

Preliminary Info

These are notes taken from Math 520 (Commutative and Homological Algebra) at the University of Illinois Chicago in Fall 2022. The class is taught by Professor Wenliang Zhang, and these notes are TeXed by Vignesh Jagathese (me!). There are still some minor errors in these notes, and by reading these and benefiting from them, all I ask in return is that you forward any errors to me.

These notes would serve as a great companion to my Fall 2019 notes from Karen Smith's Commutative Algebra course at the University of Michigan. That course leaned far more into geometric topics, with notes consisting primarily of exercises. Conversely, this course leaned more into homological algebra, and the notes have far more exposition. Hopefully between these two sets of notes, one can get a complete picture of the introductory concepts in commutative algebra.

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08/22/2022

1.1 Some Preliminaries

Homeworks are due on Fridays at 11:59PM on Gradescope. They will be assigned every other week. Also, as convention, all rings will be commutative and with (multiplicative) identity. We assert that $1 \neq 0$.

1.2 Maximal Ideals

We say that an ideal $I \subsetneq R$ is *prime* if $ab \in I \Rightarrow$ either $a \in I$ or $b \in I$. Equivalently, I is prime $\iff R/I$ is an integral domain.

We say that an ideal $I \subsetneq R$ is *maximal* if no proper ideal of R strictly contains it. Equivalently, $m \subsetneq R$ is a maximal ideal $\iff R/m$ is a field. It's a pretty elementary fact that all maximal ideals are prime.

Theorem 1.2.1. *Every commutative ring admits a maximal ideal.*

Proof. This proof will use a Zorn's Lemma trick that will be used regularly today. We first set a partial order on the set of ideals of R , ordering by inclusion. Every ascending chain under this partial ordering has an upper bound (namely, the union of all elements in the chain, which is also an ideal), so by Zorn's lemma there must exist a maximal element. Thus, R must have a maximal ideal. \square

While a ring must necessarily have a maximal ideal, there can be any number of them. We say a ring R is *local* if it has a unique maximal ideal, and *semi-local* if it has finitely many maximal ideals.

1.2.1 Examples of local rings:

- (1) Any field.
- (2) For k a field, $k[[x]]$, which is the ring of formal power series. This is a local ring with unique maximal ideal (x) . We'll prove this using a helpful criterion in a bit.

- (3) For P a prime ideal, R_P , or R localized at the complement of P . PR_P is the unique maximal ideal.

Note that a (finite) product of local rings is semi-local. For a more concrete example, consider $\mathbb{Z}/6\mathbb{Z}$. This has two maximal ideals (2) and (3), so it is semi-local (one notes that, by the Chinese Remainder Theorem, we can write $\mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, i.e. a product of fields, i.e. a product of local rings). Here is a criterion we can use to determine whether or not a ring is local.

Theorem 1.2.2. *Suppose $m \subsetneq R$ is a proper ideal.*

- (1) *If z is a unit $\forall z \in R \setminus m$, then R is local with unique maximal ideal m .*
- (2) *If m is a maximal ideal and $1 + z$ is unit $\forall z \in m$, then R is local with unique maximal ideal m .*

Proof. (2) will be left as an exercise, so we just prove (1). We want to check that every proper ideal is contained in m . Well, if there is an ideal $I \not\subseteq m$, then there is an element that lies in I but does not lie in m . All elements not in m are unit, so this element (that is contained in I) is a unit. any ideal that contains a unit cannot be proper ($s \in I \Rightarrow s^{-1}s \in I \Rightarrow 1 \in I$), so we have arrived at a contradiction. \square

Theorem 1.2.3. *$k[[x]]$ is a local ring with unique maximal ideal (x) .*

Proof. Choose $f = \sum a_i x^i, g = \sum b_i x^i \in k[[x]]$. We want to show that $fg = 1 \iff a_0 \neq 0$ and $b_0 \neq 0$. Well, if $fg = 1$, then higher order terms cancel, and we are left with $a_0 b_0 = 1$. This means that neither $a_0, b_0 = 0$, as fields are integral domains. Conversely, suppose that $a_0 \neq 0$. We'd like to construct $g = \sum b_i x^i$ such that $fg = 1$. Well, set $b_0 = \frac{1}{a_0}$. The degree 1 term of fg is $a_0 b_1 + a_1 b_0$, and if $fg = 1$, then it ought to be the case that $a_0 b_1 + a_1 b_0 = 0$. Well, a_0, a_1 are fixed, and $b_0 = \frac{1}{a_0}$, so we can solve this equation for b_1 and get $b_1 = a_0^{-1}(-a_1 b_0)$. A similar process can be done for b_i for $i > 1$. \square

1.3 Localization

$S \subset R$ is said to be *multiplicative* if $1 \in S$ and $a, b \in S \Rightarrow ab \in S$.

Theorem 1.3.1. *$S \subset R$ is a multiplicative subset, $I \subsetneq R$ an ideal. Suppose that $I \cap S = \emptyset$.*

- (1) *Among all proper ideals disjoint from S , there exists a maximal element.*
- (2) *This maximal element is a prime ideal.*

Proof. We consider the set of all proper ideals disjoint from S , denoted Σ . As $I \in \Sigma$, it is nonempty. Partially order Σ with respect to inclusion, and for any ascending chain of ideals in Σ , we know that the union of the ideals forms an upper bound. Thus by Zorn's lemma, there exists a maximal element. It is now sufficient to prove the second part of the theorem; that this maximal element (denoted P) is in fact a prime ideal.

Choose $ab \in P$. For sake of contradiction, suppose that $a, b \notin P$. This implies that $P \subsetneq P + (a), P + (b)$, suggesting by maximality that $P + (a), P + (b) \notin \Sigma$. Thus, S has nontrivial intersection with these two ideals. Choose $s_1 \in (P + (a)) \cap S, s_2 \in (P + (b)) \cap S$. As $s_1, s_2 \in S, s_1 s_2 \in S$ since S is multiplicative. Furthermore,

$$s_1 s_2 \in (P + (a))(P + (b)) \subset P + (ab) = P$$

Thus $s_1 s_2 \in P \cap S$, a contradiction, as $P \in \Sigma$ so $P \cap S = \emptyset$. \square

What follows is that, for any proper ideal $I \subset R$ disjoint from $S, S^{-1}I$ is a proper ideal of $S^{-1}R$, though we'll touch more on localization (in earnest) later.

1.4 Radical Ideals

Given $I \subset R$, we let $\sqrt{I} := \{r \in R \mid \exists n \in \mathbb{N} \text{ such that } r^n \in I\}$ denote the *radical* of I . It's easy to see that \sqrt{I} is also an ideal. If $I = \sqrt{I}$, we call I *radical*. It's an elementary fact that all primes are radical.

For one specific case of a radical ideal, consider the radical of the 0 ideal, denoted $\sqrt{0}$. These are precisely the nilpotent elements of R , so $\sqrt{0}$ is the *nilradical* of R .

Theorem 1.4.1.

$$\sqrt{0} = \bigcap_{P \in \text{Spec}(R)} P$$

Proof. First, choose $x \in \sqrt{0}$. Therefore, $x^n = 0$ for some n , suggesting that $x^n \in P \forall P \in \text{Spec}(R)$. Well, as P is prime, $x^n \in P \Rightarrow x \in P$, so $x \in \bigcap_{P \in \text{Spec}(R)} P$.

Conversely, suppose that x lie in every prime ideal, but for the sake of contradiction, that $x \notin \sqrt{0}$. Let Σ denote the set of ideals I such that $x^n \notin I \forall n \geq 1$. As $(0) \in \Sigma$, it is nonempty. We now continue with the Zorn's lemma trick again. Partially order ideals with respect to inclusion, and construct a maximal element $J \in \Sigma$. We claim that J is prime.

Choose $ab \in J$. If $a, b \notin J$, then $J \subsetneq J + (a), J + (b)$, and by maximality $J + (a), J + (b) \notin \Sigma$. Therefore, $\exists n, m \in \mathbb{N}$ such that $x^n \in J + (a), x^m \in J + (b)$, implying that $x^{n+m} \in (J + (a))(J + (b)) \subset J + (ab) = J$. Thus, $x^{n+m} \in J$, which is a contradiction. \square

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08/24/2022

As a generalization of a result from last class,

Theorem 2.0.1. *Let $I \subset R$ be an ideal.*

$$\sqrt{I} = \bigcap_{I \subseteq P, P \in \text{Spec}(R)} P$$

The proof of this is identical to the proof of the analogous result for the nilradical.

2.1 Prime Avoidance

Theorem 2.1.1. *Assume $P_1, \dots, P_n \in \text{Spec}(R)$, and $I \subset R$ an ideal. If $I \subset \bigcup_{i=1}^n P_i$, then $I \subset P_i$ for some i .*

Proof. We prove this inductively. The $n = 1$ case is clear, so we assume the result is true for $n - 1$ and prove it for n .

We prove this via contrapositive. If $I \not\subset P_i$ for any i , we want to show that it cannot be contained in the union $\bigcup_{i=1}^n P_i$. By induction, $I \not\subset \bigcup_{i \neq j} P_i$ for any $1 \leq j \leq n$. Now choose $x_j \in I$ such that $x_j \notin \bigcup_{i \neq j} P_i$. If $x_j \notin P_j$ we are done, so let's assume that $x_j \in P_j$. Now define $x := \sum_{j=1}^n \prod_{i \neq j} x_i$. Every element of the summation is in I , so $x \in I$. Conversely, P_j is prime, so $x \notin P_j$ for any j . It follows that $I \not\subset \bigcup_{i=1}^n P_i$. \square

2.2 Nakayama Style Statements

We now define an analogous construction to the Nilradical for maximal ideals. Let $\mathcal{J}(R) := \bigcap_{m \in \text{MaxSpec}(R)} m$ denote the **Jacobson Radical** of R .

Lemma 2.2.1. $x \in \mathcal{J}(R) \iff 1 - xy$ is unit $\forall y \in R$

Proof. We prove the forward case first. If $1 - xy$ is not unit for some $y \in R$, then $1 - xy$ is contained in a maximal ideal, say m .

$$x \in \mathcal{J}(R) \Rightarrow x \in m \Rightarrow xy \in m \Rightarrow (1 - xy) + xy \in m \Rightarrow 1 \in m$$

So $m = R$, a contradiction. Conversely, suppose that $x \notin m$ for some fixed maximal ideal m . This implies that $m \subsetneq m + (x)$, suggesting that (by maximality of m) $m + (x) = R$. It follows that $1 \in m + (x)$, so $\exists z \in m, y \in R$ such that $z + xy = 1$, i.e. $z = 1 - xy$. By hypothesis, $1 - xy$ is unit, so $z \notin m$, a contradiction. \square

Theorem 2.2.2. *Assume M is a finitely generated R -module, and $\varphi \in \text{End}_R(M)$. If $\varphi(M) \subset IM$, then φ satisfies a polynomial $\varphi^n + a_{n-1}\varphi^{n-1} + \dots + a_0$, where $a_i \in I$.*

Proof. Choose a minimal set of generators $\langle z_1, \dots, z_n \rangle = M$. $\varphi(z_i) \in IM$, so $\varphi(z_i) = \sum a_{ij}z_j$, where $a_{ij} \in I$. Now consider the matrix $A = [\delta_{ij}\varphi - a_{ij}]$. It follows that $A \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = 0$, as all the rows vanish. If we multiply both sides by A^\dagger , we get that

$$\text{diag}(\det(A), \dots, \det(A)) \cdot \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = 0$$

Implying that $\det(A)z_j = 0 \forall 1 \leq j \leq n$. As $\det(A)$ kills all generators of the module, it kills the module itself. Therefore, $\det(A)M = 0$, so $\det(A) = 0$. Well, $0 = \det(A) = \prod(\varphi - a_{ii})$, so φ satisfies a degree n monic polynomial with coefficients in I . \square

This so called "determinant trick" is integral to proving Nakayama's Lemma.

Lemma 2.2.3. (Nakayama's Lemma) *Suppose that $J \subset \mathcal{J}(R)$ and M is a finitely generated R -module. Then, $JM = M \Rightarrow M = 0$.*

Proof. Replace φ as above with the identity map, and I as above with J . It follows that $(1 + a_1 \cdot 1 + \dots + a_n)M = 0$. Letting $a_1 + \dots + a_n = x \in J$, it follows that $(1 + x)M = 0$. As $x \in \mathcal{J}(R)$, $1 + x$ is unit, so it follows that $M = 0$. \square

2.2.1 More Commonly Used Forms of Nakayama's Lemma

Notice that if (R, m) is local, then $\mathcal{J}(R) = m$. The following version of Nakayama's Lemma follows:

Lemma 2.2.4. *For (R, m) a local ring and M a finitely generated R -module, $M = mM \Rightarrow M = 0$.*

Lemma 2.2.5. *For (R, m) a local ring and M a finitely generated R -module, If $N \subset M$ such that $N + mM = M$, then $N = M$.*

These yield the following corollary:

Lemma 2.2.6. *If M is a finitely generated R module, then $\varphi \in \text{End}_R(M)$ being surjective implies that it is also injective.*

Proof. View M as an $R[x]$ -module where $x \cdot m = \varphi(m) \forall m \in M$ (and extend this action over the algebra). As φ is surjective, $xM = M$, implying that $(1 + xy)M = 0$. Thus, if $\varphi(m) = 0$, then $0 = (1 + xy)m = m + y \cdot \varphi(m) = m$, so $m = 0$. Thus, φ is injective. \square

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08/26/2022

NOTE: Homework 1 assigned, due Friday 9/9 at 11:59PM.

3.1 More Applications of Nakayama's Lemma

Theorem 3.1.1. *(R, m) a local ring. Suppose that M is a finitely generated R -module. If M is a projective module, then M must be free.*

Proof. Pick a minimal set of generators of $M = \langle z_1, \dots, z_n \rangle$. Define $\varphi : R^{\oplus n} \rightarrow M$ such that $\varphi(e_i) = z_i$. This map, by construction, is surjective. As M is projective, this surjection splits. Thus, \exists a map $\psi : M \rightarrow R^{\oplus n}$ such that $R^{\oplus n} = \psi(M) \oplus \ker(\varphi)$. This gives us the diagram

$$M \xrightarrow{\psi} R^{\oplus n} \xrightarrow{\varphi} M$$

If $\varphi(\sum r_i e_i) = 0$, then $\sum r_i z_i = 0$. If $r_i \notin m$, then r_i is unit. Here,

$$z_i = r_i^{-1}(-r_1 z_1 - \dots - r_n z_n) \in \sum_{j \neq i} R z_j$$

This violates minimality of $\langle z_1, \dots, z_n \rangle$, so we can suppose that $r_1, \dots, r_n \in m$. Thus, $\ker(\varphi) \subset mR^{\oplus n} = m\psi(M) \oplus m\ker(\varphi)$. But where does $\ker(\varphi)$ lie? Well, $\ker(\varphi) \cap \psi(M) = 0$. Thus, $\ker(\varphi) \subseteq m\ker(\varphi) \subseteq \ker(\varphi)$, so $\ker(\varphi) = m\ker(\varphi)$. Since $\ker(\varphi)$ is a quotient of $R^{\oplus n}$, which is finitely generated, it is clear that $\ker(\varphi)$ must also be finitely generated. Thus, by Nakayama's Lemma, $\ker(\varphi) = 0$ so φ is an isomorphism. \square

Do note that this result is true even when M is not finitely generated, though the proof is significantly more technical.

3.2 The Zariski Topology

Let's recall some things about the Zariski Topology (and not spend too much time verifying things). There exists a functor between the category of rings and the category of

topological spaces. This functor, $\text{Spec}(-)$ sends R to the set of its prime ideals, imbued with the **Zariski Topology**. To define the topology, we declare that subsets of $\text{Spec}(R) = X$ of the form $V(I) := \{P \in \text{Spec}(R) \mid I \subset P\}$ are the closed sets, for any ideal $I \subset R$. To verify that this is a topology, however, does require some work.

- $\emptyset = V(R)$.
- $V(0) = \text{Spec}(R)$.
- $\bigcap V(I_\alpha) = V(\sum I_\alpha)$ (for arbitrary intersections/summations)
- $V(I) \cup V(J) = V(I \cap J) = V(IJ)$.

So this is indeed a topology, and this functor is well defined on objects. On morphisms, for a ring homomorphism $\varphi : R \rightarrow S$, $\text{Spec}(\varphi) : \text{Spec}(S) \rightarrow \text{Spec}(R)$ is map sending $Q \mapsto \varphi^{-1}(Q)$. It's easy to verify that $\varphi^{-1}(Q)$ is a prime ideal (just choose $ab \in \varphi^{-1}(Q)$, pull back, factor, then push forward). It's also easy to verify that this is continuous under the Zariski Topology.

3.2.1 Nullstellensatz

We have an inclusion reversing bijection

$$\{\text{Closed sets in } \text{Spec}(R)\} \longleftrightarrow \{\text{Radical Ideals } I\}$$

Such that $V(I) \leftarrow I$ and $Z \mapsto \bigcap_{P \in Z} P$. Similarly, we have

$$\{\text{Irreducible, Closed sets in } \text{Spec}(R)\} \longleftrightarrow \{\text{Prime Ideals } I\}$$

Utilizing the same mappings as above. Proving that these are bijections is a bit tedious, but doable and relatively straightforward.

3.3 Localization

Say $W \subset R$ is a multiplicative set. Define a relation on $R \times W$ such that

$$(r_1, w_1) \sim (r_2, w_2) \iff \exists w \in W \text{ such that } wr_1w_2 = wr_2r_1$$

Define

$$R \times W / \sim := W^{-1}R = \left\{ \frac{r}{w} \mid r \in R, w \in W \right\}$$

there is a natural ring map $R \rightarrow W^{-1}R$ mapping $r \mapsto \frac{r}{1}$. Typically we either localize at the complement of a prime ideal, or at all nonzero elements of the ring. The former yields R_p , a local ring, and the latter yields the field of fractions of R . We can also localize at $W = \{1, r, r^2, \dots\}$, and we see that $W^{-1}R = R \left[\frac{1}{r} \right] = R_r$.

As an example, suppose we were to localize $\mathbb{Z}/6\mathbb{Z}$ at (the complement of) (3). This yields $\mathbb{Z}/3\mathbb{Z}$, a field.

