X be a Noetherian separated scheme, $\mathscr{F}$ a quasi-coherent sheaf, and $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ be an affine open cover. Then*

$$\widehat{\mathrm{H}}^i(\mathcal{U},\mathscr{F}) \cong \mathrm{H}^i(X,\mathscr{F})$$

Before proving this, we check a quick lemma:

Lemma 2.3.3. *With the same hypotheses as above, suppose $\mathscr{F}$ is a flasque sheaf. For any open cover $\mathcal{U}$, $\widehat{\mathrm{H}}^i(\mathcal{U},\mathscr{F}) = 0$ for all $i > 0$.*

Proof. If $\mathscr{F}$ is flasque, then $\mathcal{C}^\bullet(\mathcal{U},\mathscr{F})$ is a flasque resolution of $\mathscr{F}$, as for any $i : U_{i_1} \cap \cdots \cap U_{i_\ell} \hookrightarrow X$, $i_* i^{-1}\mathscr{F}$ is still flasque on X, as the restriction maps on sections will clearly stay surjective. It follows then that higher cohomologies vanish, as they do on flasque resolutions. □

Because this resolution is flasque we can see that

$$\mathrm{H}^i(X,\mathscr{F}) = \mathrm{H}^i\left(\Gamma(\mathcal{C}^\bullet(\mathcal{U},\mathscr{F})\right) = \mathrm{H}^i(C^\bullet(\mathcal{U},\mathscr{F})) = \widehat{\mathrm{H}}^i(\mathcal{U},\mathscr{F})$$

We now want to check that $\mathrm{QCoh}(X)$ has "enough flasques" in some sense.

Lemma 2.3.4. *Suppose X is a Noetherian scheme, and $\mathscr{F} \in \mathrm{QCoh}(X)$. Then $\exists$ a flasque sheaf $\mathscr{G}$ such that $\mathscr{F} \hookrightarrow \mathscr{G}$.*

Proof. Let $\mathcal{U} = \{U_i\}$ be a (finite) affine cover of X (such a thing exists because X is Noetherian, and thus quasi-compact). Clearly, $\mathscr{F} \hookrightarrow \prod_i \mathscr{F}|_{U_i}$, where $\mathscr{F}_{U_i}$ is a sheaf on the affine U_i. As $\mathscr{F}$ is quasicoherent, it follows that $\mathscr{F}_{U_i} = \widetilde{M_i}$, where $M_i = \Gamma(U_i,\mathscr{F})$. The category of modules over a ring is known to have enough injectives, so $\exists$ an injective module I^i such that $M_i \hookrightarrow I^i$. We've proven that the quasicoherent sheaf associated to an injective module is flasque. Thus, $\mathscr{F} \hookrightarrow \prod_i \mathscr{F}|_{U_i} \hookrightarrow \prod_i \widetilde{I^i}$, where $\prod_i \widetilde{I^i}$ is flasque. □

We now have enough to sketch the proof of the theorem above.

Proof. As it is clear that $\mathrm{H}^0 = \widehat{\mathrm{H}}^0 = \Gamma$, we proceed by induction. From the previous lemma, there exists $\mathscr{G}$ flasque such that $\mathscr{F} \hookrightarrow \mathscr{G}$, implying there exists a short exact sequence of sheaves

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$

$\mathscr{H}$ is locally a cokernel of a map of modules, so we know that $\mathscr{H} \in \mathrm{QCoh}(X)$. Furthermore, as each term of $\mathcal{C}^\bullet(\mathcal{U},\mathscr{F})$ is the product of quasicoherent sheaves on affine schemes, which are in turn associated to modules, we get an associated short exact sequence of complexes

$$0 \to \mathcal{C}^\bullet(\mathcal{U},\mathscr{F}) \to \mathcal{C}^\bullet(\mathcal{U},\mathscr{G}) \to \mathcal{C}^\bullet(\mathcal{U},\mathscr{H}) \to 0$$

Taking a long exact sequences in cohomology, both in sheaf cohomology and cech cohomology (the latter of which is induced via the short exact sequence above) we see that

$$\begin{tikzcd}
\widehat{\mathrm{H}}^i(\mathcal{U},\mathscr{G}) \arrow[r] \arrow[d] & \widehat{\mathrm{H}}^i(\mathcal{U},\mathscr{H}) \arrow[r] \arrow[d] & \widehat{\mathrm{H}}^{i+1}(\mathcal{U},\mathscr{F}) \arrow[r] \arrow[d] & \widehat{\mathrm{H}}^{i+1}(\mathcal{U},\mathscr{G}) \arrow[d] \\
\mathrm{H}^i(X,\mathscr{G}) \arrow[r] & \mathrm{H}^i(X,\mathscr{H}) \arrow[r] & \mathrm{H}^{i+1}(X,\mathscr{F}) \arrow[r] & \mathrm{H}^{i+1}(X,\mathscr{G})
\end{tikzcd}$$

$\mathscr{G}$ is flasque, so cohomology vanishes when $i > 0$. It follows that

$$\begin{tikzcd}
\underbrace{\widehat{\mathrm{H}}^i(\mathcal{U},\mathscr{G})}_{=0} \arrow[r] \arrow[d] & \widehat{\mathrm{H}}^i(\mathcal{U},\mathscr{H}) \arrow[r] \arrow[d] & \widehat{\mathrm{H}}^{i+1}(\mathcal{U},\mathscr{F}) \arrow[r] \arrow[d] & \underbrace{\widehat{\mathrm{H}}^{i+1}(\mathcal{U},\mathscr{G})}_{=0} \arrow[d] \\
\underbrace{\mathrm{H}^i(X,\mathscr{G})}_{=0} \arrow[r] & \mathrm{H}^i(X,\mathscr{H}) \arrow[r] & \mathrm{H}^{i+1}(X,\mathscr{F}) \arrow[r] & \underbrace{\mathrm{H}^{i+1}(X,\mathscr{G})}_{=0}
\end{tikzcd}$$

Thus the maps in the middle are isomorphisms by exactness:

$$\begin{tikzcd}
\underbrace{\widehat{\mathrm{H}}^i(\mathcal{U},\mathscr{G})}_{=0} \arrow[r] \arrow[d] & \widehat{\mathrm{H}}^i(\mathcal{U},\mathscr{H}) \arrow[r, "\cong"] \arrow[d] & \widehat{\mathrm{H}}^{i+1}(\mathcal{U},\mathscr{F}) \arrow[r] \arrow[d] & \underbrace{\widehat{\mathrm{H}}^{i+1}(\mathcal{U},\mathscr{G})}_{=0} \arrow[d] \\
\underbrace{\mathrm{H}^i(X,\mathscr{G})}_{=0} \arrow[r] & \mathrm{H}^i(X,\mathscr{H}) \arrow[r, "\cong"] & \mathrm{H}^{i+1}(X,\mathscr{F}) \arrow[r] & \underbrace{\mathrm{H}^{i+1}(X,\mathscr{G})}_{=0}
\end{tikzcd}$$