More generally, one can define localization via a universal property. For any morphism $\varphi : R \rightarrow S$ of rings, if $\varphi(w)$ is unit in $S \forall w \in W$, $W^{-1}R$ is the unique object making the following diagram commute:

$$\begin{array}{ccc}
 R & \xrightarrow{r \mapsto \frac{r}{1}} & W^{-1}R \\
 & \searrow \varphi & \downarrow \hat{\varphi} \\
 & & S
 \end{array}$$

Lemma 3.3.1. $W^{-1}(-)$ is an exact functor on R -modules. In particular, as $W^{-1}M = M \otimes_R W^{-1}R$, we can conclude that $W^{-1}R$ is a flat R -module.

Theorem 3.3.2. There is a one-to-one correspondence between $\text{Spec}(W^{-1}R)$ and $\{P \in \text{Spec}(R) \mid P \cap W = \emptyset\}$. In particular, $\text{Spec}(R_Q) \leftrightarrow \{P \in \text{Spec}(R) \mid P \subset Q\}$, suggesting that R_P is indeed local, with maximal ideal PR_P . Also, $\text{Spec}(R_f) \leftrightarrow \{P \in \text{Spec}(R) \mid f \notin P\} = \text{Spec}(R) \setminus V(f)$, suggesting that $\text{Spec}(R_f)$ is a (basic) open set in $\text{Spec}(R)$, and sets of this type form a basis for the topology of $\text{Spec}(R)$.

A property \mathcal{P} on a ring (resp. module) is a **local property** if R (resp. M) has $\mathcal{P} \iff R_P$ (resp. M_P) has \mathcal{P} for every prime ideal P . For example, being 0 is a local property, i.e. $M = 0 \iff M_P = 0 \forall P \in \text{Spec}(R) \iff M_m = 0 \forall m \in \text{MaxSpec}(R)$. To prove this, it is sufficient to verify that $M_m = 0 \forall m \in \text{MaxSpec}(R) \Rightarrow M = 0$. Assume that $M \neq 0$ and choose $z \in M$ nonzero. Consider $Rz \subset M$, a nonzero subset of M . Well, $f : R \rightarrow Rz$ is a surjection sending $1 \mapsto z$, implying that $Rz \cong R / \ker(f)$. $\ker(f)$ is a proper ideal of R , so it is contained in a maximal ideal m . $(Rz)_m \subset M_m = 0$, so $(Rz)_m$ is zero. However, this suggests that $(R/J)_m \neq 0$, a contradiction. Flatness is also an example of a local property, but we'll leave the proof of that for later.

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08/29/2022

4.1 Local Properties

Last time we showed that being zero is a local property. Here are some consequences of this.

Lemma 4.1.1. *Following from zero being a local property,*

(a) *Injectivity (and surjectivity) are local properties.*

(b) *Being flat is a local property.*

Proof. (a) follows immediately from the fact that $\ker(\varphi)$ and $\text{coker}(\varphi)$ being zero are local properties. As injectivity is locally preserved, it follows that flatness is locally preserved, as a module M is flat $\iff - \otimes M$ preserves injections. \square

Suppose that \mathcal{P} is a property for rings or for modules. The \mathcal{P} -locus for a module M is $= \{P \in \text{Spec}(R) \mid M_P \text{ has } \mathcal{P}\}$. We'd usually like these to be open, as those localizations that are not having \mathcal{P} can then be cut out by a closed set.

As an example, Say $n \geq 1$ is an integer, and M is a finitely generated R -mod. Let \mathcal{P} be that M is generated by at most n -elements. Then the \mathcal{P} -locus is $U_n := \{P \in \text{Spec}(R) \mid M_P \text{ is generated by } \leq n \text{ elements over } R_P\}$. We claim that U_n is open.

Proof. Pick any $p \in U_n$. We want to find an open neighborhood $U \ni p$ such that $U \subset U_n$. Say $\frac{z_1}{s_1}, \dots, \frac{z_n}{s_n}$ generate M_p . Then, z_1, \dots, z_n generate M_p , as $\frac{1}{s_i} \in R_p$. Now define $\varphi : R^{\oplus n} \rightarrow M$ such that $\varphi(e_i) = z_i$. So we have an exact sequence:

$$R^{\oplus n} \xrightarrow{\varphi} M \longrightarrow C = \text{coker}(\varphi) \longrightarrow 0$$

If we localize at P , then φ_P is surjective (i.e. this sequence is locally exact at any P). However, $\text{Supp}_R(C)$ As C is a quotient of a finitely generated module, it too is finitely generated. Thus, from homework we know that $\text{Supp}_R(C)$ is a closed set, so its complement is open. Thus, P lives in an open neighborhood within the complement U , which is an open neighborhood of P . $\forall Q \in U$, $C_Q = 0$. Thus, φ_Q is surjective, so $Q \in U_n$. Thus $P \in U \subset U_n$ for U open, so U_n is open. \square

Note that such a statement fails when M is not finitely generated. Consider $R = \mathbb{Z}$, $M = \bigoplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}$. In this case, $M_{(0)} = 0$, but $M_p = \mathbb{Z}/p\mathbb{Z} \neq 0$ for any prime p . Thus, $\text{Supp}_R(M) = \{(p) \mid p \text{ prime}\}$. We claim that this is not closed. If it were closed, the $\text{Supp}_R(M) = V(I)$ for some I . As \mathbb{Z} is a PID, $I = (n)$, so if this I were to exist, then p would need to divide n for all p , implying that n must be zero, but 0 is not in the support! A contradiction.

Now take $\varphi : R \rightarrow S$ a ring homomorphism. Then $\text{Spec}(\varphi)(Q) = \varphi^{-1}(Q)$. One can then ask, For $P \in \text{Spec}(R)$, What is $\text{Spec}(\varphi)^{-1}(P)$? (This is called the *fiber over* P). Well, suppose that $\text{Spec}(\varphi)(Q) = P$. Then $PS \subset Q$, so Q produces a prime S/PS . Furthermore, if $r \in R \setminus P$, then $\varphi(r) \notin Q$, so $\varphi(R \setminus P) \cap Q = \emptyset$. These two conditions are both necessary and sufficient. As P is prime, the complement is multiplicative, so its image under φ is multiplicative. As Q is disjoint from this multiplicative set, It follows that $(\varphi(R \setminus P))^{-1}Q$ is still prime in $(\varphi(R \setminus P))^{-1}S$.

Therefore, $Q \in \text{Spec}(\varphi)^{-1}(P) \iff Q \in \text{Spec}(\varphi(R \setminus P)^{-1}S/PS)$. Thus,

$$\text{Spec}(\varphi)^{-1}(P) = \text{Spec}(\varphi(R \setminus P)^{-1}S/PS) = \text{Spec}(R_P \otimes_R S/PS) = \text{Spec}(R_P \otimes_R R/P \otimes_R S)$$

$R_P \otimes R/P = \mathcal{K}(P)$, so it follows that

$$\text{Spec}(\varphi)^{-1}(P) = \text{Spec}(\mathcal{K}(P) \otimes S)$$

5

08/31/2022

5.1 Some Review

We started by reviewing the isomorphism $W^{-1}M = W^{-1}R \otimes_R M$, by showing that the R -module map $\frac{m}{w} \mapsto \frac{1}{w} \otimes m$ is an isomorphism. We also reviewed that tensor products commute with localization.

However, it is not always true that Hom commutes with localization. For instance, $\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$ so for $W = \mathbb{Z}^\times$ $W^{-1}\text{Hom}(\mathbb{Q}, \mathbb{Z}) = 0$, but $\text{Hom}_{W^{-1}\mathbb{Z}}(W^{-1}\mathbb{Q}, W^{-1}\mathbb{Z}) = \text{Hom}_{\mathbb{Q}}(\mathbb{Q}, \mathbb{Q}) = \mathbb{Q}$. It is true, however, that

Lemma 5.1.1. *If M is finitely presented, then*

$$W^{-1}\text{Hom}_R(M, N) \cong \text{Hom}_{W^{-1}R}(W^{-1}M, W^{-1}N)$$

Proof. Obviously true when $M = R$, and (slightly less obviously) true for $M = R^n$. Take a finite presentation for M and apply the left hand side as a functor (with input being M) to it, then apply the right hand side as a functor to the another copy of the presentation. Both are contravariant and right exact, so we get two new short exact sequences. As the first two elements of the short exact sequence are isomorphic, the last one is isomorphic, which is precisely the desired result. (the more general result is that, when M is finitely presented, Hom commutes with flat base change. We are just base changing by $W^{-1}R$, which is flat) \square

I didn't write down these proofs in detail since this is entirely review of past material.

It's worth asking when finitely presented and finitely generated mean different things. Finitely presented modules are always finitely generated, but not the converse (for instance, consider $R = k[x_1, \dots]$ and $M = k = R/(x_1, \dots)$. This is finitely generated, but the kernel of the surjective generator assigning morphism is (x_1, \dots) , which is infinitely generated).

6

09/02/2022

6.1 Minimal Primes

It's worth noting that R always admits minimal prime ideals. This follows from the fact that the intersection of a descending chain of prime ideals is prime.

Take $P_1 \supseteq P_2 \supseteq \dots$ a descending chain, and let $P = \bigcap P_i$. Assume that $ab \in P$. If $a, b \notin P$, there exists ℓ, m such that $a \notin P_\ell, b \notin P_m$. Let $n = \max(\ell, m)$. then $a, b \notin P_n$, but since $ab \in P, ab \in P_n$, a contradiction.

6.2 Chain Conditions

We spent a bit of time discussing ACC and DCC for modules. Over \mathbb{Z} ,

- \mathbb{Z} satisfies ACC but not DCC
- $\mathbb{Z}/2\mathbb{Z}$ satisfies ACC and DCC
- \mathbb{Z}_2 satisfies neither ACC nor DCC
- \mathbb{Z}_2/\mathbb{Z} satisfies DCC but not ACC

In general we have that

Theorem 6.2.1. M admits a composition series $\iff M$ satisfies ACC and DCC.

Proof. Suppose that M satisfies ACC and DCC. Consider the set of all proper submodules of M . ACC says that all ascending chains are bounded, so the union of proper submodules is proper. Zorn's Lemma implies that there exists a maximal element M_1 such that M/M_1 is simple (if the quotient were not simple, it would violate maximality). It follows that $M \supseteq M_1$. Apply this process to construct M_2 , and so on. As M satisfies DCC, $M \supseteq M_1 \supseteq M_2$ stabilizes, so we have a composition series.

The forward case is left as an exercise. □

Theorem 6.2.2. (Hilbert Basis Theorem) R is Noetherian $\iff R[x]$ is Noetherian $\iff R[[x]]$ is Noetherian.

7

09/07/2022

7.1 Noetherian and Artinian Rings

Theorem 7.1.1. *Every Artinian ring R is Noetherian.*

Proof. First, we check that there are finitely many maximal ideals. If there were infinitely many, then $m_1 \supset m_1m_2 \supset m_1m_2m_3 \supset \dots$ is a descending chain of maximal ideals. As R is Artinian, this chain halts; i.e. $\prod_{n \in \mathbb{N}} m_n = \prod_{i=1}^N m_i$ for some N . Thus, $m_1 \dots m_t \subseteq m_{N+1}$. Then $\exists i$ such that $1 \leq i \leq N$ such that $m_i \subset m_{N+1}$ by primality, and by maximality, $m_i = m_{N+1}$. It follows that there are only finitely many maximal ideals.

Thus, set $I = \mathcal{J}(R) = \bigcap_{i=1}^N m_i = \prod_{i=1}^N m_i$ (this follows from the fact that $I + J = R \Rightarrow I \cap J = IJ$, and the hypothesis holds for the m_i by maximality). We claim that $\exists t$ such that $I^t = 0$. We can construct a descending chain $I \supseteq I^2 \supseteq \dots$, suggesting that $\exists t$ such that $I^t = I^{t+1}$. Consider $J = (0 : I^t)$. If we show that $J = R$, then we are done. Well, $(J : I) = ((0 : I^t) : I)$, and as $((J : I) : K) = (J : IK)$, it follows that $((0 : I^t) : I) = (0 : I^{t+1}) = (0 : I^t) = J$. As $(J : I) = J$, it follows that if $J \neq R$, we can apply the minimal chain condition to construct a minimal element among all ideals that properly contain J . Call this minimal element J' and choose an element $x \in J' \setminus J$. By minimality, $J + (x) = J'$. $J' \neq xI + J$ (if so, by Nakayama $J'/J = 0$ so $J = J'$, a contradiction). Thus, $xI + J = J$, so $xI \subset J$, and thus $x \in (J : I) = J$, which is a contradiction.

Thus, we have that $I^t = m_1^t m_2^t \dots m_n^t = 0$. We'd like to use this to construct a composition series (which implies Noetherianity).

$$R \supseteq m_1 \supseteq m_1m_2 \supseteq \dots$$

This is a finite descending chain ending at 0. Each step is adding another maximal ideal product, i.e. it looks like $H \supseteq Hm_i$. Well, H/Hm_i is an R/m_i -module, and thus a vector space. To refine the sequence above into a composition series, it is enough to verify that this is finitely generated, as we can saturate the gaps in the chain with a composition series of generators. Well, H is an ideal in R that satisfies DCC, so H/Hm_i satisfies DCC. Thus, the module must be finitely generated. \square

This proof also highlights the structure theorem for Artinian Rings.

Lemma 7.1.2. *Every Artinian ring is a finite product of Artinian local rings.*

7.2 Zero Divisors

We call a prime ideal P is an *associated prime* of an R -module M if $P = \text{Ann}_R(m)$ for some $m \in M$. Equivalently, \exists a morphism of R -modules $R/P \hookrightarrow M$. If $rm = 0$ for $m \neq 0$, then r is a zero divisor on M . Otherwise, r is called M -regular. The set of associated primes of M is denoted by $\text{Ass}_R(M)$.

Theorem 7.2.1. *Assume R is noetherian, and $M \neq 0$.*

- (1) *The set $\{\text{Ann}_R(m) \mid m \in M\}$ admits maximal elements, each of which is an associated prime.*
- (2) *Zero divisors on M (i.e. those $r \in R$ such that $rm = 0$ for some $m \in M$) are precisely the set $\bigcup_{P \in \text{Ass}_R(M)} P$.*

Proof. (1) implies (2), so it is sufficient to check that (1) is true. ACC plus the standard Zorn's Lemma trick implies that we have maximal elements in the set above. Say $P = \text{Ann}_R(m)$ is a maximal element. Assume that $ab \in P$ but $a \notin P$. It is sufficient to check that $b \in P$. As $abm = 0$, $b(am) = 0$. $a \notin P$, so $am \neq 0$, so $b \in \text{Ann}_R(am)$. Everything that kills m must kill am , so it follows that $P = \text{Ann}_R(m) \subset \text{Ann}_R(am)$. By maximality, there must be equality, so $b \in P = \text{Ann}_R(m)$. \square

8

09/09/2022

2nd Homework due in two weeks, on September 23rd at 11:59PM.

8.1 More on Associated Primes

Recall that $P \in \text{Spec}(R)$ is called an associated prime if $R/P \hookrightarrow M$ (i.e. $P = \text{Ann}_R(m)$ for some $m \in M$). For example, consider $R = k[x, y]$ and $M = R/(x^2, xy)$. Then (x, y) is an associated prime of M , as $(x, y) = \text{Ann}_R(x)$. Similarly, (x) is an associated prime, as $(x) = \text{Ann}_R(y)$.

Theorem 8.1.1. *Suppose that R is Noetherian and M is an R -mod. Let W be a multiplicative set. Then $\text{Ass}_{W^{-1}R}(W^{-1}M) = \text{Ass}_R(M) \cap \text{Spec}(W^{-1}R)$ (here we identify $\text{Spec}(W^{-1}R)$ as a subset of $\text{Spec}(R)$)*

Proof. Choose $P \in \text{Ass}_R(M)$ such that P is disjoint with $W \subset R$. Thus, $P = \text{Ann}_R(m)$ for some $m \in M$. It is sufficient to check that $P = \text{Ass}_{W^{-1}R}(m/1)$. It's clear that $P \subset \text{Ass}_{W^{-1}R}(m/1)$. Now choose $r/w \in W^{-1}R$ such that $r/w \cdot m/1 = 0$. Then $\exists u \in W$ such that $urm = 0$ in M . Thus, $ur \in P = \text{Ann}_R(m)$. As $u \in W$ and $P \cap W = \emptyset$, $u \notin P$, so $r \in P$ by primality. It follows that $P \supset \text{Ass}_{W^{-1}R}(m/1)$, so we have equality.

Now let's go the other way. Choose $Q \in \text{Ass}_{W^{-1}R}(W^{-1}M)$. Thus $Q = \text{Ann}_{W^{-1}R}(m/w) = \text{Ann}_{W^{-1}R}(m)$. Say $P = Q \cap R$ (i.e. $Q = PW^{-1}R$). As R is Noetherian, $P = (g_1, \dots, g_t)$. For each g_i , $\exists u_i \in W$ such that $u_i g_i m = 0$. Set $u = \prod_{i=1}^t u_i$. Then $P = \text{Ann}_R(um)$, so $P \in \text{Ass}_R(M)$. As $P = Q \cap R$ and Q is disjoint from W , then P is disjoint from W . It follows that $P \in \text{Ass}_R(M) \cap \text{Spec}(W^{-1}R)$. \square

Lemma 8.1.2. *R is Noetherian, M an R -module, and $P \in \text{Spec}(R)$. Then, $P \in \text{Ass}_R(M) \iff PR_P \in \text{Ass}_{R_P}(M_P)$.*

Theorem 8.1.3. *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of R -modules. Then $\text{Ass}_R(M) \subset \text{Ass}_R(L) \cup \text{Ass}_R(N)$ and $\text{Ass}_R(L) \subset \text{Ass}_R(M)$.*

Note that the first inclusion can necessarily be strict. For example, consider $R = k[x]$, $L = (x)$, $M = R$, $N = R/(x) = K$. $\text{Ass}_R(R) = \{(0)\}$, but $\text{Ass}_R(N) = \{(x)\}$ and $\text{Ass}_R(L) = \{(0)\}$, so $\text{Ass}_R(M) \neq \text{Ass}_R(L) \cup \text{Ass}_R(N)$.

Theorem 8.1.4. (Prime Filtration) Suppose that R is Noetherian and M is nonzero and finitely generated. Then \exists a finite filtration of M

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

Where $M_{i+1}/M_i \cong R/P_i$ for $P_i \in \text{Spec}(R)$.

Proof. M is nonzero, so choose an associated prime P_0 . We know that $R/P_0 \hookrightarrow M$, so set $R/P_0 = M_1$. Consider M/M_1 ; if it is zero we're done. If not, then choose P_1 to be an associated prime of M/M_1 . Then $R/P_1 \hookrightarrow M/M_1$. Now choose $M_2 \subset M$ such that $M_2/M_1 \cong R/P_1$. Repeat this process until it halts (it indeed must halt due to Noetherianity). \square

Theorem 8.1.5. Suppose that R is Noetherian and M is nonzero and finitely generated as an R -module. Then,

- (a) $\text{Ass}_R(M) \subset \text{Supp}_R(M)$
- (b) $\text{Ass}_R(M)$ is finite.
- (c) The set of minimal associated primes are precisely the minimal primes in the support.

Note that part (b) shows that the inclusion in part (a) is usually strict. It's pretty rare that a module has finite support.

Proof. (a) If $P \in \text{Ass}_R(M)$, we have an injection $R/P \hookrightarrow M$. Localizing both sides at P , $(R/P)_P \hookrightarrow M_P$ is still an injection, but $(R/P)_P$ is a field, so $M_P \neq 0$. Thus, $P \in \text{Supp}_R(M)$.

- (b) Consider the prime filtration of M

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

Where $M_{i+1}/M_i \cong R/P_i$ for $P_i \in \text{Spec}(R)$. This implies that $\text{Ass}_R(M) \subset \{P_0, \dots, P_{n-1}\}$.

- (c) From part (a), it is sufficient to check that minimal primes in the support is an associated prime (minimality in $\text{Ass}_R(M)$ follows). Say $P \in \text{Supp}_R(M)$ is a minimal element. This, $\text{Ass}_{R_P}(M_P)$ is nonempty. But then

$$\text{Ass}_{R_P}(M_P) = \text{Ass}_R(M) \cap \text{Spec}(R_P) \subset \text{Supp}_R(M) \cap \text{Spec}(R_P)$$

By minimality of P in the support, this intersection is just $\{P\}$. It follows that $P \in \text{Ass}_{R_P}(M_P)$, so $P \in \text{Ass}_R(M)$. \square

This yields the following corollary:

Lemma 8.1.6. If R is Noetherian, then R admits finitely many minimal primes. Equivalently, every ideal of R has only finitely many minimal primes.

It follows from this that we can write $\text{Spec}(R) = \bigcup_{i=1}^n V(P_i)$, where P_i are the minimal primes. Furthermore, for M nonzero and finitely generated, $\text{Supp}_R(M)$ is a finite union of irreducible closed subsets of $\text{Spec}(R)$. The minimal elements of the support (or equivalently, the associated primes) are called the *isolated* associated primes of M . The non-minimal elements are called the *embedded* primes. For example, for $R = k[x, y]$ and $M = R/(x^2, xy)$, $\text{Ass}_R(M) = \{(x), (x, y)\}$ where (x) is isolated and (x, y) is embedded.

Suppose that Q is a proper ideal in R . We say that Q is *primary* if $ab \in Q$ implies that either $a \in Q$ or $b \in \sqrt{Q}$. For example, in \mathbb{Z} , the prime ideals are (p) for each prime number, but the associated primes are of the form (p^n) for p prime and $n \in \mathbb{N}$. As we'll see later, rings have a primary decomposition; over \mathbb{Z} this corresponds to taking the prime factorization of the ideal's generator (e.g. $(a) = (p_1^{n_1} \dots p_m^{n_m}) = (p_1^{n_1}) \cap \dots \cap (p_m^{n_m})$)

9

09/12/2022

9.1 More on Primary Ideals

Suppose R is a commutative ring. A proper ideal $Q \subset R$ is primary if $ab \in Q \Rightarrow a \in Q$ or $b \in \sqrt{Q}$. We discussed the examples $(p^n) \subset \mathbb{Z}$. Here are some (relatively) easy to prove lemmas about primary ideals.

Lemma 9.1.1. Q is primary if and only if in R/Q , every zero divisor is nilpotent.

Lemma 9.1.2. If Q is primary, then \sqrt{Q} must be prime.

Lemma 9.1.3. Suppose $f : R \rightarrow S$ is a ring homomorphism. If Q is primary in S , then $f^{-1}(Q)$ in R .

Theorem 9.1.4. Let R be a commutative ring and $Q \subsetneq R$. If \sqrt{Q} is maximal, then Q is primary.

Proof. Say $ab \in Q$ and $b \notin \sqrt{Q} = m$. We want to show that $a \in Q$. As $b \notin m$, $m + (b) = R$ by maximality of m . $\exists t \in m, x \in R$ such that $t + bx = 1$. As $t \in m = \sqrt{Q}$, $t^n \in Q$ for some n . Then,

$$1 = 1^n = (t + bx)^n = t^n + b(\dots)$$

So $a = a \cdot 1 = at^n + ab(\dots) \in Q$. $at^n \in Q$ as $t^n \in Q$, and $ab(\dots) \in Q$ as $ab \in Q$. It follows that their sum, a , is in Q . \square

Lemma 9.1.5. If m is a maximal ideal, then m^n is a primary ideal for any $n \in \mathbb{N}$.

This result does not hold for any old maximal ideal. To see this, let $R = \mathbb{Q}[x, y, z]/(xy - z^2)$, and $P = (x, z)$. $R/P = \mathbb{Q}[y]$ which is an integral domain, so P is prime. However, $P^2 = (x^2, xz, z^2)$. As $z^2 = xy$ in R , $P^2 = (x^2, xz, xy)$. It follows that $x \notin P^2$, but $xy \in P^2$, so if P^2 were primary, then y ought to be contained in $\sqrt{P^2} = P$, however, this is not the case. Thus, P is prime but P^2 is not primary.

9.2 Towards Primary Decomposition

Our goal is to show that primary decompositions exist for Noetherian rings, i.e. every ideal is a finite intersection of primary ideals.

9.2.1 Irreducible Ideals

An ideal I is said to be **Irreducible** if $I = I_1 \cap I_2 \Rightarrow I_1 = I$ or $I_2 = I$. For example, all prime ideals are irreducible.

Theorem 9.2.1. *Every proper ideal in a Noetherian ring R is a finite intersection of irreducible ideals.*

Proof. Let Σ be the set of proper ideals that cannot be a finite intersection of irreducible ideals. If $\Sigma \neq \emptyset$, then ACC tells us there exists a maximal element I . $I \in \Sigma$, so I is not irreducible. Thus, $\exists I_1, I_2 \subsetneq I$ such that $I = I_1 \cap I_2$. It follows that $I \subset I_1, I \subset I_2$, so by maximality of I , $I_1, I_2 \notin \Sigma$. Thus, I_1 and I_2 can be written as a finite intersection of irreducible ideals. If that is true, so can the intersection $I_1 \cap I_2$. However, $I = I_1 \cap I_2$, so we have a contradiction as we chose $I \in \Sigma$. \square

9.2.2 Primary Decomposition

Theorem 9.2.2. *Assume that R is Noetherian. Then, every irreducible ideal is primary.*

Proof. Assume I is irreducible and $ab \in I$. Suppose that $a \notin I$. We want to show that $b \in \sqrt{I}$. Thus, we need to show that $b^n \in I$. Consider the following chain

$$(I : b) \subseteq (I : b^2) \subseteq \dots$$

Via ACC, there exists t such that $(I : b^t) = (I : b^{t+1}) = \dots$. We claim that $I = (I + b^t) \cap (I + (a))$. It is clear that $I \subset (I + b^t) \cap (I + (a))$, so it is sufficient to check that $I \supset (I + b^t) \cap (I + (a))$. Choose $x = f + b^t y = g + az \in (I + b^t) \cap (I + (a))$, where $f, g \in I$. Then $bx = bf + b^{t+1}y = bg + baz$. $ba, f, g \in I$, so $b^{t+1}y = bg - bf + baz \in I$. Therefore, $y \in (I : b^{t+1}) = (I : b^t)$. Thus, $b^t y \in I$. therefore, $f \in I$ and $b^t y \in I$, so $x = f + b^t y \in I$. It follows that $I \subset (I + b^t) \cap (I + (a))$. Since I is irreducible and $I \neq I + (a)$ (as we chose $a \notin I$), it follows that $I = I + (b^t)$, so $b^t \in I$, so $b \in \sqrt{I}$. It follows that I is primary. \square

Beware! Primary does not imply irreducible in general. Consider $\mathbb{Q}[x, y]$ and $Q = (x^2, xy, y^2)$ is primary. as it is $(x, y)^2$, the power of a maximal ideal. However, Q is not irreducible, as $Q = (x^2, y) \cap (x, y^2)$. The above theorems, though, do tell us that primary decompositions exist over Noetherian rings.

Lemma 9.2.3. *P is a prime ideal, and Q_1, \dots, Q_t are all P -primary (i.e. for all $1 \leq i \leq t$, Q_i are primary and $\sqrt{Q_i} = P$). Then, $\bigcap_{i=1}^t Q_i$ is also P -primary.*

This can be used to remove redundancies in a primary decomposition. If two elements of a primary decomposition have the same radical, then we can just replace those two elements with their intersection.

Proof.

$$\sqrt{\bigcap_{i=1}^t Q_i} = \bigcap_{i=1}^t \sqrt{Q_i} = \bigcap_{i=1}^t P = P$$

It is sufficient to check that the intersection is primary. Choose $ab \in \bigcap_{i=1}^t Q_i$, but $a \notin \bigcap_{i=1}^t Q_i$. Then $\exists i$ such that $a \notin Q_i$ but $ab \in Q_i$. This means that $b \in \sqrt{Q_i} = P = \sqrt{\bigcap_{i=1}^t Q_i}$. It follows that $\bigcap_{i=1}^t Q_i$ is primary. \square

A primary decomposition $I = Q_1 \cap \cdots \cap Q_t$ is *minimal* (equivalently, *irredundant*) if:

- $\sqrt{Q_i}$ are all distinct primes.
- $\bigcap_{j \neq i} Q_j \not\subseteq Q_i$

10

09/14/2022

10.1 Uniqueness of Prime Decomposition

$I = Q_1 \cap \cdots \cap Q_t$ is a **minimal** primary decomposition if

- (1) $\sqrt{Q_i} \neq \sqrt{Q_j}$ for $i \neq j$
- (2) $\bigcap_{j \neq i} Q_j \not\subseteq Q_i$ for any i .

It's worth noting that a primary decomposition is not always truly unique. For instance, consider $I = (x^2, xy) = (x) \cap (x^2, y) = (x) \cap (x, y)^2$ in $\mathbb{Q}[x, y]$. However, we can construct two uniqueness theorems to show that primary decompositions can be unique in certain senses.

Lemma 10.1.1. *If Q is P -primary and $x \in R$, then*

- (1) $x \in Q \Rightarrow (Q : x) = R$.
- (2) $x \notin Q \Rightarrow (Q : x)$ is P -primary.
- (3) $x \notin P \Rightarrow (Q : x) = Q$.

Proof. Exercise. □

Theorem 10.1.2. (First Uniqueness Theorem) *Suppose $I = \bigcap Q_i$ is a minimal primary decomposition, where $P_i = \sqrt{Q_i}$. Then the P_i 's are precisely the prime ideals which occur in the set $\{\sqrt{(I : x)} \mid x \in R\}$*

Proof.

$$(I : x) = \left(\bigcap Q_i : x \right) = \bigcap (Q_i : x)$$

Taking a radical on both sides yields

$$\sqrt{(I : x)} = \sqrt{\bigcap (Q_i : x)} = \bigcap_{x \notin Q_i} P_i$$

By the previous lemma (and the fact that this intersection is necessarily finite). If this is prime, then $\sqrt{(I : x)} = P_i$ for some i by prime avoidance. By minimality, $\bigcap_{j \neq i} Q_j \not\subseteq Q_i$ for any i , so $\exists x_i \in \bigcap_{j \neq i} Q_j$ such that $x_i \notin Q_i$. Then, $\sqrt{(I : x_i)} = P_i$ by the previous lemma. Thus, all ideals of the form $\sqrt{(I : x)}$ look like P_i , and we can always find an x_i such that $\sqrt{(I : x_i)} = P_i$. The result follows. \square

Lemma 10.1.3. *Suppose $W \subset R$ is a multiplicative set, and $Q \subset R$ is a primary ideal.*

- (1) $W \cap P \neq \emptyset \Rightarrow W^{-1}Q = W^{-1}R$
- (2) $W \cap P = \emptyset \Rightarrow W^{-1}Q$ is $W^{-1}P$ primary, and $W^{-1}Q \cap R = Q$.

Combining (1) and (2) from this lemma, this gives us a 1-to-1 correspondence between primary ideals in $W^{-1}R$ and primary ideals disjoint from W via the maps $Q' \rightarrow Q' \cap R$ and $W^{-1}Q \leftarrow Q$.

Now suppose that R is Noetherian, so any ideal I will have a minimal primary decomposition $\bigcap_{i=1}^n Q_i$ satisfying the above results. We can rearrange the intersection such that $\sqrt{Q_1}, \dots, \sqrt{Q_t}$ are the isolated (minimal) primes, with associated minimal primes P_1, \dots, P_t . Set $W = R \setminus \bigcup_{i=1}^t P_i$. This is a multiplicative set, so we can localize by W . Then $W^{-1}I = \bigcap_{i=1}^t W^{-1}Q_i$, via the previous lemma. This gives rise to the second uniqueness theorem:

Theorem 10.1.4. (2nd Uniqueness Theorem) *When R is Noetherian and $I = \bigcap_{i=1}^n Q_i$ is a minimal prime decomposition with $\{\sqrt{Q_1}, \dots, \sqrt{Q_t}\}$ the set of isolated primes, then $Q_1 \cap \dots \cap Q_t$ is independent of the choice of the decomposition.*

the upshot of this is that, when determining a primary decomposition, the minimal primes will always be the same. It is only up to us, then, to determine the embedded primes that will build the primary decomposition.

10.2 Symbolic Powers

Let R be a Noetherian ring and $P \in \text{Spec}(R)$. We define the **n th symbolic power** $P^{(n)}$ to be $P^{(n)} := P^n R_P \cap R$. Equivalently, $P^{(n)} := \{r \in R \mid \exists s \notin P \text{ such that } sr \in P^n\}$.

Lemma 10.2.1. *Take R, P as above.*

- (1) $P^{(n)}$ is the P -primary component of P^n .
- (2) $P^{(m)} \subset P^{(n)}$ when $m \geq n$.
- (3) $P^{(m+n)}$ is the P -primary component of $P^{(m)}P^{(n)}$

P^2 is not always the same as $P^{(2)}$, though it usually is a subset. For example, consider $R = \mathbb{Q}[x, y, z]/(xy - z^2)$. Choose $P = (x, z)$. Then, $P^2 = (x^2, xz, z^2) = (x^2, xz, xy)$, so $x \notin P^2$. However, $y \notin P$ but $yx = z^2 \in P^2$, so $x \in P^{(2)}$. This shows that $P^2 \subsetneq P^{(2)}$ (i.e.

there are cases where we have strict containment).

For another example, consider $R = \mathbb{C}[x_{ij}]_{1 \leq i, j \leq 3}$, which corresponds to the space of 3×3 matrices, isomorphic to \mathbb{C}^9 . Let P be the prime ideal generated by the determinants of the 2×2 minors. We claim that, similarly to the first example, $P^2 \subsetneq P^{(2)}$. Set $\delta = \det(x_{ij})$. It is clear that $\deg(\delta) = 3$, so $\delta \notin P^2$ for degree reasons, as P^2 has generators entirely of degree 4. $x_{11} \notin P$ for similar degree reasons, but we claim that $x_{11}\delta \in P^2$, so $\delta \in P^{(2)}$. Let X be the matrix $\{x_{ij}\}_{1 \leq i, j \leq 3}$ with the 2nd and 3rd rows scaled by x_{11} . It follows that $\det(X) = x_{11}^2 \delta$, which can be written as $x_{11} \det(M)$ for M a 2×2 matrix with 2×2 minors. Dividing both sides by x_{11} , it follows that $x_{11}\delta = \det(M) \in P^2$, so $\delta \in P^{(2)}$, and $P^2 \subsetneq P^{(2)}$.

Theorem 10.2.2. (Zariski-Nagata) Suppose that $R = k[x_1, \dots, x_n] = k[[x_1, \dots, x_n]]$, where k is a perfect field, and P is a prime ideal. Then,

$$P^{(n)} = \{f \in R \mid (\partial_1^{a_1} \cdot \partial_n^{a_n})f \in P, \text{ where } \sum a_i \leq n - 1\}$$

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09/16/2022

11.1 More on Symbolic Powers

For $P \in \text{Spec}(R)$, $P^{(n)} := P^n R_P \cap R$ is the P -primary component of P^n . Symbolic powers are a major area of active research, with a variety of unsolved conjectures:

Theorem 11.1.1. (Eisenbud-Mazur) Let $R = \mathbb{C}[[x_1, \dots, x_n]]$, $P \in \text{Spec}(R)$, and $m = (x_1, \dots, x_n)$. Then, $P^{(2)} \subset mP$.

This is unsolved, so don't think about it too hard. One can also define a symbolic power of an arbitrary ideal I as follows:

$$I^{(n)} = \bigcap_{P \in \text{Ass}(R/I)} I^n R_P \cap R$$

11.2 Dimension Theory

For R Noetherian, we say the dimension of R is the length of the longest ascending chain of prime ideals. Here are some examples:

- If R is Artinian, then it is a product of Artinian local rings. Prime ideals are all maximal, so $\dim(R) = 0$.
- If R is a PID, then every **non-zero** prime ideal is maximal, so $\dim(R) = 1$.

Do note that Krull dimension of a Noetherian ring need not be finite. For instance, there is a famous example by Nagata illustrating this. Take an ascending chain of positive integers $a_1 < a_2 < a_3 < \dots$. For each a_i , take a_i variables $x_{i_1}, \dots, x_{i_{a_i}}$. Define $m = (x_{i_1}, \dots, x_{i_{a_i}})$. Define $S = K[x_{ij}]$ for all (infinitely many x_{ij}). Set $W = S \setminus \bigcup_{i=1}^{\infty} m_i$. $W^{-1}S$ is Noetherian, though it is hard to show. Notice that m_i are precisely the maximal ideals in $W^{-1}S$. For each m_i , there is a chain of prime ideals $(x_{i_1}) \subsetneq (x_{i_1}, x_{i_2}) \subsetneq \dots \subsetneq m_i$, so $\text{len}(m_i) = a_i - 1$. As there are infinitely many maximal ideals, there is no "longest" ascending chain, as one can always choose a_i large enough to surpass any supposed bound. Thus, $\dim(W^{-1}S) = \infty$, but $W^{-1}S$ is Noetherian.