And via the inductive hypothesis, we have another isomorphism:

$$\begin{tikzcd}
\underbrace{\widehat{\mathrm{H}}^i(\mathcal{U},\mathscr{G})}_{=0} \arrow[r] \arrow[d] & \widehat{\mathrm{H}}^i(\mathcal{U},\mathscr{H}) \arrow[r, "\cong"] \arrow[d, "\cong"] & \widehat{\mathrm{H}}^{i+1}(\mathcal{U},\mathscr{F}) \arrow[r] \arrow[d] & \underbrace{\widehat{\mathrm{H}}^{i+1}(\mathcal{U},\mathscr{G})}_{=0} \arrow[d] \\
\underbrace{\mathrm{H}^i(X,\mathscr{G})}_{=0} \arrow[r] & \mathrm{H}^i(X,\mathscr{H}) \arrow[r, "\cong"] & \mathrm{H}^{i+1}(X,\mathscr{F}) \arrow[r] & \underbrace{\mathrm{H}^{i+1}(X,\mathscr{G})}_{=0}
\end{tikzcd}$$

So it follows that $\widehat{\mathrm{H}}^{i+1}(\mathcal{U},\mathscr{F}) \cong \mathrm{H}^{i+1}(X,\mathscr{F})$, and the inductive case is (almost) completed. We do need to be careful, since for $i = 0$, $\mathrm{H}^0(X,\mathscr{G})$ does not vanish, so we need to explicitly check that $\widehat{\mathrm{H}}^1(\mathcal{U},\mathscr{F}) \cong \mathrm{H}^1(X,\mathscr{F})$. This is easy to check explicitly:

$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathrm{H}}^0(\mathcal{U},\mathscr{F}) \arrow[r] \arrow[d] & \widehat{\mathrm{H}}^0(\mathcal{U},\mathscr{G}) \arrow[r] \arrow[d] & \widehat{\mathrm{H}}^0(\mathcal{U},\mathscr{H}) \arrow[r] \arrow[d] & \widehat{\mathrm{H}}^1(\mathcal{U},\mathscr{F}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{H}^{i+1}(X,\mathscr{G}) \arrow[r] & \mathrm{H}^0(X,\mathscr{G}) \arrow[r] & \mathrm{H}^0(X,\mathscr{H}) \arrow[r] & \mathrm{H}^1(X,\mathscr{F}) \arrow[r] & 0
\end{tikzcd}$$

With the final zero following from the fact that $\mathscr{G}$ is flasque, so first cohomology vanishes. We're just taking sections in the first three terms, so

$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathrm{H}}^0(\mathcal{U},\mathscr{F}) \arrow[r] \arrow[d, "\cong"] & \widehat{\mathrm{H}}^0(\mathcal{U},\mathscr{G}) \arrow[r] \arrow[d, "\cong"] & \widehat{\mathrm{H}}^0(\mathcal{U},\mathscr{H}) \arrow[r] \arrow[d, "\cong"] & \widehat{\mathrm{H}}^1(\mathcal{U},\mathscr{F}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{H}^{i+1}(X,\mathscr{G}) \arrow[r] & \mathrm{H}^0(X,\mathscr{G}) \arrow[r] & \mathrm{H}^0(X,\mathscr{H}) \arrow[r] & \mathrm{H}^1(X,\mathscr{F}) \arrow[r] & 0
\end{tikzcd}$$

So the last isomorphism follows. □

2.4 Cohomology of Line Bundles on Projective Space

The first (and easiest) practical application for sheaf cohomology is computing sheaf cohomology on $\mathbb{P}^n$, specifically over line bundles. This is a computation you should try to put to memory, since it ends up being very useful later in life to compute cohomology on many projective schemes.

As an example, lets compute the cohomology of $\mathbb{P}^1_{\mathbb{C}}$ first. $\mathbb{P}^1_{\mathbb{C}}$ can be covered by two affine open sets, namely $D_+(x_0) = \operatorname{Spec}\left(\mathbb{C}\left[x_0, x_1, \frac{1}{x_0}\right]_0\right) = \operatorname{Spec}\left(k\left[\frac{x_1}{x_0}\right]\right)$ and $D_+(x_1) = \operatorname{Spec}\left(k\left[\frac{x_0}{x_1}\right]\right)$. Viewing $\frac{x_1}{x_0} = t$, we see that we can cover $\mathbb{P}^1_{\mathbb{C}}$ by patches that are the spectrum of $k[t]$ and $k[t^{-1}]$ respectively. Call these U_0 and U_1 respectively. In general,

$$\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\ell) = \mathbb{C}\left[\widetilde{x_0, x_1, \frac{1}{x_0}}\right]_\ell = \mathbb{C}[t] \cdot x_1^\ell$$

We now are ready to compute $\mathrm{H}^i(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(\ell))$. For $\ell = 0$, we have the Cech complex

$$0 \to \Gamma(U_0, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}) \oplus \Gamma(U_1, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}) \to \Gamma(U_0 \cap U_1, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}) \to 0$$

Computing sections we see get

$$0 \to \mathbb{C}[t] \oplus \mathbb{C}[t^{-1}] \to \mathbb{C}[t, t^{-1}] \to 0$$

Where the middle map is the alternating differential, where $(f, g) \mapsto f - g$. It follows that the kernel of this middle map is precisely when $f = g$, and the image is the entire target (because any polynomial in t, t^{-1} can be split into terms of positive and terms of negative degree). It follows that $\mathrm{H}^0(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}) = \mathbb{C}$ and $\mathrm{H}^1(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}) = 0$. What about for a general $\mathcal{O}_{\mathbb{P}^n_A}(\ell)$?

Theorem 2.4.1. *Suppose A is Noetherian, and $S = A[x_0, \dots, x_n]$. Then*

$$\mathrm{H}^i\left(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(\ell)\right) = \begin{cases} 0 & i \neq 0, n \\ S_{\deg \ell} & i = 0 \\ 0 & i = n, \ell > -n-1 \\ A^{\binom{n+1-\ell+n}{n}} & i = n, \ell \leq -n-1 \end{cases}$$

In particular, $\mathrm{H}^i\left(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-n-1)\right) = A$. All of the top cohomologies arise from Serre Duality, which we will prove later (in this case, $\omega_{\mathbb{P}^n_A} = \mathcal{O}_{\mathbb{P}^n_A}(-n-1)$). In light of this, to compute $\mathrm{H}^n\left(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(\ell)\right)$, $\exists$ a perfect pairing of finitely generated A modules

$$\mathrm{H}^0\left(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(\ell)\right) \otimes \mathrm{H}^n\left(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-n-1-\ell)\right) \to \mathrm{H}^n\left(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-n-1)\right) \cong A$$

Proof. We've already computed H^0, as this is the same as just taking sections. We will prove the H^n case with Serre Duality. We can also see this explicitly in some sense. To compute the top cohomology, we just look at the end of the sequence

$$\bigoplus_{i=0}^{n} S\left[\frac{1}{x_0\cdots\widehat{x_i}\cdots x_n}\right]\to S\left[\frac{1}{x_0\cdots x_n}\right]\to 0$$