This means that we can only apply dimension theory to a certain sub-class of rings; even Noetherian rings can break it! We'll start by proving results over rings of finite type over a field, and compute dimension via Noether Normalization. Then, we'll generalize to local rings, which are finite dimensional provided that the unique maximal ideal is finitely generated. Then we'll talk about Hilbert functions to discuss a theory of dimension, then compare it to the Krull case and see that they give the same answer.

11.2.1 Integral Extensions

Suppose that $R \subset S$ are rings. We call S a *ring extension* of R . We say that S is *integral* if any $s \in S$ satisfies a monic equation in $R[t]$ (more generally, if there is an element $s \in S$ that satisfies such an equation, we say that s is integral over R).

Lemma 11.2.1. *If R is a UFD and F is the field of fractions of R , then $c \in F$ is integral over $R \iff c \in R$.*

Proof. For the reverse direction, if $c \in R$ it clearly satisfies the monic equation $t - c \in R[t]$. It suffices to verify the forward direction. Given any $c = \frac{a}{b} \in F$, require that $\gcd(a, b) = 1$. Suppose it satisfies the monic polynomial

$$\left(\frac{a}{b}\right)^n + r_1 \left(\frac{a}{b}\right)^{n-1} + \cdots + r_n = 0$$

Clearing denominators, we get that

$$a^n + r_1 b a^{n-1} + \cdots + r_n b^n = 0$$

If this were to equal zero, then for the terms to cancel out $b|a^n$, suggesting that $b|a$. As $\gcd(a, b) = 1$, it follows that $b = 1$, so $\frac{a}{b} = a \in R$. \square

We say that an integral domain R is *integrally closed* (or *normal*) if an element in the fraction field of R is integral over $R \iff$ is it contained in R to begin with.

Lemma 11.2.2. *Consider $R = \mathbb{Q}[x, y]/(x^2 - y^3)$. This is not integrally closed.*

Proof. Notice that $\frac{x}{y}$ satisfies the polynomial $T^3 - x = 0$, as $\left(\frac{x}{y}\right)^3 = \frac{x^3}{y^3} = \frac{x^3}{x^2} = x$. \square

Theorem 11.2.3. *Suppose $R \subset S$ is a ring extension. Then,*

- (1) $s \in S$ is integral over $R \iff \exists R \subset C \subset S$ such that C is a finitely generated R -module, and $s \in C$.
- (2) Set \tilde{R} to be the collection of all $s \in S$ integral over R . Then, \tilde{R} is a subring of S .

Proof. (1) Let's check the forward direction first. Suppose that s satisfies $s^n + r_1 s^{n-1} + \cdots + r_n = 0$. Now consider the subring of S generated by R and s (i.e. take the polynomial ring $R[T]$ and consider the ring map $R[T] \rightarrow S$ such that $T \mapsto s$). We claim that the image of this morphism, denoted $R[s]$, is the desired subring.

Notice that $T^n + r_1 T^{n-1} + \dots + r_n$ is contained in the kernel, as s satisfies the polynomial. this implies that $\frac{R[T]}{(T^n + r_1 T^{n-1} + \dots + r_n)} \twoheadrightarrow R[s]$, so $R[s]$ is a finitely generated R -module.

For the reverse direction, assume there is a C such that $R \subset C \subset S$ such that C is a finitely generated R -module and $s \in C$. We want to show that s satisfies a monic polynomial $R[t]$. Pick some finite set of generators $C = \langle c_1, \dots, c_n \rangle$. $sc_i = \sum r_{ij} c_j$ defines a matrix of the generators. Thus, $(sI_{n \times n} - [r_{ij}]) [c_1 \dots c_n] = 0$. As this kills all generators, it follows that $\det(sI_{n \times n} - [r_{ij}]) = 0$, but this is a monic polynomial with variable s and coefficients in R , so we can conclude that s is integral over R .

- (2) It suffices to check that \tilde{R} is closed under algebraic operations, i.e. we must verify that $s_1 s_2, s_1 \pm s_2 \in \tilde{R}$. As s_1 is integral over R , then $R[s_1]$ is a finitely generated R -module. s_2 is integral over R , so it is integral over $R[s_1]$. This implies that $R[s_1, s_2]$ is finitely generated over $R[s_1]$, so it is finitely generated over R . It follows that any algebraic combination of s_1, s_2 will be integral over R , so by (1), each algebraic combination of s_1, \dots, s_2 are in this subring, so we are integral over R . □

\tilde{R} is called the *integral closure* of R in S .

Lemma 11.2.4. *Suppose that W is a multiplicative set in R .*

- (1) *If $R \subset S$ is an integral extension, then $W^{-1}R \subset W^{-1}S$ is too.*
(2) *If \tilde{R} is the integral closure of R in S , then $W^{-1}\tilde{R}$ is the integral closure of $W^{-1}R \subset W^{-1}S$.*

Proof. (1) Given any $\frac{s}{w} \in W^{-1}S, s \in S \Rightarrow s^n + r_1 s + \dots + r_n = 0$, so $\frac{s^n}{w^n} + r_1 \frac{s^{n-1}}{w^n} + \dots + \frac{r_n}{w^n} = 0$. Therefore,

$$\left(\frac{s}{w}\right)^n + \frac{r_1}{w} \left(\frac{s}{w}\right)^{n-1} + \dots + \frac{r_n}{w^n} = 0$$

Each coefficient is in $W^{-1}R$, so we can conclude.

- (2) For the forward direction, $W^{-1}\tilde{R}$ is in the integral closure of $W^{-1}R \subset W^{-1}S$. Take $\frac{s}{w}$ to be integral over $W^{-1}R$. For $s \in S, w \in W$,

$$\left(\frac{s}{w}\right)^n + \frac{r_1}{w_1} \left(\frac{s}{w}\right)^{n-1} + \frac{r_n}{w_n} = 0$$

Multiplying by $(w w_1 w_2 \dots w_n)^n$, we get

$$(w_1 \dots w_n s)^n + r_1 w_1^{n-1} \dots w_n s^{n-1} + \dots = 0$$

Redefining coefficients we see that

$$(w_1 \dots w_n s)^n + r_1 w_1^{n-1} \dots w_n s^{n-1} + \dots = 0$$

$$(w_1 \dots w_n s)^n + r'_1 (w_1 \dots w_n s)^{n-1} + \dots + r'_n = 0$$

So $w w_1, \dots, w_n s$ is integral over R . Therefore,

$$\frac{s}{w} = \frac{w_1 \dots w_n s}{w_1 \dots w_n w} \in W^{-1} \tilde{R}$$

□

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09/19/2022

12.1 Integral Extensions

Last time, we proved that for an integral extension $R \rightarrow S$ and ideal $J \subset S$ and multiplicative set $W \subset R$, $W^{-1}R \rightarrow W^{-1}S$ and $R/(J \cap R) \rightarrow S/J$ are still integral.

Lemma 12.1.1. $R \subset S$ is an integral extension and S is an integral domain. Then R is a field $\iff S$ is a field.

Lemma 12.1.2. For the forward case, $\forall s \neq 0$ in S , $s^n + r_1s^{n-1} + \dots + r_n = 0$ for $r_n \neq 0$. It follows that $s(s^{n-1} + r_1s^{n-2} + \dots + r_{n-1}) = -r_n \neq 0$. As R is a field, $-r_n$ has an inverse. Thus $s(s^{n-1} + r_1s^{n-2} + \dots + r_{n-1})(-r_n^{-1}) = 1$, so s is invertible. It follows that S is a field.

Conversely, $\forall r \neq 0$ in $R \subset S$, r has an inverse $r^{-1} \in S$. r^{-1} is integral over R , so it satisfies the polynomial $(r^{-1})^n + r_1(r^{-1})^{n-1} + \dots + r_n = 0$. Multiplying by r^{n-1} , we have that $r^{-1} + r_1 + \dots + r^{n-1} = 0$, so $r^{-1} = -(r_1 + \dots + r^{n-1}) \in R$.

Lemma 12.1.3. Suppose that $R \subset S$ is an integral extension.

- (1) If $\mathfrak{n} \subset S$ is a maximal ideal, then so is $\mathfrak{n} \cap R$ over R .
- (2) For every maximal ideal $\mathfrak{m} \subset R$, there is a maximal ideal $\mathfrak{n} \subset S$ such that $\mathfrak{n} \cap R = \mathfrak{m}$.

Proof. For $\mathfrak{m} = \mathfrak{n} \cap R$. $R/\mathfrak{m} \rightarrow S/\mathfrak{n}$ is an integral extension, and since S/\mathfrak{n} is a field, the previous result says that R/\mathfrak{m} is a field, so \mathfrak{m} is maximal. This proves (1).

For (2), it suffices to show that $\mathfrak{m}S$ is proper. If it were not proper, then $1 = \sum m_i s_i$ for $m_i \in \mathfrak{m}, s_i \in S$. Considering the subring $C = R[s_1, \dots, s_n]$, since each s_i is integral over R , C is a finitely generated R -module and $\mathfrak{m}C = C$. Thus, $\exists r \in R$ such that $r \equiv 1 \pmod{\mathfrak{m}}$, and $rc = 0$. Thus, $1 \in C \Rightarrow r = 0$, so $1 \in \mathfrak{m}$, a contradiction. \square

Theorem 12.1.4. Suppose that $R \subset S$ is an integral extension, and $P \in \text{Spec}(R)$.

- (1) **(Lying Over)** $\exists Q \in \text{Spec}(S)$ such that $Q \cap R = P$.
- (2) **(Incomparability)** If $Q_1, Q_2 \in \text{Spec}(S)$ and $Q_1 \cap R = Q_2 \cap R$, then neither is contained in the other.

Proof. As $R \subset S$ is integral, $R_P \subset (R \setminus P)^{-1}S$ is integral, where P is the unique maximal ideal of R_P . It follows that $\exists (R \setminus P)^{-1}Q \in \text{Spec}((R \setminus P)^{-1}S)$ such that $(R \setminus P)^{-1}Q \cap R_P = PR_P$ by the previous lemma. It follows from this that $Q \cap R = P$.

For the incomparability theorem, In $R_P \subset (R \setminus P)^{-1}S$, each $(R \setminus P)^{-1}Q$ lying over PR_P is a maximal ideal. Under the map $R_P/PR_P \hookrightarrow (R \setminus P)^{-1}S/(R \setminus P)^{-1}Q$, this is still integral, and as $(R \setminus P)^{-1}S/(R \setminus P)^{-1}Q$ is an integral domain and R_P/PR_P is a field, it follows that $(R \setminus P)^{-1}S/(R \setminus P)^{-1}Q$ is a field, so $(R \setminus P)^{-1}Q$ is maximal. Thus, we don't have containment in the localization, so we don't have containment in general. \square

Theorem 12.1.5. (Going Up Theorem) $R \subset S$ is an integral extension. Suppose $P_1 \subsetneq P_2 \subsetneq \dots$ is a chain of prime ideals in R . Then \exists a chain of prime ideals in S of the form $Q_1 \subsetneq Q_2 \subsetneq \dots$ such that $Q_i \cap R = P_i$ for all $i \in \mathbb{N}$.

Proof. By inducting on the length of the chain $\{P_i\}$, it suffices to prove the case where we have the chain $P_1 \subsetneq P_2$. By lying over, $\exists Q_1 \in \text{Spec}(S)$ such that $Q_1 \cap R = P_1$. $R/P_1 \rightarrow S/Q_1$ is an integral extension, and P_2/P_1 is a prime ideal in R/P_1 . Lying over implies that $\exists Q_2/Q_1 \in \text{Spec}(S/Q_1)$ such that $(Q_2/Q_1) \cap R/P_1 = P_2/P_1$, and so the claim is proven. \square

Theorem 12.1.6. $R \subset S$ is an integral extension of Noetherian rings. Then, $\dim(R) = \dim(S)$.

Proof. Given any chain $Q_0 \subsetneq Q_1 \subsetneq \dots$ of prime ideals in S , the chain $(Q_0 \cap R) \subsetneq (Q_1 \cap R) \subsetneq \dots$ in R . It follows that $\dim(S) \leq \dim(R)$, as any chain can be dropped down. The Going Up Theorem allows us to lift any chain, so $\dim(S) \geq \dim(R)$. It follows that $\dim(S) = \dim(R)$. \square

Assume that R is of finite type over a field (i.e. $R = k[x_1, \dots, x_n]/J$). Noether Normalization tells us that there are $y_1, \dots, y_d \in R$ that are algebraically independent over k , and R is integral over $k[y_1, \dots, y_d]$. It follows that we only really need to study the krull dimension of polynomial rings in d variables.

Lemma 12.1.7. Suppose k is a field, and $f \in k[x_1, \dots, x_n]$, where $\deg(f) > 0$. Then, $\exists y_1, \dots, y_{n-1} \in k[x_1, \dots, x_n]$ such that $k[y_1, \dots, y_{n-1}, x_n] = k[x_1, \dots, x_n]$, and $f = cx_n^m + \sum_{i=0}^{m-1} g_i x_n^i$ for $g_i \in k[y_1, \dots, y_{n-1}]$ and $c \neq 0$.

Proof. Let's write

$$f = \sum c_\alpha x^\alpha$$

For $x_\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ (for simplicity). Consider $\Sigma = \{(\alpha_1, \dots, \alpha_n) \mid c_\alpha \neq 0\}$. It's clear that $|\Sigma| < \infty$. $\forall \alpha, \beta \in \Sigma$, consider $(\alpha_1 - \beta_1)t + (\alpha_2 - \beta_2)t^2 + \dots + (\alpha_{n-1} - \beta_{n-1})t^{n-1} + (\alpha_n - \beta_n) = 0$. There are finitely many such equations, so only finitely many solutions t to these equations.

Pick a large enough integer ℓ such that ℓ is NOT a solution to any equation of the form above, for any $\alpha, \beta \in \Sigma$. Set $y_i = x_i - x_n^{\ell^i}$. So, $x_i = y_i + x_n^{\ell^i}$. Thus, $k[y_1, \dots, y_{n-1}, x_n] = k[x_1, \dots, x_n]$ as desired. Furthermore, $f = \sum c_\alpha \left(\prod_{i=1}^{n-1} (y_i + x_n^{\ell^i})^{\alpha_i} \right) x_n^{\alpha_n}$, so set $m = \max\{\alpha_1 + \ell + \alpha_2 \ell^2 + \dots + \alpha_{n-1} \ell^{n-1} + \alpha_n\}$. \square

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09/21/2022

13.1 More Towards Noether Normalization

Lemma 13.1.1. *Suppose $R = k[x_1, \dots, x_n]$ for k any field, and choose $f \in R$ such that $\deg(f) > 0$. Then there are y_1, \dots, y_{n-1} and $y_n = f$ such that:*

- (1) y_1, \dots, y_n are algebraically independent.
- (2) R is integral over $k[y_1, \dots, y_n] = A$
- (3) $(f) \cap A = (y_n)$

Proof. By last class's lemma, there are y_1, \dots, y_{n-1} such that $k[y_1, \dots, y_{n-1}, x_n] = k[x_1, \dots, x_n]$ and $f = cx_n^m + \{\text{lower degree terms}\}$. Then y_1, \dots, y_{n-1}, x_n are algebraically independent. Replacing x_n with f , we see that $y_1, \dots, y_{n-1}, f = y_n$ are still algebraically independent. It follows immediately from this construction that R is integral over $k[y_1, \dots, y_n] = A$.

For part 3, assume that $g \in k[y_1, \dots, y_n]$ and $g \in (f) \cap A$. This implies that $g = hf \in A$ for $h \in R$. We'd like to show that f divides g in A . R is integral over A , so there is an equation $h^t + a_1h^{t-1} + \dots + a_t = 0$, where every $a_i \in A$. This implies that $(hf)^t + a_1f(hf)^{t-1} + \dots + a_t f^t = 0$. This is an equation with variables still in A . As A is a polynomial ring over a field, A is a UFD. f divides $a_1f(hf)^{t-1} + \dots + a_t f^t$, so it divides $(hf)^t = g^t$, so $f|g$, as A is a UFD. It follows that $(f) \cap A \subset (y_n)$, and equality is clear. \square

Theorem 13.1.2. *$R = k[x_1, \dots, x_n]$ for k a field. Choose a proper ideal $I \subsetneq R$. Then there are $y_1, \dots, y_n \in R$ and t where $0 \leq t \leq n$ such that:*

- (1) R is integral over $A = k[y_1, \dots, y_n]$
- (2) y_1, \dots, y_n are algebraically independent over k .
- (3) $I \cap A = (y_{t+1}, \dots, y_n)$.

This is a strengthening of the previous result; whereas the previous lemma shows this is true for I principal, this shows the result holds for any proper ideal I .

Proof. If $I = (0)$, the result is clear. Thus, let's induct on n (the number of variables). If $n = 1$, $k[x_1]$ is a PID so all ideals are principal. Then we can conclude by the previous lemma. We now move to the inductive step. Pick any non-constant polynomial $f \in I$. Then, the previous lemma says there are $z_1, \dots, z_{n-1}, z_n = f$ such that the $R = k[z_1, \dots, z_n]$, z_1, \dots, z_n are algebraically independent, and $(f) \cap k[z_1, \dots, z_n] = (z_n)$, all via the previous lemma.

Now consider the ideal $I \cap k[z_1, \dots, z_{n-1}]$ in $k[z_1, \dots, z_{n-1}]$. The inductive hypothesis tells us that we can find new variables $y_1, \dots, y_{n-1} \in k[z_1, \dots, z_{n-1}]$ such that $k[z_1, \dots, z_{n-1}]$ over $k[y_1, \dots, y_{n-1}]$, the y_i are algebraically independent, and

$$I \cap k[z_1, \dots, z_{n-1}] \cap k[y_1, \dots, y_{n-1}] = (y_{t+1}, \dots, y_{n-1})$$

Setting $y_n = f = z_n$, then we are done, as we have all the y -vars we need. Part (1) and (2) are immediate, as we are taking an integral extension of an integral extension, which is still an integral extension (and similarly for algebraic independence). For part (3), it is clear that $(y_{t+1}, \dots, y_n) \subset I \cap A$. We just need to show \supset . Choose $g \in I \cap A$. Then $g = \sum c_\alpha y^\alpha$ for $y^\alpha = \prod_{i=1}^n y_i^{\alpha_i}$. Write $g = g_1 + y_n g_2$. $g_1 \in k[y_1, \dots, y_{n-1}]$, so $g - g_1 - y_n g_2 \in I \cap k[y_1, \dots, y_{n-1}] = (y_{t+1}, \dots, y_{n-1})$. $g, y_n g_2 \in I$. Thus, $g \in (y_{t+1}, \dots, y_n)$. \square

Theorem 13.1.3. $\dim(k[x_1, \dots, x_n]) = n$.

Proof.

$$0 \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \dots \subsetneq (x_1, \dots, x_n)$$

This is an ascending chain of prime ideals. Thus, $\dim(k[x_1, \dots, x_n]) \leq n$. For the other direction, suppose that we have an ascending chain $0 = P_0 \subsetneq P_1 \subsetneq \dots$. Apply the previous theorem to $P_1 \subsetneq R$. There are y_1, \dots, y_n such that $k[x_1, \dots, x_n]$ is integral over $k[y_1, \dots, y_n] = A$, y_1, \dots, y_n are algebraically independent, and $P_1 \cap A = (y_{t+1}, \dots, y_n)$. $k[x_1, \dots, x_n]/P_1$ is integral over $A/P_1 \cap A = k[y_1, \dots, y_t]$. As $P_1 \neq 0$, $P_1 \cap A \neq 0$ by the incomparability theorem. It follows that $t < n$. This tells us that $\text{trdeg}(\text{Frac}(R/P_1)) = t = \text{trdeg}(k(y_1, \dots, y_t)) < n$, it follows that the transcendence degree strictly decreases along the sequence $R/P_1, R/P_2, \dots$. As the maximum such degree is less than n , it follows that $\dim(k[x_1, \dots, x_n]) \leq n$, so we have equality. \square

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09/23/2022

14.1 Finally Proving Noether Normalization

Theorem 14.1.1. (Noether Normalization) Suppose R is a finitely generated k -algebra (k any field) and $I \subset R$. Then there are $y_1, \dots, y_d \in R$ such that:

- (1) y_1, \dots, y_d are algebraically independent over k .
- (2) R is integral over $k[y_1, \dots, y_d] = A$.
- (3) $I \cap A = (y_{t+1}, \dots, y_d)$ for some t .

Proof. R is a quotient of a polynomial ring, defined via the quotient $\varphi : k[x_1, \dots, x_N] \twoheadrightarrow R$ that maps x_i to each generator of R as a k -algebra. Applying the previous theorem to $\ker(\varphi) \subset k[x_1, \dots, x_N]$ to find z_1, \dots, z_N in $k[x_1, \dots, x_N]$ such that they are algebraically independent, R is integral over $k[z_1, \dots, z_N]$, and $\ker(\varphi) \cap k[z_1, \dots, z_N] = (z_{d+1}, \dots, z_N)$. It follows that $\varphi(z_1), \dots, \varphi(z_d)$ are algebraically independent over k . Now consider $k[z_1, \dots, z_d]$. Apply the theorem now to $\varphi^{-1}(I) \cap k[z_1, \dots, z_d]$ as an ideal inside $k[z_1, \dots, z_d]$. This gives us $w_1, \dots, w_d \in k[z_1, \dots, z_d]$ such that they are algebraically independent, R is integral over $k[w_1, \dots, w_d]$, and $\varphi^{-1}(I) \cap k[w_1, \dots, w_d] = (w_{t+1}, \dots, w_d)$. Set $y_i = \varphi(w_i)$. \square

14.2 Going Down

Assume that $R \hookrightarrow S$ is a ring extension. We say the **Going Down Theorem** holds if for any pair $P_1 \supseteq P_2$ in $\text{Spec}(R)$ and $Q_1 \in \text{Spec}(S)$ such that $Q_1 \cap R = P_1$. Then there is a $Q_2 \in \text{Spec}(S)$ such that $Q_1 \supseteq Q_2$ and $Q_2 \cap R = P_2$. Going Down holds when:

- $R \subset S$ where S (and therefore R) is an integral domain, R is integrally closed, and S is integral over R . (This situation holds for an integral domain R that's of finite type over k , i.e. it holds for R over its Noether Normalization).
- If S is a flat R -algebra.

We won't prove that this holds in the first case here (the proof is highly technical and, relatively speaking, a sort of black box). The second case is reasonably straightforward to prove and has put to homework.

14.3 Graded Rings

We spent some time after discussing Going Down reviewing facts about graded rings. On that note, for R any ring and $I \subset R$ a nontrivial, proper ideal. Then $\mathcal{R}(I) = R[It] := \bigoplus_{n \geq 0} I^n$ (where $I^0 = R$) is called the **Rees Algebra** (or, blow-up algebra) of R . It also has a grading with respect to which copy of I you are in. To keep track of degree, we adjoin a formal variable t to I and look at the algebra generated by that, determining degree of a term by computing it's t -degree, hence the use of the notation $R[It]$. For any R -module M , an analogous construction $M[It] := \bigoplus_{n \geq 0} I^n M$ is naturally a graded module over $R[It]$.

For another example of a graded ring, with R, I as before, we can define $\text{gr}_I(R) = \bigoplus_{n \geq 0} I^n / I^{n+1}$ to be the **graded ring associated to I** . We can similarly define the graded $\text{gr}_I(R)$ -module $\text{gr}_I(M) = \bigoplus_{n \geq 0} I^n M / I^{n+1} M$.

Theorem 14.3.1. (Artin-Rees Lemma) *Suppose R is Noetherian and $I \subsetneq R$. Suppose that M is a finitely generated R -module, and $N \subset M$. Then $\exists c \in \mathbb{N}$ such that $\forall n > c$, $(I^n M) \cap N = I^{n-c}(I^c M \cap N)$.*

Proof. R is Noetherian, so I is finitely generated. the result is trivial for $I = 0, R$, so suppose I is non-trivial and proper. Let $I = (g_1, \dots, g_n)$. $R[It] = R[g_1 t, \dots, g_n t]$ is then a finitely generated algebra over R , so it is Noetherian via the Hilbert Basis Theorem. M is finitely generated, so say that $M = \sum_{j=1}^{\ell} R \cdot z_j$. One can check that $M[It] = \sum_{j=1}^{\ell} R[It] \cdot z_j$. This is clearly Noetherian, so all submodules of $M[It]$ are finitely generated. It follows that, as $\bigoplus_{n \geq 0} I^n M \cap N$ is a graded submodule of $M[It]$, $\bigoplus_{n \geq 0} I^n M \cap N$ is finitely generated by $\alpha_1, \dots, \alpha_s$. Without loss of generality, suppose that these generators are homogeneous (otherwise, let our generating set comprise of their homogeneous components). Let $\deg(\alpha_i) = d_i$, and set $c = \max_i(d_i)$.

Then $\forall n > c$, each element in $I^n M \cap N$ (the degree n piece of $\bigoplus_{n \geq 0} I^n M \cap N$) has the form $\sum h_i \alpha_i$. Via degree concerns, it follows that $\deg(h_i) = n - d_i$, so $h_i \in I^{n-d_i}$. This implies that

$$I^n M \cap N \subset \sum I^{n-d_i}(I^{d_i} M \cap N)$$

Since $c \geq d_i$,

$$\sum I^{n-d_i}(I^{d_i} M \cap N) \subset I^{n-c}(I^c M \cap N) \Rightarrow \sum I^{n-d_i}(I^{d_i} M \cap N) \subset I^{n-c}(I^c M \cap N)$$

Thus, $I^n M \cap N \subset I^{n-c}(I^c M \cap N)$. To go the other way,

$$I^{n-c}(I^c M \cap N) \subseteq I^{n-c} I^c M = I^n M \subset I^{n-c} N \subset N$$

Giving us $I^n M \cap N \supset I^{n-c}(I^c M \cap N)$, so we have equality. □

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09/26/2022

15.1 More on Graded Rings

Lemma 15.1.1. Let $R = \bigoplus_{n \in \mathbb{N}} R_n$ be a graded ring. Then, the following are equivalent:

- (1) R is Noetherian.
- (2) R_0 is Noetherian and $R_+ = \bigoplus_{n > 0} R_n$ is a finitely generated ideal of R .
- (3) R_0 is Noetherian and R is a finitely generated R_0 -algebra.

Lemma 15.1.2. Let $R = \bigoplus_{n \in \mathbb{N}} R_n$ be a graded ring, and let $M = \bigoplus M_n$ is a graded Noetherian R -module. Then, M_n is a finitely generated R_0 -module.

Proof. $\forall \ell$, define $M_{\geq \ell} := \bigoplus_{n \geq \ell} M_n$. This is a graded submodule of M , and each $M_{\geq \ell}$ is finitely generated, so $M_\ell = M_{\geq \ell} / M_{\geq \ell-1}$ is finitely generated. Well, $M_{\geq \ell} / M_{\geq \ell-1}$ is annihilated by R_+ , so it is an R_0 module. \square

Lemma 15.1.3. If R_0 is Artinian and $M = \bigoplus M_n$ is Noetherian, then $\ell_{R_0}(M_n) < \infty$.

Proof. Follow proof of last lemma. \square

Theorem 15.1.4. (Graded Nakayama Lemma) Let $R = \bigoplus R_n, M = \bigoplus M_n$ where M is a finitely generated R -module. If $R_+ \cdot M = M$, then $M = 0$.

Proof. Say M is finitely generated by z_1, \dots, z_t (we can assume that these are homogeneous). Set $\delta = \min_i \deg(z_i)$. Then if $s < \delta$, then $M_s = 0$ and $M_\delta \neq 0$.

$$M_\delta \subseteq M = R_+ M = R_{>0} \cdot \bigoplus_{n \geq \delta} M_n \subseteq \bigoplus_{n \geq \delta+1} M_n$$

A contradiction. \square

15.2 Hilbert Series

Let $R = \bigoplus R_n$ where R_0 is Artinian, and let $M = \bigoplus M_n$ be a finitely generated graded R -module. Define the *Hilbert Series*, denoted $P(M, t)$ as follows:

$$P(M, t) := \sum_{n=0}^{\infty} \ell_{R_0}(M_n) t^n$$

Do note that if M has negative degree pieces, then there are only finitely many such pieces. Thus, we can shift the degrees up so that there are no terms of negative degree. Alternatively, you can just ignore the negative degree components, but remember to add the (finite) quantity $\ell_{R_0}(M_{<0})$ to the series. For example, $R = k[x_0, \dots, x_d]$, $M = R$. $R_0 = k$, which is a field, so Artinian. Then

$$\ell_{R_0}(R_n) - \dim_k(R_n) = \binom{n+d-1}{n}$$

This is polynomial in n , with degree $d-1$. It follows that $\ell_{R_0}(R_{\leq n})$ is a polynomial of degree d , and in particular,

$$P(M, t) = \frac{1}{(1-t)^d}$$

In general Hilbert series tend to look like this.

Theorem 15.2.1. *Suppose $R = \bigoplus_{n \geq 0} R_n$ is Noetherian and R_0 is Artinian, and $M = \bigoplus M_n$ is a finitely generated graded R -module. Then $P(M, t)$ is a rational function of the form*

$$P(M, t) = \frac{f(t)}{\prod_{i=1}^r (1-t)^{d_i}}$$

where $f(t) \in \mathbb{Z}[t]$ and $d_i = \deg(x_i)$ where R is generated by x_1, \dots, x_r over R_0 .

16

09/28/2022

16.1 More on Hilbert Polynomials

Let's compute some more examples. Consider $R = k[x, y, z]$ and $M = R/(xy - z^2)$ for k a field. We want to compute $\dim(M_n)$.

- $M_1 = \text{span}(x, y, z)$, so $\dim(M_1) = 3$.
- $M_2 = \text{span}(x^2, xy, y^2, xz, z^2)$, but $xy = z^2$. Thus, $\dim(M_2) = 5$.
- ...
- $M_n = \text{span}(x^n, x^{n-1}y, \dots, xy^{n-1}, y^n, x^{n-1}z, x^{n-2}yz, \dots, xy^{n-2}z, y^{n-1}z)$, so $\dim(M_n) = n + 1 + n = 2n + 1$.

Thus,

$$P(M, t) = \sum_{n=0}^{\infty} (2n + 1)t^n = \sum_{n=0}^{\infty} 2(n + 1)t^n - \sum_{n=0}^{\infty} t^n = \frac{2}{(1 - t)^2} - \frac{1}{1 - t} = \frac{1 + t}{(1 - t)^2}$$

16.2 Depth

Theorem 16.2.1. (R, m) is a local ring, and M is a finitely generated R -module. Then,

$$\text{gr}_m R = \bigoplus_{n \geq 0} m^n / m^{n+1}$$

It follows that

$$P(\text{gr}_m(M), t) = \sum_{n=0}^{\infty} \dim_k(m^n M / m^{n+1} M) t^n$$

is a rational function of the form $f(t)/(1 - t)^d$. We assume that $(1 - t)$ does not divide $f(t)$ if $d > 0$, making d canonically defined in terms of M . We call $d(M) = d$ the **depth** of M . Recall that the depth of a module is (as of now) only defined for a finitely generated module over a local ring.

We call $\chi_M(n) = \ell(M/m^{n+1}M)$ the **Samuel Function** of M . We see that when $f(t) = \sum_{i=0}^s a_i t^i$, we have

$$\chi_M(n) = \ell(M/m^{n+1}M) = \sum_{i=0}^n \dim_k(m^i M/m^{i+1}M) = a_0 \binom{d+n}{d} + \cdots + a_s \binom{d+n-s}{d}$$

It follows that $\chi_M(n)$ is always of degree d and has the same leading coefficient as $\dim_k(m^n M/m^{n+1}M)$.

Lemma 16.2.2. *Assume that*

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

is an exact sequence of finitely generated R -modules, where R is Noetherian and local. Then, $d(M) = \max(d(L), d(N))$.

Proof. We may assume that $N = M/L$, so $N/m^n N \cong M/(L + m^n M)$. Well,

$$\chi_N(n) = \ell(N/m^n N) = \ell(M/(L + m^n M))$$

$$\chi_M(n) = \ell(M/m^n M) = \ell(M/(L + m^n M)) + \ell((L + m^n M)/m^n N)$$

Letting $\ell((L + m^n M)/m^n N) =: \varphi(n)$, we see that $\chi_M(n) = \chi_N(n) + \varphi(n)$.

$$\varphi(n) = \ell((L + m^n M)/m^n N) = \ell(L/(L \cap m^n M))$$

We want to compare $\varphi(n)$ with $\chi_L(n) = \ell(L/m^n L)$. Since all functions here have positive values, adding them together causes no cancellation of terms. Thus, $d(M) = \max(d(N), \deg(\varphi))$. It is thus sufficient to check that $\deg(\varphi) = d(L)$. Applying the Artin-Rees Lemma to L , we see that $\exists c$ such that $\forall n \geq c$,

$$m^n M \cap L = m^{n-c}(m^c M \cap L) \subset m^{n-c}L$$

Thus, $\chi_L(n) \geq \varphi(n) \geq \chi_L(n-c)$, so it follows that these must have the same degree. \square

16.2.1 Relationship with Krull Dimension

Suppose that R is Noetherian and M is finitely generated. Recall that $\text{Supp}_R(M) = V(\text{Ann}_R(M)) = \text{Spec}(R/\text{Ann}_R(M))$. The **krull dimension** of M , denoted $\dim(M)$, is $\dim(\text{Supp}(M)) = \dim(R/\text{Ann}_R(M))$.

Lemma 16.2.3. $d(M) = \dim(M)$.

This does not necessarily hold for general M . For instance, consider the case where $R = \mathbb{Z}$, $M = \bigoplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}$. Then $\text{Supp}_R(M) = \{(p)\}$, so $\dim(\text{Supp}_R(M)) = 0$. However, $\text{Ann}_R(M) = 0$, so $\dim(M) = \dim(R/\text{Ann}_R(M)) = \dim(\mathbb{Z}) = 1$.

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09/30/2022

17.1 More on Depth

Suppose (R, m) is Noetherian and local, and M is a finitely generated R -module. We'd shown that, if $P(\text{gr}_m(M).t) = \frac{f(t)}{(1-t)^d}$, then $\chi_M(n)$ is a polynomial in n of degree d . We've also shown that $d = \dim(M)$.

Now define $\delta(M)$ to be the smallest n such that there are $x_1, \dots, x_n \in m$ such that $\ell(M/\sum x_i M) < \infty$. This is equivalent to checking that $(x_1, \dots, x_n) + \text{Ann}_R(M)$ is m -primary. For example, let $R = k[[x, y]]$ and let $M = R/(x^2, xy)$. Suppose $\delta(M) \leq 0$. Then $(y) \supseteq (y^2) \supseteq \dots$ is an infinite descending chain, so M is not finite length. So $\delta(M) > 0$. Suppose that $\delta(M) = 1$. Then $M/yM = k[[x]]/(x^2)$, so $\ell(M/yM) = 2$, and in particular, the length is finite. It follows that $\delta(M) = 1$.

Theorem 17.1.1. *(R, m) is Noetherian and local, with M a finitely generated R -module. Then $\dim(M) = \delta(M) = d(M)$.*

Proof. We'd like to show $\dim(M) \geq \delta(M) \geq d(M) \geq \dim(M)$ to conclude:

- $d(M) \geq \dim(M)$:

First, we check that this is true for R , i.e. $d(R) \geq \dim(R)$. As we know $d(R)$ is finite, we can induct on it. If $d(R) = 0$, then $P(R, t) = f(t) \in \mathbb{Z}[t]$. It follows that $m^n/m^{n+1} = 0$ for $n \gg 0$. It follows that $m^n = m^{n+1}m \cdot m^n$, so by Nakayama, $m^n = 0$. It follows immediately from this that R is 0-dimensional.