Where H^n, varying over degree, is just the cokernel of this map. The cokernel comprises of polynomials where each monomial can be written in all but one of the coefficients. This correspondence can be explicitly expressed:

$$\begin{aligned}\bigoplus_{\ell\in\mathbb{Z}}\mathrm{H}^n\left(\mathbb{P}^n_A,\mathcal{O}_{\mathbb{P}^n_A}(\ell)\right)&\cong\operatorname{coker}\left(\bigoplus_{i=0}^{n} S\left[\frac{1}{x_0\cdots\widehat{x_i}\cdots x_n}\right]\to S\left[\frac{1}{x_0\cdots x_n}\right]\right)\\&\cong\frac{S\left[\frac{1}{x_0\cdots x_n}\right]}{\sum_{i=0}^{n} S\left[\frac{1}{x_0\cdots\widehat{x_i}\cdots x_n}\right]}\\&\cong\frac{1}{x_0\cdots x_n}A[x_0^{-1},\ldots,x_n^{-1}]\end{aligned}$$

Which is just $\mathrm{H}^n\left(\mathcal{O}_{\mathbb{P}^n_A}(-n-1-\ell)\right)$, i.e. a free A module on basis $\frac{f}{x_0\cdots x_n}$ where f is a monomial in $x_0^{-1},\ldots,x_n^{-1}$ of degree ℓ. Thus we can explicitly see the perfect pairing duality. An element of $\mathrm{H}^0(\mathbb{P}^1_{\mathbb{C}},\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}})$ is represented by $x_0^{a_0}\cdots x_n^{a_n}$ where $a_i\geq 0$ and $\sum a_i=\ell$, and an element of $\mathrm{H}^n\left(\mathcal{O}_{\mathbb{P}^n_A}(-n-1-\ell)\right)$ is represented by $\frac{1}{x_0\cdots x_n}x_0^{-b_0}\cdots x_n^{-b_n}$, where again $b_i\geq 0$ and $\sum b_i=\ell$. Thus the pairing sends

$$\left(x_0^{a_0}\cdots x_n^{a_n},\frac{1}{x_0\cdots x_n}x_0^{-b_0}\cdots x_n^{-b_n}\right)\mapsto x_0^{a_0-b_0}\cdots x_n^{a_n-b_n}\cdot\frac{1}{x_0\cdots x_n}$$

All that is left to do is verify the middle terms vanish. Induct on n and use the inclusion $\mathbb{P}^{n-1}_A$ *into*$\mathbb{P}^n_A$ where the image is precisely $V(x_n)=\mathbb{P}^n_A\backslash D_+(x_n)$. This yields a short exact sequence

$$0\longrightarrow\mathcal{O}_{\mathbb{P}^n_A}(-1)\xrightarrow{\cdot x_n}\mathcal{O}_{\mathbb{P}^n_A}\longrightarrow\mathcal{O}_{\mathbb{P}^{n-1}_A}\longrightarrow 0$$

Varying across degree, the section version of the short exact sequence is

$$0\longrightarrow A[x_0,\ldots,x_{n-1}]_{x_n}\xrightarrow{\cdot x_n}A[x_0,\ldots,x_n]\longrightarrow A[x_0,\ldots,x_{n-1}]\longrightarrow 0$$

As twisting is exact, we can view the first short exact sequence over all twists as

$$0\longrightarrow\bigoplus_{\ell\in\mathbb{Z}}\mathcal{O}_{\mathbb{P}^n_A}(\ell-1)\xrightarrow{\cdot x_n}\bigoplus_{\ell\in\mathbb{Z}}\mathcal{O}_{\mathbb{P}^n_A}(\ell)\longrightarrow\bigoplus_{\ell\in\mathbb{Z}}\mathcal{O}_{\mathbb{P}^{n-1}_A}(\ell)\longrightarrow 0$$

This map is already exact on H^0 (as it is just taking sections) so we start computing the long exact sequence in cohomology at H^1:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{\ell\in\mathbb{Z}} \mathrm{H}^1(\mathcal{O}_{\mathbb{P}^n_A}(\ell-1)) \arrow[r, "\cdot x_n"] & \bigoplus_{\ell\in\mathbb{Z}} \mathrm{H}^1(\mathcal{O}_{\mathbb{P}^n_A}(\ell)) \arrow[r] & \underbrace{\bigoplus_{\ell\in\mathbb{Z}} \mathrm{H}^1(\mathcal{O}_{\mathbb{P}^{n-1}_A}(\ell))}_{=0} \arrow[dll] \\
& \bigoplus_{\ell\in\mathbb{Z}} \mathrm{H}^2(\mathcal{O}_{\mathbb{P}^n_A}(\ell-1)) \arrow[r, "\cdot x_n"] & \bigoplus_{\ell\in\mathbb{Z}} \mathrm{H}^2(\mathcal{O}_{\mathbb{P}^n_A}(\ell)) \arrow[r] & \cdots \arrow[dll] \\
& \bigoplus_{\ell\in\mathbb{Z}} \mathrm{H}^n(\mathcal{O}_{\mathbb{P}^n_A}(\ell-1)) \arrow[r] & \bigoplus_{\ell\in\mathbb{Z}} \mathrm{H}^n(\mathcal{O}_{\mathbb{P}^n_A}(\ell)) \arrow[r] & \underbrace{\bigoplus_{\ell\in\mathbb{Z}} \mathrm{H}^n(\mathcal{O}_{\mathbb{P}^{n-1}_A}(\ell))}_{=0}
\end{tikzcd}$$

As the terms of the right column vanish via the inductive hypothesis, we see that multiplication map is an isomorphism, i.e.

$$\bigoplus_{\ell\in\mathbb{Z}} \mathrm{H}^i(\mathcal{O}_{\mathbb{P}^n_A}(\ell-1)) = x_n \bigoplus_{\ell\in\mathbb{Z}} \mathrm{H}^i(\mathcal{O}_{\mathbb{P}^n_A}(\ell))$$

For $i \neq 0, n$. Now notice that, when summing over ℓ, the -1 shift is not recognized. Thus we can rewrite this as

$$\bigoplus_{\ell\in\mathbb{Z}} \mathrm{H}^i(\mathcal{O}_{\mathbb{P}^n_A}(\ell)) = x_n \bigoplus_{\ell\in\mathbb{Z}} \mathrm{H}^i(\mathcal{O}_{\mathbb{P}^n_A}(\ell))$$

From here, we just conclude via Nakayama's lemma that $\bigoplus_{\ell\in\mathbb{Z}} \mathrm{H}^i(\mathcal{O}_{\mathbb{P}^n_A}(\ell)) = 0$. $\square$

This construction yields the following corollary:

Lemma 2.4.2. *Suppose A is Noetherian and X is a projective scheme over A. Let $\mathcal{L} = \pi_* \mathcal{O}_X(1)$ be a very ample line bundle on X, and $\mathscr{F}$ a coherent sheaf on X, and $\mathscr{F}(\ell) = \mathscr{F} \otimes \mathcal{L}^{\otimes n}$ (because $\mathcal{L}$ is very ample, this agrees with the standard definition of the twist of $\mathscr{F}$).*

(1) $\mathrm{H}^i(X, \mathscr{F})$ is a finitely generated A-module.

(2) $\exists \ell_0 \gg 0$, dependent on the sheaf $\mathscr{F}$, such that $\mathrm{H}^i(X, \mathscr{F}(\ell)) = 0 \; \forall \ell \geq \ell_0, i > 0$.

This is generalizing results that we know over H^0 to higher cohomology.