We now move to the inductive step. Assume $d(R) > 0$. We can assume that $\dim(R) > 0$. Consider $P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_t$, for $t = \dim(R)$. This is a saturated chain of prime ideals. Pick $x \in P_1 \setminus P_i$. Set $A = R/(P_0 + (x))$. This gives us the short exact sequence

$$0 \longrightarrow R/P_0 \xrightarrow{\cdot x} R/P_0 \longrightarrow A \longrightarrow 0$$

It follows that $d(A) \leq d(R/P_0) \leq d(R)$, so the degree of the polynomial associated with A is lower than the degree of the polynomial associated with R/P_0 . Thus, $d(A) < d(R/P_0) \leq d(R)$. By the inductive hypothesis, $d(A) \geq \dim(A)$, and in A ,

every prime ideal corresponds to a prime ideal in R containing $P_0 + (x)$. Given any chain of prime ideals in A , we can thus consider its preimage in R by just adding a P_0 to the left. Thus, $\dim(A) \leq \dim(R) - 1$. Taking the chain $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_t$ in R that we started with, then $\overline{P_1} \subsetneq \cdots \subsetneq \overline{P_t}$ is a saturated chain of prime ideals in A , where $\overline{P_i} = P_i / (P_0 + (x))$. Thus, $\dim(A) \geq t - 1 = \dim(R) - 1$. Thus,

$$\dim(R) - 1 = \dim(A) \leq d(A) \leq d(R) - 1$$

Adding 1 to both sides, we get $d(R) \geq \dim(R)$ as desired.

Using this, we want to convince ourselves that such an inequality holds for any finitely generated R -module M , not just R itself. For M , take a prime filtration

$$0 \subsetneq M_0 \subsetneq \cdots \subsetneq M$$

Such that $M_{i+1}/M_i = R/P_i$. Now recall that if $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is a short exact sequence, we proved that $\text{Supp}(M) = \text{Supp}(L) \cup \text{Supp}(N)$. It follows that $\dim(M) = \max(\dim(L), \dim(N))$. Considering now the short exact sequence $0 \rightarrow M_i \hookrightarrow M_{i+1} \twoheadrightarrow M_{i+1}/M_i \rightarrow 0$, As $M_{i+1}/M_i = R/P_i$, it follows that $d(M) = \max_i(d(R/P_i)) \geq \max_i(\dim(R/P_i)) = \dim(M)$.

- $\delta(M) \geq d(M)$:

Induct on $\delta(M)$ again. if $\delta(M) = 0$, then $\ell(M) < \infty$, so χ_M is bounded by $\ell(M)$, and in particular, it is constant, so $d(M) = 0$.

For the inductive step, now suppose that $\delta(M) = s > 0$. Then we can choose $x_1, \dots, x_s \in \mathfrak{m}$ such that $\ell(M/\sum x_i M) < \infty$. Set $M_i = M/(x_1 M + \cdots + x_i M)$. As $\ell(M_s) < \infty$, it follows that $\delta(M_s) = 0$.

$$\ell(M_1/m^{n+1}M_1) = \ell(M/(x_1 M + m^{n+1}M)) = \ell(M/m^{n+1}M) - \ell\left(\frac{x_1 M + m^{n+1}M}{m^{n+1}M}\right)$$

Proven via the taking the short exact sequence

$$0 \rightarrow \frac{x_1 m + m^{n+1}M}{m^{n+1}M} \rightarrow \frac{M}{m^{n+1}M} \rightarrow \frac{M}{x_1 M + m^{n+1}M} \rightarrow 0$$

Thus,

$$\ell(M/m^{n+1}M) - \ell\left(\frac{x_1 m + m^{n+1}M}{m^{n+1}M}\right) = \ell(M/m^{n+1}M) - \ell\left(\frac{x_1 M}{x_1 M \cap m^{n+1}M}\right)$$

Now consider $\theta : M \twoheadrightarrow \frac{x_1 M}{x_1 M \cap m^{n+1}M}$, defined by sending $z \mapsto x_1 z$. This implies that

$$M/\ker(\theta) \cong \frac{x_1 M}{x_1 M \cap m^{n+1}M}$$

The key insight now is that $x_1 M \subset x_1 M$ (obviously) and $x_1 M \subset m^{h+1}M$, so it follows that $m^n M \subset \ker(\theta)$, so

$$M/m^n M \twoheadrightarrow M/\ker(\theta) \cong \frac{x_1 M}{x_1 M \cap m^{n+1}M}$$

Therefore,

$$\ell\left(\frac{M}{m^n M}\right) = \ell\left(\frac{x_1 M}{x_1 \cap m^{n+1} M}\right) \implies \ell\left(\frac{M_1}{m^{n+1} M_1}\right) \geq \ell\left(\frac{M}{m^{n+1} M}\right) - \ell\left(\frac{M}{m^n M}\right)$$

Which is $\geq \chi_M(n) - \chi_M(n-1)$. It follows from this calculation that $d(M_1) = d(M) - 1$. Repeating this process, we get that $d(M_s) = d(M) - S$. Since $\delta(M_s) = 0$, $d(M_s) = 0$, so $d(M) \leq S = \delta(M)$. We'll do the last step next time.

□

18

10/03/2022

18.1 Finishing The Proof of Dimensional Equivalence

All that was left to show was that $\dim(M) \geq \delta(M)$. We can induct on $\dim(M)$, which is known to be finite, as it at most $d(M)$. If $\dim(M) = 0$, then $\text{Supp}(M) = \{m\}$, implying that $V(m) = V(\text{Ann}_R(m))$, so $m^n \subset \text{Ann}_R(m)$ for some n , and $m^n M = 0$. Thus, M has a composition series $M \supseteq mM \supseteq m^2M \supseteq \cdots \supseteq 0$. It follows that $\ell_R(M) < \infty$, so $\delta(M) = 0$.

For the inductive step, we now assume that $\dim(M) > 0$, so $\dim(R/\text{Ann}_R(M)) > 0$. We can now consider all minimal prime ideals of $\text{Ann}_R(M)$, denoted P_1, \dots, P_t , such that $\dim(R/P_i) = \dim(R/\text{Ann}_R(M))$ for any i . m is different from P_1, \dots, P_t , so by prime avoidance, $m \not\subseteq \bigcup P_i$ (if not, $\exists i$ such that $m \subseteq P_i$, so $m = P_i$ by maximality, a contradiction). Pick $x \in m$ such that $x \notin P_1, \dots, P_t$, and set $M_1 = M/xM$.

$$\dim(M_1) = \dim(\text{Supp}(M_1)) = \dim(V(\text{Ann}_R(M)) + (x)) = \dim(R/(\text{Ann}_R(M) + x))$$

This has dimension 1 less than the dimension of M , as we are avoiding the minimal primes of maximal chains, implying that all maximal chains will shorten by 1. It follows that

$$d(M/xM) \leq \dim(M) - 1$$

So by the inductive hypothesis, $\dim(M/xM) \geq \delta(M/xM)$. We claim that $\delta(M/xM) + 1 \geq \delta(M)$. This is because if

$$\ell_R(M_1/((x_1, \dots, x_s)M_1)) < \infty$$

then clearly

$$\ell_R(M/((x_1, \dots, x_s, x)M)) < \infty$$

Thus,

$$\delta(M) \leq \delta(M_1) + 1 \leq \dim(M_1) + 1 \leq \dim(M)$$

From this and yesterday's work, we can conclude that

$$d(M) = \delta(M) = \dim(M)$$

18.2 Height

Let R be a Noetherian ring and $P \in \text{Spec}(R)$. The **Height** of P is the supremum of the length of all strictly decreasing chains of prime ideals starting from P . In general, the height of an arbitrary ideal I is $\min(\text{ht}(P) \mid I \subset P)$. In a Noetherian ring, this is a finite number, and is denoted by $\text{ht}(P)$. From this, we get a very clear result:

Lemma 18.2.1. $\text{ht}(P) = \dim(R_P)$

Which we will use to prove the following:

Theorem 18.2.2. (Krull) Suppose R is Noetherian and $I = (a_1, \dots, a_r) \subset R$. If P is a minimal prime of I , then $\text{ht}(P) \leq r$.

Proof. Localize at P . We just need to check that $\dim(R_P) \leq r$. In R_P , PR_P is the only ideal containing IR_P , so $\sqrt{IR_P} = PR_P$, so because PR_P is maximal, we can conclude that IR_P is PR_P -primary. This implies that $\ell_{R_P}(R_P/IR_P) < \infty$, so $\delta(R_P) \leq r$. This implies that $\dim(R_P) \leq r$ as desired. \square

Note that this bound can be strict. For example, consider $R = k[x, y]$ and $I = (x^2, xy)$. We know $P = (x)$ to be a minimal prime of I . Our theorem tells us that $\text{ht}(P) \leq 2$, but in this case, $\text{ht}(P) = 1$.

Theorem 18.2.3. Suppose R is Noetherian, and $P \in \text{Spec}(R)$ has height h . Then P is a minimal prime of an h -generated ideal (a_1, \dots, a_h) , and for any $r_1, \dots, r_s \in P$, The height of $P/(r_1, \dots, r_s) \geq h - s$. More specifically,

$$\text{ht}(P/(a_{j_1}, \dots, a_{j_i})) = h - i$$

i.e. when killing a subset of the minimal prime generators, the inequality is sharp.

Proof. $\dim(R_P) = h$, so $\delta(R_P) = h$. Find $a'_1, \dots, a'_h \in PR_P$ such that $\ell_{R_P}(R_P/(a'_1, \dots, a'_h)) < \infty$. Thus, $\sqrt{(a'_1, \dots, a'_h)R_P} = PR_P$. We can write $a'_i = \frac{a_i}{s_i}$ for $a_i \in P, s_i \notin P$. It follows that P is minimal over (a_1, \dots, a_n) . If not (say $\exists Q$ strictly in between them and prime), then

$$PR_P \supsetneq QR_P \subset \sqrt{(a'_1, \dots, a'_n)R_P} = PR_P$$

A contradiction. Thus the first part is proved. For the second part, set $\bar{R} = R/(r_1, \dots, r_s)$ and $\bar{P} = P/(r_1, \dots, r_s)$. Set $\text{ht}_{\bar{R}}(\bar{P}) = t$. Then by the first part, $\exists \bar{c}_1, \dots, \bar{c}_t \in \bar{P}$ such that \bar{P} is minimal over $(\bar{c}_1, \dots, \bar{c}_t)$. Pick $c_1, \dots, c_t \in P$ such that c_i is a preimage of \bar{c}_i under the canonical projection $R \twoheadrightarrow R/(r_1, \dots, r_s)$. Thus, P is minimal over $(r_1, \dots, r_s, c_1, \dots, c_t)$, so $\text{ht}(P) \leq s + t$, so $t = \text{ht}(\bar{P}) \geq h - s$.

In the specific case where the ideal $\bar{P} = R/(a_1, \dots, a_i)$ is minimal over $(\bar{a}_{t+1}, \dots, \bar{a}_n)$ in $\bar{R} = R/(a_1, \dots, a_i)$, $\text{ht}(\bar{P}) \leq h - i$. Combining this with the previous result gives us that $\text{ht}(\bar{P}) = h - i$. \square

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10/05/2022

19.1 Dimension Theory when R_0 is a field.

We've discussed dimension theory for rings that are of finite type over a field and for local rings. What if these cases both held at once? Do the two theories hold? This overlap occurs precisely when $R_0 = k$ is a field, and with some work, we'll show that these do in fact agree.

Theorem 19.1.1. *Suppose $R = \bigoplus_{n \geq 0} R_n$ is a Noetherian graded ring.*

- (1) *If J is a homogeneous ideal and P is a minimal prime of J , then P is homogeneous.*
- (2) *If P is prime of height h and homogeneous, then there is a descending chain of homogeneous prime ideals starting from P such that*

$$P = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_n$$

Lemma 19.1.2. *Let k be a field, and set $R = k[x_1, \dots, x_n]/J$, where J is a homogeneous ideal. R is certainly of finite type over k . Let $m = (x_1, \dots, x_n)$ and $A = R_m$. Over A , the dimension theory viewing A as a ring of finite type over k is identical to the dimension theory viewing A as a local ring. More explicitly,*

- (1) $\dim_k(R_n) = \chi_A(n) - \chi_A(n-1)$
- (2) $\dim(R) = \text{ht}(m) = \dim(A) = \deg_n(\chi_A(n) - \chi_A(n-1)) + 1$
- (3) *If (R, m) is a Noetherian graded ring, then $\dim(R) = \dim(\text{gr}_m(R))$.*

Proof. These proofs are omitted due to their relative length and lack of insight. □

19.2 Regular Local Rings

Suppose (R, m) is a Noetherian local ring where M is a finitely generated R -module where $\dim(M) = d$. If $x_1, \dots, x_d \in m$ satisfies $\ell_R(M/(x_1 + \cdots + x_d)M) < \infty$, then x_1, \dots, x_d are a **system of parameters** of M . (If $M = R$, we just call these a system of parameters).

As M/mM is a finite dimensional R/m -vector space, it follows that $\ell_R(M/mM) < \infty$. Thus we can always pick a set of generators of m , denoted z_1, \dots, z_t . Then $\ell(M/\sum z_i M) < \infty$ and $d \leq t$. $t \leq \dim_{R/m}(m/m^2)$. Thus, $d \leq \dim_{R/m}(m/m^2)$. If $d = \dim_{R/m}(m/m^2)$, then (R, m) is a **regular local** ring (we call this regular because, as it turns out, this criterion is equivalent to saying that m is generated by a regular sequence). m can indeed be generated by $\dim(R)$ elements in m .

19.2.1 Examples

- $R = k[[x, y]]$ is regular, as $m = (x, y)$ is generated by 2 elements and $\dim(R) = 2$.
- Let $A = k[x, y]/(x^2, xy)$. $\dim(A) = 1$, but $\dim_{A/m}(m/m^2) = 2$, so this ring is NOT regular.

Now say that (R, m) is a Noetherian local (though not necessarily complete) ring, and x_1, \dots, x_d are a system of parameters. Then $\dim(R/(x_1, \dots, x_i)) = d - i$. (\leq is always true, you only get $=$ because this is a system of parameters). However, it is possible that $\text{ht}(x_1, \dots, x_i) < i$. As an example, let $R = k[[x, y, z]]/(xy, xz)$. This is a local ring of dimension 2. $y, x + z$ is a system of parameters, and indeed $R/(y, x + z) = k[[x, z]]/(xz, x + z) \cong k[[x]]/(x^2)$. However, $\text{ht}((y)) = 0$.

20

10/07/2022

20.1 Systems of Parameters over Regular Local Rings

Let (R, m) be a local ring and M an R -module such that $\dim(M) = d$. If $x_1, \dots, x_d \in m$ satisfies $\ell_R(M / \sum x_i M) < \infty$, then x is a system of parameters.

Lemma 20.1.1. *Suppose (R, m) is local and Noetherian. Then there exists a system of parameters $x_1, \dots, x_d \in m$ such that \forall subsets $S \subset \{x_1, \dots, x_d\}$, $\text{ht}((x_i \in S)) = \#S$*

Proof. As R is Noetherian, there are finitely many minimal primes. We may assume $\dim(R) > 0$, so m is NOT a minimal prime. Thus by prime avoidance, m is not contained in the union of all minimal primes. Pick $x_1 \in m$ such that x is not contained in the union of all the minimal primes. Since R is Noetherian, (x_1) has finitely many minimal primes. Via prime avoidance again, choose $x_2 \in m$ such that x is not contained in the minimal primes of R and the minimal primes of x_1 . Repeating this process yields our system of parameters. \square

(R, m) is Noetherian and local. Recall that for such R , R is **regular** if m can be generated by $\dim(R)$ elements (that is, $\text{ht}(m) = \dim_{R/m}(m/m^2)$). Pick any minimal set of generators x_1, \dots, x_d of m , for $d = \dim(R)$. Then, $R/(x_i)$ is clearly still regular and of dimension $d - 1$ by minimality.

Theorem 20.1.2. *Let (R, m) be a Noetherian regular local ring for $d = \dim(R)$. Then the following are equivalent:*

- (1) x_1, \dots, x_i is a subset of a minimal set of generators of m .
- (2) The image of $x_1, \dots, x_i \in m/m^2$ are linearly independent over R/m .
- (3) $R/(x_1, \dots, x_i)$ is a $d - i$ dimensional regular local ring.

Proof. (1) \Rightarrow (2) by Nakayama's lemma, (x_1, \dots, x_d) is a minimal set of generators of $m \iff$ their images in m/m^2 are a basis. It follows that any subset of elements of a basis are linearly independent. We now check that (2) \Rightarrow (3). We can extend x_1, \dots, x_i to $x_1, \dots, x_i, x_{i+1}, \dots, x_d$ to get a minimal set of generators of m . Well, $m = (x_1, \dots, x_d)$, so in

$R/(x_1, \dots, x_i)$, $m = (x_{i+1}, \dots, x_d)$. It follows from regularity that $R/(x_1, \dots, x_i)$ is a $d - i$ dimensional regular local ring. To conclude we check that (3) \Rightarrow (1). If $R/(x_1, \dots, x_i)$ is regular local, then its maximal ideal $m/(x_1, \dots, x_i)$ can be generated by $d - i$ elements. Set $x_{i+j} := y_j$. Then $m = (x_1, \dots, x_d)$. \square

Theorem 20.1.3. *If (R, m) is a regular local ring, then R is an integral domain.*

To prove this theorem, we need a more general form of prime avoidance. It turns out, if I have a set of ideals $\{I_j\}$ where at most 2 are NOT primes, then if $I \subset \bigcup I_j$ then $I \subset I_j$. (The prove this, we just change the base case slightly but otherwise replicate the previous prime avoidance proof). We now prove the above theorem:

Proof. Induct on $\dim(R)$. If $d = 0$, then $m = 0$ so R is a field, which is integral. For $d = 1$, m is not minimal, so pick a minimal prime P . $\forall y \in P, y \in P \subset m = (x)$. This implies that $y = ax$. $m \neq p$ implies that $x \notin P$, so by primality $a \in P$. $\forall y \in P, y \in xP$, so $P \subset xP \subset P$, so $P = xP$. By Nakayama's lemma, $P = 0$, so in particular, 0 is the unique minimal prime.