Proof. $\exists$ a minimal resolution of $\mathscr{F}$ by direct sums of twists of $\mathcal{O}_X$:

$$0 \to \bigoplus_{j\in\mathbb{Z}} \mathcal{O}_X(-j)^{\oplus b_{n,j}} \to \cdots \to \bigoplus_{j\in\mathbb{Z}} \mathcal{O}_X(-j)^{\oplus b_{1,j}} \to \bigoplus_{j\in\mathbb{Z}} \mathcal{O}_X(-j)^{\oplus b_{0,j}} \to \mathscr{F} \to 0$$

Where each $b_{i,j}$ is finite. Why is this? Well, $\mathscr{F} = \widetilde{M}$ locally, where M is a finitely generated module. Hilbert's Syzygy Theorem tells us that we can result M by a finite free resolution. Lift this back up into the space of sheaves and you're good to go for the first part. For the second, notice that for $j \gg 0$, $b_{i,j} = 0$ for any i. It follows that we can twist this resolution by a sufficiently high ℓ such that every copy of $\mathcal{O}_X$ has positive degree. Part 2 then follows from this. $\square$

In fact, because a sufficiently high twist of an ample line bundle is very ample, the result above actually holds for any ample line bundle $\mathcal{L}$.

2.5 Ext Sheaves

To construct a formal, general statement for Serre Duality, we must first introduce different notions of Ext, Local and Global Ext. Local Ext relates to sheaf Hom, denoted $\mathcal{H}om$, and Global Ext relates to $\operatorname{Hom}_{\mathcal{O}_X}$.

2.5.1 Local Ext

We'd like to glue Ext (at the level of modules/rings) modules along affine patches. The issue with this, and the larger issue with $\mathcal{E}xt$, is that there is no guarantee that we can construct projective resolutions in general to pass through the functor $\mathcal{H}om_{\mathcal{O}_X}(-,-)$ on the left. This means that we necessarily need to resolve the right module via injective resolution. In particular, for any $\mathscr{G}$, $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})$ should precisely be the i-th cohomology of the resolution $\mathcal{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{I}^\bullet)$, where $0 \to \mathscr{G} \to \mathscr{I}^\bullet$ is an injective resolution.

Lemma 2.5.1. *$\mathscr{I}$ is an injective $\mathcal{O}_X$-module and $U \subset X$ is open. Then, $\mathscr{I}|_U$ is an injective $\mathcal{O}_X$-Module.*

This is a relatively easy check. In general, when we proved that the category of sheaves (and in particular, the category of $\mathcal{O}_X$-Modules) has enough injectives, we took a sheaf $\mathscr{F}$, injected it into the product of the stalks $\prod_{x \in X} \mathscr{F}_x$, then concluded from the fact that the category of modules has enough injectives to pass to another injection, where the final result is an injective sheaf.

Restricting to an open subset merely throws away some of these stalks, so it's fairly clear that if we repeat this process on each open subset, we end up with an injective sheaf on that subset. Thus, logically, we can construct injective sheaves where this lemma holds fairly canonically. Thus, it should logically follow that the lemma holds for all injective sheaves. This lemma suggests the following corollary:

Lemma 2.5.2.

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})|_U = \mathcal{E}xt^i_{\mathcal{O}_U}(\mathscr{F}|_U,\mathscr{G}|_U)$$

Thus, $\mathcal{E}xt$ modules are compatible with restriction. Later, we'll show that, provided $\mathscr{F}$ is coherent and $\mathscr{G}$ is quasicoherent, $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})$ is itself a quasicoherent sheaf. In particular, $\mathcal{E}xt$ is compatible with taking stalks:

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})_x = \mathcal{E}xt^i_{\mathcal{O}_{X,x}}(\mathscr{F}_x,\mathscr{G}_x)$$

2.5.2 Global Ext

In contrast, Global Ext are the right derived functors of $\operatorname{Hom}_{\mathcal{O}_X}(\mathscr{F},-)$. It's readily apparent that

$$\Gamma(X,-) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X,-) = \operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X,-)$$

And that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X,-)$ is the identity functor on $\mathcal{O}_X$-Modules (similar to how $\operatorname{Hom}_R(R,-)$ is the identity functor on R-modules). Thus,

Lemma 2.5.3.
- $\mathcal{E}xt^0(\mathcal{O}_X, \mathscr{G}) = \mathscr{G}$
- $\mathcal{E}xt^i(\mathcal{O}_X, \mathscr{G}) = 0$
- $\mathrm{Ext}^i(\mathcal{O}_X, \mathscr{G}) = \mathrm{H}^i(X, \mathscr{G})\ \forall i \geq 0.$

Lemma 2.5.4. *Suppose we have a short exact sequence of sheaves*

$$0 \longrightarrow \mathscr{G}' \longrightarrow \mathscr{G} \longrightarrow \mathscr{G}'' \longrightarrow 0$$

We get a naturally derived long exact sequence in cohomology:

$$\ldots \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}') \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}) \longrightarrow \mathrm{Ext}^i_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}'') \longrightarrow \ldots$$

Similarly, with a short exact sequence

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

We get a long exact sequence

$$\ldots \longleftarrow \mathrm{Ext}^i_{\mathcal{O}_X}(\mathscr{F}', \mathscr{G}) \longleftarrow \mathrm{Ext}^i_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}) \longleftarrow \mathrm{Ext}^i_{\mathcal{O}_X}(\mathscr{F}'', \mathscr{G}) \longleftarrow \ldots$$

Proof. The first part of this lemma follows immediately from properties of derived functors; we know that they induce a long exact sequence in cohomology. The second long exact sequence, however, does not immediately follow. This is because projective resolutions do not necessarily exist in the category of sheaves; thus Ext is not a derived functor when resolving on the left. To see why this still works, let $0 \to \mathscr{G} \to \mathscr{I}^\bullet$ be an injective resolution of $\mathscr{G}$. passing through the functor $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathscr{I}^\bullet)$, we get a short exact sequence

$$0 \longleftarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}', \mathscr{I}^\bullet) \longleftarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{I}^\bullet) \longleftarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}'', \mathscr{I}^\bullet) \longleftarrow 0$$

Because $\mathscr{I}^\bullet$ is a complex of injectives. Equivalently, if you prefer, you can think of $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathscr{I}^\bullet)$ as a functor on the derive category, and $\mathscr{I}^\bullet$ is injective in the derived category, so it should induce an exact contravariant Hom functor. Nevertheless, taking a long exact sequence in cohomology via this short exact sequence of complexes, we arrive at the result. □

Lemma 2.5.5. *Suppose $\mathcal{L}_\bullet \to \mathscr{F} \to 0$ is an exact sequence where each $\mathcal{L}_i$ is locally free of finite rank. Then,*

$$\mathcal{E}xt^i(\mathscr{F}, \mathscr{G}) = \mathcal{H}_i(\mathcal{H}om(\mathcal{L}_\bullet, \mathscr{G}))$$

Where $\mathcal{H}_i$ denotes taking the homology groups of the complex of sheaves $\mathcal{H}om(\mathcal{L}_\bullet, \mathscr{G})$.

Note that when we are working over a Noetherian scheme, the hypothesis holds only when $\mathscr{F}$ is coherent. This is a relatively easy axiom check, so I leave it to you to verify. Suppose that $\mathscr{E}$ is locally free of finite rank. We define $\mathscr{E}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathcal{O}_X)$. We have the following nice duality result:

Lemma 2.5.6.

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathscr{F}\otimes\mathscr{E},\mathscr{G}) = \mathrm{Ext}^i_{\mathcal{O}_X}(\mathscr{F},\mathscr{E}^\vee\otimes\mathscr{G})$$

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathscr{F}\otimes\mathscr{E},\mathscr{G}) = \mathcal{E}xt^i_{\mathcal{O}_X}(\mathscr{F},\mathscr{E}^\vee\otimes\mathscr{G}) = \mathcal{E}xt^i_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})\otimes\mathscr{E}^\vee$$

Proof. This follows nicely from the (easy to verify) fact that, if $\mathscr{I}$ is injective, then $\mathscr{E}^\vee\otimes\mathscr{I}$ is still injective (i.e. tensoring by the dual of a free sheaf of locally finite rank preserves injectivity). The local case is similar, though checking that you can actually pull out the bundle completely needs to be explicitly verified by hand. □

The goal of all this work is to eventually get to Serre duality, which holds (in the level of generality that this course is comfortable with) on smooth projective schemes over Noetherian rings. Thus, we should use some of the machinery above to state a proposition in this restricted case about relating Ext and $\mathcal{E}xt$ sheaves.

Lemma 2.5.7. *Suppose that X is a projective scheme over a Noetherian ring A, and $\mathcal{L} = \mathcal{O}_X(1)$ is very ample over A. Further suppose $\mathscr{F},\mathscr{G} \in \mathrm{Coh}(X)$. Then $\exists n_0 \gg 0$ (depending on $\mathscr{F},\mathscr{G},i$) such that $\forall n \geq n_0$,*

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathscr{F},\mathscr{G}(n)) = \Gamma(X,\mathcal{E}xt^i_{\mathcal{O}_X}(\mathscr{F},\mathscr{G}(n))$$

$\forall i \geq 0$.

When $\mathscr{F} = \mathcal{O}_X$, we see that this follows from the first lemma at the beginning of this section, so in some sense this is a generalization of those common sensibilities.

2.6 Serre Duality

We now have the technology to properly state Serre Duality. This duality comes in different forms, so we'll start off with a "simple" version then generalize. For k a field, let $\omega_{\mathbb{P}^n_k} = \mathcal{O}_{\mathbb{P}^n_k}(-n-1) = \bigwedge^n \Omega_{\mathbb{P}^n_k/k}$ be the dualizing sheaf. Then,

Theorem 2.6.1. ***(Serre Duality, easy)*** *Let $\mathscr{E}$ be a locally free sheaf over $\mathbb{P}^n_k$, for k a field, and let X be a projective scheme over k.*

$$\mathrm{H}^i(\mathbb{P}^n_k,\mathscr{E})^* = \mathrm{H}^{n-i}(\mathbb{P}^n_k,\mathscr{E}^\vee\otimes\omega_{\mathbb{P}^n_k})$$

Where * denotes taking the natural k-vector space dual. Most of the time you cite Serre Duality, you use it on locally free sheaves. That being said, we can replace $\mathscr{E}$ with a coherent sheaf $\mathscr{F}$ over $\mathbb{P}^n_k$.

Theorem 2.6.2. ***(Serre Duality, a bit more advanced)*** *$\mathrm{H}^n(X,\omega_{\mathbb{P}^n_k}) = k$, and we have a perfect pairing*

$$\mathrm{Ext}^i(\mathscr{F},\omega_{\mathbb{P}^n_k})\otimes\mathrm{H}^{n-i}(\mathbb{P}^n_k,\mathscr{F}) \to \mathrm{H}^n(\mathbb{P}^n_k,\omega_{\mathbb{P}^n_k}) = k$$

Determining an isomorphism

$$\mathrm{Ext}^i(\mathscr{F},\omega_{\mathbb{P}^n_k}) = \mathrm{H}^{n-i}(\mathbb{P}^n_k,\mathscr{F})^*$$

Proof. Let's first check this pairing in special cases. When $i = 0$, this reduces to

$$\mathrm{Hom}(\mathscr{F}, \omega_{\mathbb{P}^n_k}) \otimes \mathrm{H}^n(\mathbb{P}^n_k, \mathscr{F}) \to \mathrm{H}^n(\mathbb{P}^n_k, \omega_{\mathbb{P}^n_k}) = k$$

For $\varphi \in \mathrm{Hom}(\mathscr{F}, \omega_{\mathbb{P}^n_k})$, taking the top cohomology $\mathrm{H}^n(\mathbb{P}^n_k, -)$ determines the perfect pairing.

Now notice that we know this already works for $\mathcal{O}_{\mathbb{P}^n_k}(\ell)$, and thus, it must work for any finite direct sums of copies of $\mathcal{O}_{\mathbb{P}^n_k}(\ell)$, varying over ℓ if necessary. Thus, because $\mathscr{F}$ is coherent, it has a finite presentation of $\mathcal{O}_X$ modules

$$\mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0$$

where each $\mathscr{E}_i = \oplus_j \mathcal{O}_{\mathbb{P}^n_k}(\ell_{i,j})^{\oplus m_{i,j}}$. $\mathrm{Hom}(-, \omega_{\mathbb{P}^n_k})$ is left exact, and since $\mathrm{H}^n(\mathbb{P}^n_k, -)$ is right exact (because the higher cohomology above H^n vanishes), it follows that $\mathrm{H}^n(\mathbb{P}^n_k, -)^*$ is left exact. Thus, both are left exact and contravariant, so passing the presentation through both, we see that

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\mathscr{F}, \omega_{\mathbb{P}^n_k}) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathscr{E}_0, \omega_{\mathbb{P}^n_k}) \arrow[r] \arrow[d, hook, two heads] & \mathrm{Hom}(\mathscr{E}_1, \omega_{\mathbb{P}^n_k}) \arrow[d, hook, two heads] \\
0 \arrow[r] & \mathrm{H}^n(\mathbb{P}^n_k, \mathscr{F}) \arrow[r] & \mathrm{H}^n(\mathbb{P}^n_k, \mathscr{E}_0) \arrow[r] & \mathrm{H}^n(\mathbb{P}^n_k, \mathscr{E}_1)
\end{tikzcd}$$

The last two maps being isomorphisms immediately implies the first first map is an isomorphism.

We've now like to show that such a result holds for any i. To do this, we use the case of $i = 0$ and trace down to $i = 1$, then $i = 2$, then so on until $i = n$. □

Now let's generalize further. A dualizing sheaf (and more generally, a dualizing complex) actually exists in much greater generality. Let X be a proper scheme over k. A dualizing sheaf ω_x° is a coherent sheaf with a trace isomorphism $t : \mathrm{H}^n(X, \omega_x^\circ) \to k$ such that, $\forall \mathscr{F} \in \mathrm{Coh}(X)$, there exists a natural pairing

$$\mathrm{Hom}(\mathscr{F}, \omega_X^\circ) \otimes \mathrm{H}^n(X, \mathscr{F}) \to \mathrm{H}^n(\omega_X^\circ)$$

That determines an isomorphism $\mathrm{H}^n(X, \mathscr{F})^* = \mathrm{Hom}(\mathscr{F}, \omega_X^\circ)$. Essentially, a dualizing sheaf is a sheaf that is defined to recover Serre Duality in the $i = 0$ case. We'll soon show that dualizing sheaves are essentially canonically defined, and furthermore, that the existence of a dualizing sheaf tells us a lot about the singularities of X.