Now we move onto the inductive case $d \geq 2$. Say $\{P_1, \dots, P_t\}$ are the minimal primes of R . By (generalized) prime avoidance, m is not contained in the union of m^2, P_1, \dots, P_t . Choose $x \in m, x \notin m^2, P_1, \dots, P_t$. It's image in m/m^2 is thus nonzero. Therefore, $R/(x)$ is a regular local ring by the previous theorem, and is of dimension $d - 1$. Thus by the inductive step, we know that $R/(x)$ is an integral domain, so (x) is a prime ideal. Therefore, (x) contains some minimal prime P_j . $\forall y \in P_j, y \in (x)$, so $y = ax$ for some a . We now proceed similarly to how we did in the $d = 1$ case. $x \notin P_j$ so $a \in P_j$. Thus $\forall y \in P_j, y \in xP_j$, so $P_j \subset xP_j \subset P_j$, so $P_j = xP_j$. Thus, $P_j = 0$. \square

Theorem 20.1.4. *Assume (R, m, k) (for $k = R/m$) is a Noetherian local ring, and let $d = \dim(R)$. Then R is regular $\iff \text{gr}_m(R) = k[x_1, \dots, x_d]$.*

Proof. For the forward case, we assume R is regular. Then $d = \dim_k(m/m^2)$, so

$$\text{gr}_m(R) = \bigoplus_{n=0}^{\infty} m^n/m^{n+1} = k \oplus m/m^2 \oplus m^2/m^3 \oplus \dots$$

Define the map

$$\varphi : k[x_1, \dots, x_d] \rightarrow \text{gr}_m(R)$$

such that $\varphi(x_i) = g_i^* \in m/m^2$. This map is clearly surjective, so

$$\text{gr}_m(R) \cong k[x_1, \dots, x_d] / \ker(\varphi)$$

If $\ker(\varphi)$ contains any nontrivial polynomial f , then

$$k[x_1, \dots, x_d] / (f) \twoheadrightarrow k[x_1, \dots, x_d] / \ker(\varphi) \cong \text{gr}_m(R)$$

Well, this shows that $\dim \text{gr}_m(R) \leq d - 1$, but $\dim \text{gr}_m(R) = \dim R = d$, a contradiction. Thus, this $\ker(\varphi) = 0$, we have an isomorphism.

For the reverse case, if $\text{gr}_m(R) \cong k[x_1, \dots, x_d]$, then $m/m^2 \cong \bigoplus_{i=1}^d kx_i$ by a grading argument. Thus, $\dim_k(m/m^2) = d$. Thus by Nakayama's lemma, m can be generated by d elements, so R is regular. \square

From here, we hit a bit of a roadblock. We'd *like* to show that the localization of a regular ring at a prime ideal is regular, but this was an open problem for quite some time. Serre eventually proved this using heavy homological algebra machinery. Starting from next class, we'll be focusing more towards homological algebra with the focus towards proving this, and maybe later, Cohen Structure Theorem.

21

10/12/2022

21.1 Regular Sequences

A **regular sequence** $r_1, \dots, r_n \in R$ over an R -module M is defined recursively r_n is not a zero divisor in $M/(r_1, \dots, r_{n-1})M$. We'd hope that regular sequences "behave" as formal variables in some sense. If (R, m) is local or $R = \bigoplus R_n$ is graded and r_1, \dots, r_n is a regular sequence of positive degree terms (i.e. for $I = (r_1, \dots, r_n)$, $\bigcap_{n \in \mathbb{N}} I^n = 0$), then we'd like to show that any permutation of r_1, \dots, r_n is a regular sequence.

Suppose $\text{gr}_I(R)$ is generated by $r_1^*, \dots, r_n^* \in I/I^2$ over R/I . Define a map $\varphi : R[x_1, \dots, x_n] \rightarrow \text{gr}_I(R)$ such that $x_i \mapsto r_i^*$. This is a degree preserving ring homomorphism.

Theorem 21.1.1. *The following are equivalent:*

- (1) $(R/I)[x_1, \dots, x_n] \cong \text{gr}_I(R)$, i.e. $I[x_1, \dots, x_n] = \ker(\varphi)$
- (2) Given any homogeneous polynomial $F \in R[x_1, \dots, x_n]$ of degree d , If $F \in I^{d+1}$, then the coefficients of F are in I .

Sequences of the form r_1, \dots, r_n (being coefficients of F as in (2)) are called **quasi-regular**.

Theorem 21.1.2. *Suppose R is a commutative ring and r_1, \dots, r_n is a sequence of elements in R .*

- (1) *If r_1, \dots, r_n is a regular sequence, then it is a quasi-regular sequence.*
- (2) *When (R, m) is a local graded ring and r_1, \dots, r_n are homogeneous of positive degree, Then r_1, \dots, r_n being quasi-regular implies that it is indeed regular.*

Proofs of these were left as an exercise for the reader due to how technical they are, and general lack of insightful ideas proof wise. It's certainly good practice, though. It follows from these theorems that

Lemma 21.1.3. r_1, \dots, r_n is a regular sequence $\iff \text{gr}_I(R) \cong (R/I)[x_1, \dots, x_n]$ for $I = (r_1, \dots, r_n)$. The second description is indexing independent, so it follows that any permutation of a regular sequence is still a regular sequence.

22

10/14/2022

22.1 Flatness

Lemma 22.1.1. *A nontrivial flat module over a local ring is faithfully flat. Thus, flat local morphisms of local rings are faithfully flat morphisms.*

Proof. First, suppose that M is a nonzero flat R -module and (R, m) is a local ring. By Nakayama's Lemma, $mM \neq M$. Let N be any nontrivial R -module, and choose $x \in N$ nonzero. If $I = \text{Ann}_R(x)$, it follows that $Rx = R/I$, and since $IM \subset mM \neq M$, $M/IM = M \otimes_R R/I \neq 0$. There is a natural injection $Rx = R/I \hookrightarrow N$, and as M is flat, it follows that $R/I \otimes_R M \hookrightarrow N \otimes_R M$ is still an injection. $R/I \otimes_R M \neq 0$, so $N \otimes_R M$ contains a nontrivial submodule, and is thus itself nontrivial. It follows from this that M is faithfully flat. \square

22.2 Back to Regular Sequences

Theorem 22.2.1. *Suppose that r_1, \dots, r_n is a sequence in R . Let $J_i = (r_1, \dots, r_i)$. This sequence is regular if and only if $J_i : r_{i+1} = J_i \forall i$. Similarly, if r_1, \dots, r_n is regular, then $J_i^t : r_{i+1} = J_i^t$ for some $t \in \mathbb{N}$.*

If we're given a sequence, it would be nice to have an easy way to check whether it is regular. You can use the above theorem but this is tedious; fortunately there is a better way. If x is regular in R , then $r \mapsto xr$ is injective, so $0 \rightarrow R \rightarrow R$ is exact, where $R \rightarrow R$ is this multiplication map. In particular, $0 \rightarrow R \rightarrow R \rightarrow 0$, is cyclic, i.e. is exact in positive degree.

Now say that x, y is a regular sequence in R . This gives us the sequence

$$0 \longrightarrow R \xrightarrow{\begin{bmatrix} -y \\ x \end{bmatrix}} R \oplus R \xrightarrow{[x, y]} R \longrightarrow R/(x, y) \longrightarrow 0$$

This is a mapping $0 \rightarrow \{\text{degree } 2\} \rightarrow \{\text{degree } 1\} \rightarrow \{\text{degree } 0\} \rightarrow 0$. Now assume that $[x, y] \begin{bmatrix} a \\ b \end{bmatrix} = 0$. This means that $ax + by = 0$. In $R/(x)$, y is regular, so $b = 0$ in $R/(x)$. Thus $b = b_1x$, so $a + b_1y = 0$. It follows that $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b_1y \\ b_1x \end{bmatrix} = b_1 \begin{bmatrix} y \\ x \end{bmatrix}$.

23

10/17/2022

23.1 Koszul Complexes

Suppose that (A_\bullet, d_\bullet^A) and (B_\bullet, d_\bullet^B) are complexes of R -modules. We can define the tensor product of complexes as follows:

$$(A_\bullet \otimes_R B_\bullet)_n := \bigoplus_{i+j=n} A_i \otimes_R B_j$$

Defining $(A_\bullet \otimes_R B_\bullet)_\bullet$. For the differential maps, we have

$$d_n^{A \otimes B} = \bigoplus_{i+j=n} (d_i^A \otimes \text{Id}_{B_j} + (-1)^{i-1} \text{Id}_{A_i} \otimes d_j^B)$$

We'll use this notion of a tensor product of complexes to define the Koszul Complex.

Now suppose that R is a commutative ring and M is an R -module. let x_1, \dots, x_n be a sequence of elements in R . Define $k(x_i) := 0 \rightarrow R \rightarrow R \rightarrow 0$ where $R \rightarrow R$ is defined by $r \mapsto x_i r$. This is naturally a complex, so we can define

$$k(x) := \bigotimes_{i=1}^n k(x_i)$$

to be the *Koszul Complex* of $x = (x_1, \dots, x_n)$, with $k(x, M) := k(x) \otimes_R M$ being the corresponding complex for the R -module M . Let $k(x_1, \dots, x_j) = k(j)$ for simplicity (so $k(x) = k(n)$).

Theorem 23.1.1. *There is an exact sequence of complexes:*

$$0 \longrightarrow k(x_1, \dots, x_{n-1}) \longrightarrow k(x_1, \dots, x_n) \longrightarrow k(x_1, \dots, x_{n-1})[-1] \longrightarrow 0$$

Where $A[-1]_n = A_{n-1}$. This induces a long exact sequence of homology of these complexes:

$$\dots \longrightarrow H_j(k(n-1)) \longrightarrow H_j(k(n)) \longrightarrow H_{j-1}(k(n-1)) \xrightarrow{(-1)^{j-1}x_n} H_{j-1}(k(n-1)) \longrightarrow \dots$$

With the connecting homomorphism defined as above. Finally, we have that $x_n = H_j(k(n)) = 0$.

Proof. Most of the proof here follows from looking at one complex. Notice that, by construction, $k(n-1) \otimes k(x_n) = k(n)$. It follows that

$$\begin{aligned} k(n)_j &= (k(n-1)_j \otimes R) \oplus (k(n-1)_{j-1} \otimes R) \\ k(n)_{j-1} &= (k(n-1)_{j-1} \otimes R) \oplus (k(n-1)_{j-2} \otimes R) \end{aligned}$$

Where

$$\begin{array}{ccc} k(n-1)_j \otimes R & & k(n-1)_{j-1} \otimes R \\ \downarrow d_j & \swarrow & \downarrow d_{j-1} \\ k(n-1)_{j-1} \otimes R & \xleftarrow{(-1)^{j+1}x_n} & k(n-1)_{j-2} \otimes R \end{array}$$

This defines the differential

$$d(a_j, a_{j-1}) = (d(a_j) + (-1)^j a_{j-1}, d_{j-1}(a_{j-1}))$$

Yielding the sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & k(n-1)_j & \longrightarrow & k(n-1)_j \oplus k(n-1)_{j-1} & \longrightarrow & k(n-1)_{j-1} \longrightarrow 0 \\ & & \downarrow d_j & & \downarrow d_j & \downarrow d_{j+1} & \downarrow d_{j-1} \\ 0 & \longrightarrow & k(n-1)_{j-1} & \longrightarrow & k(n-1)_{j-1} \oplus k(n-1)_{j-2} & \longrightarrow & k(n-1)_{j-2} \longrightarrow 0 \end{array}$$

From this we see that the top and bottom sequences are exact, and by exactness of the initial maps we know that this is indeed a commutative diagram. This morphism of short exact sequences of complexes induces a long exact sequence in homology like we are after. A diagram chase shows that $H_{j-1}(k(n-1)) \rightarrow H_{j-1}((k(n-1)))$ in the long exact sequence is precisely the map multiplying by $(-1)^j x_n$. It is now sufficient to check that $x_n H_j(k(n)) = 0 \forall j$.

Say $[a_j, a_{j-1}] \in H_k(k(n))$. Thus, $0 = d[a_j, a_{j-1}] = (d_j(a_j) + (-1)^j x_n a_{j-1}, d_{j-1}(a_{j-1}))$, so each individual coordinate is 0. Thus, $d_j(a_j) = (-1)^{j-1} x_n a_{j-1}$ and $d_{j-1}(a_{j-1}) = 0$. Thus,

$$x_n(a_j, a_{j-1}) = (x_n a_j, x_n a_{j-1}) = d(0, (-1)^{j-1} a_j)$$

It follows from this that $x_n(a_j, a_{j-1}) = 0 \in H_j(k(n))$. □

Notice that $k(x_1) \otimes k(x_2) \cong k(x_2) \otimes k(x_1)$. Thus, the ordering x_1, \dots, x_n is not particularly important, i.e. the ordering of the tensor is irrelevant to the construction of $k(n)$. Therefore, x_n killing all homology groups implies that any x_i can kill all homology groups. It follows that $x_i H_j(k(n)) = 0 \forall i, j$.

There are some homology groups that we understand regardless of the sequence. Notice that

$$H_n(k(n)) = \ker \left(R \xrightarrow{\begin{bmatrix} \pm x_1 \\ \vdots \\ \pm x_n \end{bmatrix}} \underbrace{R \oplus \cdots \oplus R}_{\text{rank}=n} \right) = \{r \in R \mid x_i \cdot r = 0 \forall i = 1, \dots, n\}$$

or over modules,

$$H_n(k(x, M)) = \ker \left(M \xrightarrow{\begin{bmatrix} \pm x_1 \\ \vdots \\ \pm x_n \end{bmatrix}} \underbrace{M \oplus \cdots \oplus M}_{\text{rank}=n} \right) = \{m \in M \mid x_i \cdot m = 0 \forall i = 1, \dots, n\}$$

We can also compute the 0th homology.

$$H_0(k(n)) = \text{coker} \left(\underbrace{R \oplus \cdots \oplus R}_{\text{rank}=n} \xrightarrow{[\pm x_1 \cdots \pm x_n]} R \right) = R/(x_1, \dots, x_n)$$

Similarly one can conclude that $H^0(k(x, M)) \cong M/(x_1, \dots, x_n)M$. Notice that from our construction of the $k(x_i)$, $k(x)$ is a free R -module of rank $\binom{n}{j}$. Say that the basis is of the form $\{e_{a_1, \dots, a_j} \mid 1 \leq a_1 < \cdots < a_j \leq n\}$. On the bases, the differential acts as follows:

$$d_j(e_{a_1, \dots, a_j}) = \sum (-1)^{j-1} e_{\hat{a}_1, \dots, \hat{a}_j, \dots, a_j}$$

From this we can prove the following theorem:

Theorem 23.1.2. *Suppose $x_1, \dots, x_n \in R$ is a sequence. Then if x_1, \dots, x_n is a regular sequence, then $H_j(k(n)) = 0 \forall j > 0$. Furthermore, if (R, m) is Noetherian and local (or, R is \mathbb{N} -graded, Noetherian, and each x_i is homogeneous of positive degree) and $H_1(k(n)) = 0$, then x_1, \dots, x_n is a regular sequence.*

Proof. We prove the first part first, and check this by inducting on n . For $n = 1$, $0 \rightarrow R \rightarrow R \rightarrow 0$ is our sequence, and $H_1(k(1)) = \ker(x : R \rightarrow R) = 0$. We now move on to the inductive step. We get a long exact sequence

$$\dots \longrightarrow H_j(k(n-1)) \longrightarrow H_j(k(n)) \longrightarrow H_{j-1}(k(n-1)) \xrightarrow{(-1)^{j-1}x_n} H_{j-1}(k(n-1)) \longrightarrow \dots$$

If $j \geq 2$, then $H_j(k(m)) = 0 \forall j \geq 2, m < n$. Thus,

$$\dots \longrightarrow \underbrace{H_j(k(n-1))}_{=0} \longrightarrow H_j(k(n)) \longrightarrow \underbrace{H_{j-1}(k(n-1))}_{=0} \xrightarrow{(-1)^{j-1}x_n} H_{j-1}(k(n-1)) \longrightarrow \dots$$

So the remaining terms are 0 by exactness. If $j = 1$, then our sequence is of the form

$$0 \longrightarrow H_1(k(n)) \longrightarrow R/(x_1, \dots, x_{n-1}) \xrightarrow{tx_n} R/(x_1, \dots, x_{n-1})$$

Because x_n is regular on $R/(x_1, \dots, x_{n-1})$, it follows that $H_1(k(n)) = 0$. We now move on to the second part Using the same sequence as above and setting $H_1(k(n)) = 0$, that implies that x_n is regular on $R/(x_1, \dots, x_{n-1})$. Via our long exact sequence,

$$\dots \quad H_1(k(n-1)) \xrightarrow{\pm x_n} H_1(k(n-1)) \longrightarrow \underbrace{H_1(k(n))}_{=0} \longrightarrow H_0(k(n-1)) \longrightarrow \dots$$

We know via exactness that $x_n H_1(k(n-1)) = H_1(k(n-1))$. As $H_1(k(n-1))$ is finitely generated, and R has one of the hypotheses specified above, we can use Nakayama's Lemma to conclude that $H_1(k(n-1)) = 0$. Inducting on n , we see that x_1, \dots, x_n is a regular sequence. \square

24

10/19/2022

24.1 More on Koszul Complexes

Let x_1, \dots, x_n be a sequence in R , used to construct the Koszul Complex $k(x)$. Recall that we have an initial morphism $0 \rightarrow R \rightarrow R^{\oplus n}$ in the complex defined by the alternating map $r \mapsto (x_1 r, -x_2 r, \dots, \pm x_{n-1} r, \mp x_n r)$. It follows that $H_n(k(n)) \cong \{r \in R \mid x_i r = 0 \forall i\}$. We also proved that $H_0(k(n)) = R/(x_1, \dots, x_n)$, and that $H_1(k(n))$ characterizes regular sequences.

Let $I \subset R$ be an ideal, and M an R -module such that $M \neq IM$. We say the *depth* of M in I is the length of a maximal M -sequence in I . Denote this as $\delta(I, M)$. This definition leads us to asking a couple questions. Is the maximal M -sequence finite? Are all maximal length sequences of equal length?

Theorem 24.1.1. (Depth Sensitivity) *Suppose R is Noetherian, $I = (y_1, \dots, y_n)$. M is a nonzero finitely generated module and $M \neq IM$. If $t = \max\{i \mid H_i(y, M) \neq 0\}$, then any maximal M sequence in I has length $n - t$.*

In particular, if y_1, \dots, y_n is an M -sequence, then $H_i(k(y, M)) = 0 \forall i \geq 1$, and $H_0(k(y, M)) = M/IM \neq 0$, so $t = 0$. $n - 0 = n$, and y_1, \dots, y_n is a maximal M -sequence, so the result is true in this case.