Lemma 2.6.3. *ω_X°, if it exists, is unique up to choice of trace isomorphism.*

Proof. Suppose we have two dualizing sheaves (ω_X, t) and (ω'_X, t'). Then

$$\mathrm{Hom}(\omega_X, \omega'_X) \cong \mathrm{H}^n(\omega_X) \cong k$$

$$\mathrm{Hom}(\omega'_X, \omega_X) \cong \mathrm{H}^n(\omega'_X) \cong k$$

Thus the only morphisms between ω_X and ω'_X is associated to an element $a \in k$. Choose a nonzero a; this has a multiplicative inverse a^{-1}, which under the isomorphism above, corresponds to a map $\omega'_X \to \omega_X$ that is the inverse of the map associated to a. Thus, we have an isomorphism. □

Theorem 2.6.4. *If X is projective over k, then X has a dualizing sheaf. In particular, if* $\operatorname{codim}_{\mathbb{P}^n}(X) = c$*, then* $\omega_X = \mathcal{E}xt^c(\mathcal{O}_X, \omega_{\mathbb{P}^n})$.

These constructions are part of a much deeper package. In general, if we have a proper morphism $\pi : Y \to X$ where X has a dualizing sheaf ω_X, we can construct a functor $\pi^!$ such that $\pi^! \omega_Y = \omega_X$. In the case we have a projective morphism $X \to \mathbb{P}^n$ (i.e. when X is a projective scheme) this construction specializes to the one in the theorem. To prove the theorem, we need to check some quick lemmas.

Lemma 2.6.5. *$\mathcal{E}xt^c(\mathcal{O}_X, \omega_{\mathbb{P}^n})$ is the first non-vanishing $\mathcal{E}xt$ sheaf, i.e. using notation as above, $\mathcal{E}xt^i(\mathcal{O}_X, \omega_{\mathbb{P}^n}) = 0$ if $i < c$.*

Proof. When checking coherent sheaves are 0, the only way one can conceivably do this in general is to twist the coherent sheaf until it is globally generated, then check that it has no global sections. In this case, we know from a previous lemma that

$$\Gamma(\mathbb{P}^n, \mathcal{E}xt^i(\mathcal{O}_X, \omega_{\mathbb{P}^n})(\ell)) = \operatorname{Ext}^i(\mathcal{O}_X, \omega_{\mathbb{P}^n}(\ell))$$

Thus we just need to show that $\operatorname{Ext}^i(\mathcal{O}_X, \omega_{\mathbb{P}^n}(\ell)) = 0$ for some ℓ sufficiently large to get global generation upstairs. Well, via Serre Duality on projective space,

$$\operatorname{Ext}^i(\mathcal{O}_X, \omega_{\mathbb{P}^n}(\ell)) = \mathrm{H}^{n-i}(X, \mathcal{O}_X(-\ell))^*$$

Where $\mathrm{H}^{n-i}(X, \mathcal{O}_X(-\ell))$ vanishes for $n - i > \dim(X)$, i.e. precisely when $i < \operatorname{codim}(X)$. □

Lemma 2.6.6. *Let* $\dim(X) = n$*, $X \subseteq \mathbb{P}^N$ be a projective scheme, and* $c = \operatorname{codim}_{\mathbb{P}^N}(X)$*, and $\mathscr{F}$ a functorial isomorphism. Then,*

$$\operatorname{Hom}(\mathscr{F}, \omega_X^\circ) \cong \operatorname{Ext}^c_{\mathbb{P}^N}(\mathscr{F}, \omega_{\mathbb{P}^N})$$

Proof. Let $0 \to \omega_{\mathbb{P}^N} \to \mathscr{I}^\bullet$ be an injective resolution for $\omega_{\mathbb{P}^N}$. Passing this through $\operatorname{Hom}_{\mathbb{P}^N}(\mathscr{F}, -)$ yields a complex of $\mathcal{O}_X$-Modules. Notice that any such morphism $\mathscr{F} \to \mathscr{I}^j$ has to factor through $\mathcal{H}om_{\mathbb{P}^N}(\mathcal{O}_X, \mathscr{I}^j)$. This is because $\mathscr{I}^j$ has torsion from the ideal sheaf $\mathcal{I}_X$, and $\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^N}/\mathcal{I}_X$, and thus $\mathcal{H}om_{\mathbb{P}^N}(\mathcal{O}_X, \mathscr{I}^j)$ is the module cataloguing all the $\mathcal{I}_X$ torsion. Therefore, $\mathcal{H}om_{\mathbb{P}^N}(\mathcal{O}_X, \mathscr{I}^j)$ is a natural submodule of $\mathscr{I}^j$ that $\mathscr{F}$ must factor through, as it too is a $\mathcal{O}_X$ Module.

Now let $\mathscr{K}^\bullet = \mathcal{H}om_{\mathbb{P}^n}(\mathcal{O}_X, \mathscr{I}^\bullet)$. This is a complex of injective $\mathcal{O}_X$-Modules, and in particular a complex of flasque $\mathcal{O}_X$=Modules. Therefore, the cohomology is precisely $\mathcal{E}xt^i(\mathcal{O}_X, \omega_{\mathbb{P}^N})$, checked via chasing through definitions. Thus, $\mathscr{K}^\bullet$ is exact up to the cth step via the previous lemma. Thus we write $\mathscr{K}^\bullet = \mathscr{K}_1^\bullet \oplus \mathscr{K}_2^\bullet$ where $\mathscr{K}_1^\bullet$ is defined on degrees $0 \leq j \leq c$ and $\mathscr{K}_2^\bullet$ is defined on degrees $j \geq c$; here $\mathscr{K}_1^\bullet$ is exact by construction. Thus restricting to $\mathscr{K}_2^\bullet$, we can see via some homological trickery that $\operatorname{Hom}(\mathscr{F}, \omega_X^\circ) = \operatorname{Ext}^0(\mathscr{F}, \omega_X^\circ)$ corresponds to the corresponding Ext groups on $\mathbb{P}^N$ after a shift by c (i.e. a shift by cutting out the $\mathscr{K}_1^\bullet$ portion of the complex above). Thus, $\operatorname{Hom}(\mathscr{F}, \omega_X^\circ) = \operatorname{Ext}^0(\mathscr{F}, \omega_X^\circ) \cong \operatorname{Ext}^c_{\mathbb{P}^N}(\mathscr{F}, \omega_{\mathbb{P}^N})$ as desired. □

Using this machinery, we can generalize Serre Duality for a reasonably subclass of projective schemes:

Theorem 2.6.7. ***(Serre Duality, for suitable projective schemes)*** *Let X be a Cohen-Macaulay projective scheme over a field k, that is equidimensional of dimension n. Say X embeds in $\mathbb{P}^N$ for some N. Let $\omega_X^\circ := \mathcal{E}xt_{\mathbb{P}^N}^{N-n}(\mathcal{O}_X, \omega_{\mathbb{P}^N})$ (and $\omega_{\mathbb{P}^N} = \mathcal{O}_{\mathbb{P}^N}(-N-1)$) be the dualizing sheaf of X, with corresponding trace map $t : \mathrm{H}^n(X, \omega_X^\circ) \to k$. Then, $\forall \mathscr{F} \in \mathrm{Coh}(X)$,*

$$\mathrm{H}^i(X, \mathscr{F})^* \cong \mathrm{Ext}_{\mathcal{O}_X}^{n-i}(\mathscr{F}, \omega_X^\circ)$$

The proof is sketched below. The trade map induces a perfect pairing as in the $\mathbb{P}^N$ case, which then induces an isomorphism $\mathrm{H}^n(X, \mathscr{F}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}, \omega_X^\circ)$. Using the same tricks as in the $\mathbb{P}^N$ case, we work backwards from the top cohomology class to recover the result for lower cohomology classes.
Let's compute ω_X° is some reasonable cases. Suppose that $X \hookrightarrow \mathbb{P}^N$ is a hypersurface of degree d, cut out by a polynomial f. Using the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^N}(-d) \xrightarrow{\cdot f} \mathcal{O}_{\mathbb{P}^N} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

Applying $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N}}(-, \mathcal{O}_{\mathbb{P}^N}(-N-1))$, we see that

$$\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_{\mathbb{P}^N}(-d), \mathcal{O}_{\mathbb{P}^N}(-N-1)) = \mathcal{O}_{\mathbb{P}^N}(-N-1+d)$$

$$\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_{\mathbb{P}^N}, \mathcal{O}_{\mathbb{P}^N}(-N-1)) = \mathcal{O}_{\mathbb{P}^N}(-N-1)$$

Giving us the the following long exact sequence in cohomology:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^N}(-N-1) \xrightarrow{\cdot f} \mathcal{O}_{\mathbb{P}^N}(-N-1+d) \longrightarrow \mathcal{E}xt^1(\mathcal{O}_X, \omega_{\mathbb{P}^N}) \longrightarrow 0$$

As $\mathcal{O}_{\mathbb{P}^N}(-N-1+d) = \mathcal{O}_{\mathbb{P}^N}(-N-1)(d) = \omega_{\mathbb{P}^N}(d) = \omega_{\mathbb{P}^n}(X)$, we can realize this complex as

$$0 \longrightarrow \omega_{\mathbb{P}^N} \longrightarrow \omega_{\mathbb{P}^N}(X) \longrightarrow \omega_X^\circ \longrightarrow 0$$

Implying that $\omega_X^\circ = \omega_{\mathbb{P}^N}(X)|_X$. We can also compute ω_X° when X is a smooth projective scheme. Let $Z \subseteq X$ be a closed subscheme of X. We have the following sequence:

$$0 \longrightarrow \mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{\varphi} \Omega_{X/k}|_Z \longrightarrow \Omega_{Z/k} \longrightarrow \Omega_{Z/X} \longrightarrow 0$$

Where φ is injective if and only if $\mathcal{I}_Z$ is locally generated by a regular sequence on X, i.e. Z is a local complete intersection. In this case, we see that

$$\bigwedge^N \Omega_{X/k}|_Z \cong \bigwedge^n \Omega_{Z/k} \otimes \bigwedge^{N-n} \mathcal{I}_Z/\mathcal{I}_Z^2$$

As $X \subseteq \mathbb{P}^N$ is a closed subscheme (and in particular, via smoothness, a local complete intersection) we have that

$$\omega_{\mathbb{P}^N}|_X \cong \omega_X \otimes \bigwedge^{N-n} \mathcal{I}_X/\mathcal{I}_X^2$$

and therefore,

$$\omega_X \cong \omega_{\mathbb{P}^N}|_X \otimes \bigwedge^{N-n}\left(\mathcal{I}_X/\mathcal{I}_X^2\right)^*$$

One can check that, in the case where X is a hypersurface, this computation recovers the first one. Furthermore, the hypersurface case immediately implies the case where X is a global complete intersection, i.e. $I_X = (f_1, \ldots, f_c)$ where the f_i form a regular sequence. Just consider the corresponding Koszul Complex and take a long exact sequence in $\mathcal{E}xt$ within $\mathbb{P}^N$. It follows from a small amount of work that the isomorphism $\omega_X \cong \omega_{\mathbb{P}^N}|_X \otimes \bigwedge^{N-n}\left(\mathcal{I}_X/\mathcal{I}_X^2\right)^*$ still holds. This discussion can be encapsulated in the following theorem:

Theorem 2.6.8. *Suppose $X \subseteq \mathbb{P}^N$ is a closed local complete intersection over k of codimension c. Then, Ω_X° is an invertible sheaf and*

$$\omega_X^\circ \cong \omega_{\mathbb{P}^N} \otimes \bigwedge^c \left(\mathcal{I}_X/\mathcal{I}_X^2\right)^*$$

In particular, all the various definitions of a dualizing sheaf are the same thing, i.e.

$$\mathcal{E}xt^{N-n}(\mathcal{O}_X, \omega_{\mathbb{P}^N}) = \omega_X^\circ = \omega_X = \bigwedge^n \Omega_{X/k}$$

We also have the following corollary:

Lemma 2.6.9. *Suppose $X \subseteq \mathbb{P}_K^N$ is a smooth projective variety of dimension n. Then.*

$$\mathrm{H}^p\left(X, \bigwedge^q \Omega_{X/k}\right) \cong \mathrm{H}^{n-p}\left(X, \bigwedge^{n-q} \Omega_{X/k}\right)^*$$

Proof. This is an easy application of Serre Duality along with the equivalence above.

$$\mathrm{H}^p\left(X, \bigwedge^q \Omega_{X/k}\right) \cong \mathrm{Ext}^{n-p}\left(\bigwedge^q \Omega_{X/k}, \bigwedge^n \Omega_{X/k}\right) \cong \mathrm{H}^{n-p}\left(X, \left(\bigwedge^q \Omega_{X/k}\right)^\vee \otimes \bigwedge^n \Omega_{X/k}\right)^*$$

It follows from the perfect pairing arising from the exterior algebra $\bigwedge \Omega_{X/k}$ that

$$\left(\bigwedge^q \Omega_{X/k}\right)^\vee \otimes \bigwedge^n \Omega_{X/k} \cong \bigwedge^{n-q} \Omega_{X/k}$$

□

Notably, this is useful in Hodge Theory, and determines an equivalence between pieces of the Hodge Decomposition of a smooth projective variety over $\mathbb{C}$.

2.7 A bit on Curves

The remainder of the class we'll briefly touch on some of the later topics in chapter 4 and 5 of Hartshorne. We'll begin by introducing some notions about curves.

When we refer to a curve, we properly mean a complete (or alternatively, projective) nonsingular integral scheme of dimension 1 over an algebraically closed field k. Curves have a lot of invariants, like genus. We say that the ***geometric genus*** of X, denoted $p_g(X)$, is $h^0(X, \Omega_{X/k}$, i.e. the (dimensional) count of 1 forms on the space. We say that the ***arithmetic genus*** of X, denoted $p_a(X)$, is $h^1(X, \mathcal{O}_X)$. We immediately see as a consequence of Serre Duality that

Lemma 2.7.1. *If X is a curve, then*

$$p_a(X) = p_g(X)$$

i.e. the arithmetic genus is the same as the geometric genus.

Proof. By Serre Duality,

$$\mathrm{H}^1(X, \mathcal{O}_X) = \mathrm{H}^0(X, \omega_X)^* = \mathrm{H}^0(X, \Omega_{X/k})^*$$

□

We can thus call either number the genus of X with no ambiguity. As another (dual) consequence of Serre Duality, $h^0(X, \mathcal{O}_X) = h^1(X, \omega_X)$, where $h^0(X, \mathcal{O}_X) = 1$ since X is an integral curve.

Over curves, divisors are just finite formal sums of points $D = \sum a_i P_i$ where $P_i \in X$ and $a_i \in \mathbb{Z}$. $\deg(D) = \sum a_i$, and to any divisor D we have an associated line bundle $\mathcal{O}_X(D)$ such that

$$\Gamma(U, \mathcal{O}_X(D)) = \{f \in k(X) \mid \mathrm{div}_U(f) + D \geq 0\}$$

Recall that since X is smooth, Weil Divisors are the same as Cartier Divisors, and to each Cartier Divisor we can associated a line bundle. In this case, this line bundle is precisely $\mathcal{O}_X(D)$ as above. The cohomology of line bundles and genus of curves are intimately related

Theorem 2.7.2. ***(Riemann-Roch)*** *Let D be a divisor on a curve X of genus g. Then,*

$$h^0(\mathcal{O}_X(D)) - h^0(\mathcal{O}_X(K_X - D)) = \deg(D) + 1 - g$$

For a given divisor D, we'll let $|D|$ denote the set of effective divisors which are linearly equivalent to D (i.e., divisors E such that $E \geq 0$ and $E - D = \mathrm{div}_X(f)$ for some f). This is the set of ***linear systems*** associated to D, and is canonically identified with $\mathbb{P}\,\mathrm{H}^0(X, \mathcal{O}_X(D)) := \frac{\mathrm{H}^0(X, \mathcal{O}_X(D) \setminus \{0\}}{k^\times}$. This is a vector space, with dimension denoted $\ell(D) := \dim_k |D|$. When $|D|$ is nonempty, this is just $h^0(X, \mathcal{O}_X(D)) - 1$. Using this notation, we can rewrite Riemann Roch as

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g$$

By Serre Duality, $h^0(\mathcal{O}_X(K_X - D)) = h^1(\mathcal{O}_X(D))$, so we can recover the Euler characteristic $\chi(\mathcal{O}_X(D))$. Thus the following chain of equalities immediately follow from Riemann Roch:

$$\begin{aligned} h^0(\mathcal{O}_X(D)) - h^0(\mathcal{O}_X(K_X - D)) &= h^0(\mathcal{O}_X(D)) - h^1(\mathcal{O}_X(D)) \\ &= \ell(D) - \ell(K - D) \\ &= \deg(D) + 1 - g \\ &= \chi(\mathcal{O}_X(D)) \end{aligned}$$

We can now proceed with the proof of Riemann Roch.

Proof. First, we observe that this works if $D = 0$. Inductively, if $D \neq 0$, we write $D = D' + P$, where P is a closed point. Considering the short exact sequence

$$0 \to \mathcal{O}_X(-P) \to \mathcal{O}_X \to k(P) \to 0$$

Twisting by $\mathcal{O}_X(D)$, we get

$$0 \to \mathcal{O}_X(D') \to \mathcal{O}_X(D) \to k(P) \to 0$$

Euler Characteristic is additive on short exact sequences, and clearly $\chi(k(P)) = 1$. Thus,

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(D')) + 1$$

Inductively, we see that $\chi(\mathcal{O}_X(D')) = \deg(D') + 1 - g$, where $\deg(D') + 1 = \deg(D)$. The result then follows. $\square$

Examples

- For a simple application, let $D = K_X$. The formula above reduces to $h^0(K_X) - h^1(K_X) = \deg(K_X) + 1 - g$. $h^0(K_X) = h^1(\mathcal{O}_X) = g$ and $h^1(K_X) = h^0(\mathcal{O}_X) = 1$ by Serre duality, so we see that $g - 1 = \deg(K_X) + 1 - g$, so $\deg(K_X) = 2g - 2$.

- Now suppose that $X \subseteq \mathbb{P}^2_k$ is a curve of degree d. By multiplying the coordinate ring $\mathcal{O}_{\mathbb{P}^2}$ by the defining polynomial of X, denoted f, we get a map $\mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2}$, inducing the short exact sequence

  $$0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_X \to 0$$

  $\chi(\mathcal{O}_{\mathbb{P}^2}) = 1$. To compute the Euler characteristic of $\mathcal{O}_{\mathbb{P}^2}(-d)$, notice that it has no sections as $d > 0$, so $-d < 0$. Furthermore, it clearly has no h^1. To compute h^2, we see that this is just $\binom{d-3+2}{2} = \binom{d-1}{2}$, so this is the Euler Characteristic. $\mathcal{O}_X$ has Euler Characteristic $1 - g$, when X is of genus g, so by additivity of Euler Characteristic,

  $$\binom{d-1}{2} - 1 + 1 - g = 0 \Rightarrow g = \binom{d-1}{2}$$

  Thus, in $\mathbb{P}^2$ the degree completely determines the genus of the curve, and vice versa.

- Using similar tricks, a curve $X \subseteq \mathbb{P}^1_k \times \mathbb{P}^1_k$ of type (a, b) for $a, b > 0$ has genus $g = (a-1)(b-1)$. Via the logic of the previous example, we get the short exact sequence
$$0 \to \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-a,-b) \to \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \to \mathcal{O}_X \to 0$$
determined by the defining equation f. As X is a divisor in $\mathbb{P}^1 \times \mathbb{P}^1$, so one can associate $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-a,-b)$ to the line bundle $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-X)$. To see this, observe that $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-a,-b) = \mathcal{O}_{\mathbb{P}^1}(-a) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-b)$, where in general, with line bundles $\mathcal{L}_1, \mathcal{L}_2$ on $\mathbb{P}^1$ on the corresponding projections $\pi_1, \pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, we define
$$\mathcal{L}_1 \boxtimes \mathcal{L}_2 := \pi_1^*\mathcal{L}_1 \otimes_{\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}} \pi_2^*\mathcal{L}_2$$
Such formalism extends to when you replace the copies of $\mathbb{P}^1$ with any scheme, and when $\mathcal{L}_i$ are each any sheaf. We'll invoke the following theorem of Künneth:

  Theorem 2.7.3. ***(Künneth's formula)*** *X and Y are projective schemes over k, and $\mathscr{F} \in \mathrm{Coh}(X)$ and $\mathscr{G} \in \mathrm{Coh}(X)$. Then,*
$$\mathrm{H}^n(X \times Y, \mathscr{F} \boxtimes \mathscr{G}) = \bigoplus_{i+j=n} \mathrm{H}^i(X, \mathscr{F}) \otimes_k \mathrm{H}^j(Y, \mathscr{G})$$
Use this to decompose the homology classes of X over $\mathbb{P}^1 \times \mathbb{P}^1$ into its component parts, then use Riemann Roch to conclude the genus formula.