Proof. Pick a maximal M -sequence $x_1, \dots, x_s \in I$. This is maximal in the sense that every element in I is a zero divisor in $M/(x_1, \dots, x_s)M$. We prove the result by inducting on s .

For the base case, suppose $s = 0$. In this case, there are no regular elements, so all elements of I are zero divisors on M . But, we know that the set of zero divisors on M is precisely the union of all the associated primes. By prime avoidance, $I \subset P$, for P some associated prime in particular. Let $P = \text{Ann}_R(z)$ for some $z \in M$. Thus, $y_i z = 0 \forall i = 1, \dots, n$. Thus, $z \in H_n(k(y, M)) = \{m \in M \mid y_i m = 0 \forall i\}$, so $H_n(k(y, M)) \neq 0$. Thus, no homology terms vanish and the case holds for $s = 0$.

Recall that $y_j \cdot H_i(k(y, M)) = 0 \forall i, j$, so since $x_1 \in I$, $x_1 \cdot H_i(k(y, M)) = 0$ for any i . It follows that the map $H_i(k(y, M)) \xrightarrow{\cdot x_1} H_i(k(y, M))$ has full kernel, so $H_i(k(y, M_1)) \twoheadrightarrow H_i(k(y, M))$ by exactness. Thus $H_{i+1}(k(y, M)) \neq 0$, so for $i > t$,

$$\begin{aligned} \dots \longrightarrow \underbrace{H_{i+1}(k(y, M))}_{=0} \longrightarrow H_{i+1}(k(y, M_1)) \longrightarrow \underbrace{H_i(k(y, M))}_{=0} \xrightarrow{\cdot x_1} H_i(k(y, M)) \longrightarrow \dots \\ \Rightarrow H_j(k(y, M_1)) = 0 \quad \forall j > t + 1 \end{aligned}$$

Thus, $t + 1 = \sup\{i \mid H_i(k(y, M_1)) \neq 0\}$. By induction, any max M_1 -sequence in I has length $n - t - 1$, so $s - 1 = n - t - 1$, so $s = n - t$. \square

This has the following useful corollary:

Lemma 24.1.2. *Say $I = (y_1, \dots, y_n)$, R is Noetherian, M is a nonzero finitely generated module with $M \neq IM$. Then y_1, \dots, y_n is an M -sequence if and only if $\delta(I, M) = n$.*

Theorem 24.1.3. *Suppose that R is Noetherian, $I \subset R$ is a nontrivial ideal, and M is a finitely generated R -module such that $M \neq IM$. For $n > 0$, the following are equivalent:*

- (1) *For any finitely generated R -module N such that $\text{Supp}(N) \subset V(I)$, $\text{Ext}_R^i(N, M) = 0$ for $i < n$.*
- (2) *$\text{Ext}_R^i(R/I, M) = 0$ for $i < n$.*
- (3) *For SOME finitely generated R -module N such that $\text{Supp}(N) \subset V(I)$, $\text{Ext}_R^i(N, M) = 0$ for $i < n$.*
- (4) *There exists an M -sequence of length n in I .*

Proof. (1) \Rightarrow (2) \Rightarrow (3) is clear. We'd like to show that (3) \Rightarrow (4) \Rightarrow (1). Let's first check that (3) \Rightarrow (4). By (3), $\text{Hom}_R(N, M) = 0$ for N chosen as above, as $\text{Hom} = \text{Ext}_R^0$. First, we want to show that I contains an M -regular element. If it doesn't, then like before I is contained in the union of all associated primes of M , so by prime avoidance it is contained entirely in $\mathfrak{1}$, denoted P . It follows that $R/P \hookrightarrow M$ by construction, so $\kappa(P) \hookrightarrow M_P$. Thus, $\text{Hom}_{R_P}(\kappa(P), M) \neq 0$. $P \in V(I) = \text{Supp}(N)$, implying that $N_P \neq 0$, and thus $N_P \otimes_{R_P} \kappa(P) \neq 0$.

Thus \exists a surjection $N_P \otimes \kappa(P) \twoheadrightarrow \kappa(P)$, where $N_P \otimes \kappa(P)$ is a finite dimensional $\kappa(P)$ vector space. Thus we have the sequence of maps

$$N_P \longrightarrow N_P \otimes_{R_P} \kappa(P) \longrightarrow \kappa(P) \longrightarrow M_P$$

So $\text{Hom}_{R_P}(N_P, M_P) \neq 0$. Since N is finitely generated and R is Noetherian,

$$\text{Hom}_{R_P}(N_P, M_P) \cong \text{Hom}_R(N, M)_P$$

so it follows that $\text{Hom}_R(N, M) \neq 0$, a contradiction. This logic also holds for $n = 1$. Now assume that $n > 1$, and set $M_1 = M/rM$, for some M -regular $r \in I$. This gives us a short exact sequence

$$\dots \longrightarrow \text{Ext}_R^i(N, M) \xrightarrow{r} \text{Ext}_R^i(N, M) \longrightarrow \text{Ext}_R^i(N, M_1) \longrightarrow \dots$$

Giving us a long exact sequence on Ext

$$\dots \longrightarrow \text{Ext}_R^i(N, M) \xrightarrow{r} \text{Ext}_R^i(N, M) \longrightarrow \text{Ext}_R^i(N, M_1) \longrightarrow \dots$$

It follows that $\text{Ext}_R^i(N, M_1) = 0$ for $i < n - 1$, and that $r_1, \dots, r_n \in I$ is regular.

To conclude we check that (4) \Rightarrow (1). Assume that $x_1, \dots, x_n \in I$ are an M -sequence. Set $M_1 = M/x_1M$. As $\text{Supp}(N) = V(\text{Ann}_R(N)) \subset V(I)$, it follows that $I \subset \sqrt{\text{Ann}_R(N)}$, implying that $x_1^\ell \in \text{Ann}_R(N)$. For $n = 1$, x_1^ℓ is both injective and zero on $\text{Hom}_R(N, M)$, so $\text{Hom}_R(N, M) = 0$. For $n \geq 2$, x_2, \dots, x_n is a regular M_1 -sequence, so by induction $\text{Ext}^i(N, M_1) = 0 \forall i < n - 1$. Taking a long exact sequence in Ext, we get that

$$\begin{array}{ccccccc} & & & & \dots & \longrightarrow & \underbrace{\text{Ext}_R^{i-1}(N, M_1)}_{=0} \\ & & & & & \swarrow & \\ & & & & & & \\ \text{Ext}_R^i(N, M) & \xrightarrow{x_1} & \text{Ext}_R^i(N, M) & \longrightarrow & \text{Ext}_R^i(N, M_1) & & \\ & & & & \swarrow & & \\ \text{Ext}_R^{i+1}(N, M) & \longrightarrow & \dots & & & & \end{array}$$

It follows that $0 \longrightarrow \text{Ext}_R^i(N, M) \xrightarrow{x_1} \text{Ext}_R^i(N, M)$ is exact for $i < n$. As x_1^ℓ is both injective and zero on $\text{Ext}_R^i(N, M)$, it follows that $\text{Ext}_R^i(N, M) = 0$. \square

25

10/21/2022

25.1 Bounding Depth

We had last showed that for R Noetherian, $I \subset R$ a nontrivial ideal, M a finitely generated R -module such that $M \neq IM$, and $n > 0$, the following are equivalent:

- (1) For any finitely generated R -module N such that $\text{Supp}(N) \subset V(I)$, $\text{Ext}_R^i(N, M) = 0$ for $i < n$.
- (2) $\text{Ext}_R^i(R/I, M) = 0$ for $i < n$.
- (3) For SOME finitely generated R -module N such that $\text{Supp}(N) \subset V(I)$, $\text{Ext}_R^i(N, M) = 0$ for $i < n$.
- (4) There exists an M -sequence of length n in I .

With these hypotheses, we'd like to verify the following corollary:

Lemma 25.1.1. $\delta(I, M) = \inf\{i \mid \text{Ext}_R^i(R/I, M) \neq 0\}$

Proof. Let $n = \delta(I, M)$. (4) \iff (1) from the previous theorem shows us that $\text{Ext}_R^i(R/I, M) = 0 \forall i < n$. It remains to prove that, for $i = n$, $\text{Ext}_R^n(R/I, M) \neq 0$. As we have a depth of n , there exists a maximal M -sequence $x_1, \dots, x_n \in I$. Set $M_i = M/(x_1, \dots, x_i)M$. Because this is an M -sequence, we have a short exact sequence

$$0 \longrightarrow M_{n-1} \xrightarrow{\cdot x_n} M_{n-1} \longrightarrow M_n \longrightarrow 0$$

Giving us a long exact sequence on Ext :

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Hom}_R(R/I, M_{n-1}) & \xrightarrow{\cdot x_n} & \text{Hom}_R(R/I, M_{n-1}) & \longrightarrow & \text{Hom}_R(R/I, M_n) \\
& & & & & \swarrow & \\
& & \text{Ext}_R^1(R/I, M_{n-1}) & \xleftarrow{\cdot x_n} & \text{Ext}_R^1(R/I, M_{n-1}) & \longrightarrow & \dots
\end{array}$$

But, the last map is 0, as $x_n \cdot R/I = 0$. It follows that we can put a zero at the end of the sequence. Furthermore, since $x_n \in I$ is M_{n-1} -regular, $\text{Hom}(R/I, M_{n-1}) = 0$. Thus we have

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Hom}_R(R/I, M_{n-1}) & \xrightarrow{\cdot x_n} & \underbrace{\text{Hom}_R(R/I, M_{n-1})}_{=0} & \longrightarrow & \text{Hom}_R(R/I, M_n) \\
& & & & & \nearrow & \\
& & \text{Ext}_R^1(R/I, M_{n-1}) & \xrightarrow{\cdot x_n} & 0 & &
\end{array}$$

It follows that $\text{Hom}_R(R/I, M_n) = \text{Ext}_R^1(R/I, M_{n-1})$. We can repeat this argument to get the chain of isomorphisms

$$\text{Hom}_R(R/I, M_n) \cong \text{Ext}_R^1(R/I, M_{n-1}) \cong \text{Ext}_R^2(R/I, M_{n-2}) \cong \dots \cong \text{Ext}_R^n(R/I, M)$$

Thus we are reduced to proving that $\text{Hom}_R(R/I, M_n) \neq 0$. Well, suppose that $\text{Hom}_R(R/I, M_n) = 0$. By (4), this implies that there is an M_n -regular element $x_{n+1} \in I$. It follows that $x_1, \dots, x_n, x_{n+1} \in I$ is an M -sequence, but this is a contradiction, as we've chosen x_1, \dots, x_n to be a maximal M -sequence. It follows that $\text{Hom}_R(R/I, M_n) \neq 0$. \square

25.2 Introduction to Cohen Macaulay Rings

The above results suggest that we can bound depth in a concrete way.

Theorem 25.2.1. *Suppose (R, m) is a Noetherian local ring, and M and N are finitely generated, non-zero R -modules. Assume that $\delta(m, M) = n$ and $\dim(N) = \delta$. Then $\text{Ext}_R^i(M, N) = 0$ for all $i < n - \delta$.*

Proof. We induct on δ . If $\delta = 0$, then $\text{Supp}(N) = \{m\}$. Then $\text{Ext}_R^i(N, M) = 0 \iff \text{Ext}_R^i(R/m, M) = 0$ for $i < n = n - \delta$, via the previous theorem. Since n is precisely our depth and $M \neq mM$, it follows that $\text{Ext}_R^i(N, M) = 0$ for any $i < n$.

Now onto the inductive step. Assume that $\delta > 0$. Consider the prime filtration of N :

$$N = N_0 \supsetneq N_1 \supsetneq \dots \supsetneq N_t \supsetneq 0 = N_{t+1}$$

Where $N_j/N_{j+1} \cong R/P_j$ for $P_j \in \text{Spec}(R)$. It follows that $N_t \cong R/P_t$. This gives us a short exact sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & N_t & \longrightarrow & N_{t-1} & \longrightarrow & N_{t-1}/N_t \longrightarrow 0 \\
& & \Downarrow & & & & \Downarrow \\
& & R/P_t & & & & R/P_{t-1}
\end{array}$$

If $\text{Ext}_R^i(R/P_t, M) = 0$ and $\text{Ext}_R^i(R/P_{t-1}, M) = 0$, then $\text{Ext}_R^i(N_{t-1}, M) = 0$ for any i . (this is seen by taking the long exact sequence in homology). One can also make this argument for

$$\begin{array}{ccccccc}
0 & \longrightarrow & N_{t-1} & \longrightarrow & N_{t-2} & \longrightarrow & N_{t-2}/N_{t-1} \longrightarrow 0 \\
& & \Downarrow & & & & \Downarrow \\
& & R/P_{t-1} & & & & R/P_{t-2}
\end{array}$$

Thus we can conclude that, if $\text{Ext}_R^i(R/P, M) = 0$ then $\text{Ext}_R^i(N, M) = 0$ for $i < n - \delta$. We are thus reduced to proving that $\text{Ext}_R^i(R/P, M) = 0$ for $i < n - \delta$. and $\dim(R/P) = \delta$. Well, since $\delta > 0$, $P \subsetneq m$, so choose $x \in m$ such that $x \notin P$. Consider the short exact sequence

$$0 \longrightarrow R/P \xrightarrow{\cdot x} R/P \longrightarrow R/(P + (x)) \longrightarrow 0$$

Killing x lowers dimension by at least 1, and as killing an element can lower dimension by at most 1, it follows that $\dim(R/(P + (x))) = \dim(R/P) - 1 = \delta - 1$. This allows us to use our inductive hypothesis to show that $\text{Ext}_R^i(R/(P + (x)), M) = 0 \forall i < n - (\delta - 1) = n - \delta + 1$. Apply $\text{Hom}_R(-, M)$ to this short exact sequence. This is contravariant and flips the arrows. This then determines a long exact sequence in Ext ,

$$\begin{array}{ccccccc} \dots & \longrightarrow & \text{Ext}_R^i(R/(P + (x)), M) & \longrightarrow & \text{Ext}_R^i(R/P, M) & \xrightarrow{\cdot x} & \text{Ext}_R^i(R/P, M) \\ & & & & & \swarrow & \\ & & \text{Ext}_R^{i+1}(R/(P + (x)), M) & \longrightarrow & \dots & & \end{array}$$

When $i < n - \delta$, this reduces to the sequence

$$0 \longrightarrow \text{Ext}_R^i(R/P, M) \xleftarrow{\cdot x} \text{Ext}_R^i(R/P, M) \longrightarrow 0$$

Thus, $x \text{Ext}_R^i(R/P, M) = \text{Ext}_R^i(R/P, M)$, so by Nakayama's lemma, $\text{Ext}_R^i(R/P, M) = 0 \forall i < n - \delta$. \square

This implies the following corollary:

Lemma 25.2.2. *(R, m) is a Noetherian local ring, and M is a nonzero finitely generated R-module. Let $P \in \text{Ass}_R(M)$. Then $\delta(m, M) \leq \dim(R/P) \leq \max(\dim(R), \dim(M))$.*

Proof. $P \in \text{Ass}_R(M) \iff R/P \hookrightarrow M \Rightarrow \text{Hom}_R(R/P, M) \neq 0$. The theorem then implies that $n = \delta(m, M) \leq \delta = \dim(R/P)$. \square

If the upper bound is achieved, we have a special name for this ring. More specifically, if (R, m) is Noetherian and local, with M being a nonzero finitely generated R -module, then M is a **Cohen-Macaulay** R -module if $\delta(M) = \dim(M)$. Similarly, if R is a Cohen-Macaulay module over itself (i.e. $\dim(R) = \delta(R)$), it is a Cohen Macaulay ring.

25.2.1 Examples

- All Artinian modules are Cohen Macaulay, as $\dim(M) = \delta(M) = 0$.
- Regular local rings are Cohen Macaulay, as a minimal set of generators of the maximal ideal are a regular sequence, and by construction, there are $\dim(R)$ many generators. It follows that $\delta(R) = \dim(R)$.
- NON-EXAMPLE: Consider $R = k[[x, y, u, v]] / ((x, y) \cap (u, v))$. This is a dimension 2 ring. Consider the short exact sequence

$$0 \longrightarrow R \longrightarrow \frac{k[[x,y,u,v]]}{(x,y)} \oplus \frac{k[[x,y,u,v]]}{(u,v)} \longrightarrow \frac{k[[x,y,u,v]]}{(x,y,u,v)} \longrightarrow 0$$

Taking a long exact sequence in Ext of this short exact sequence, we see that sufficiently many terms vanish to get the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R \left(\frac{k[[x,y,u,v]]}{(x,y,u,v)}, \frac{k[[x,y,u,v]]}{(x,y,u,v)} \right) \longrightarrow \mathrm{Ext}_{k[[x,y,u,v]]}^1 \left(\frac{k[[x,y,u,v]]}{(x,y,u,v)}, R \right) \longrightarrow 0$$

Implying that these are isomorphic. Well,

$$0 \neq \mathrm{Hom}_R \left(\frac{k[[x,y,u,v]]}{(x,y,u,v)}, \frac{k[[x,y,u,v]]}{(x,y,u,v)} \right) = \mathrm{Ext}_{k[[x,y,u,v]]}^1 \left(\frac{k[[x,y,u,v]]}{(x,y,u,v)}, R \right)$$

so we have nonvanishing Ext. Hence R is not Cohen-Macaulay.

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10/24/2022

26.1 More on Cohen-Macaulay Modules

Recall that if (R, m) is a Noetherian local ring, then a finitely generated R -module M is CM if $\delta(m, M) = \dim(M)$. For simplicity, if we are over a local ring we say that $\delta(M) = \delta(m, M)$.

Theorem 26.1.1. *(R, m) is a Noetherian local ring, and M is a finitely generated R -module.*

- (1) *If M is CM, then for each $P \in \text{Ass}(M)$, then $\dim(R/P) = \dim(M) = \delta(M)$.*
- (2) *Assume $x_1, \dots, x_r \in M$ is an M -sequence and set $M' = M/(x_1, \dots, x_r)M$. Then, M is CM $\iff M'$ is CM.*
- (3) *If M is CM, then every localization M_P is CM over R_P . Furthermore, if $M_P \neq 0$, then $\delta_{R_P}(M_P) = \delta(P, M)$.*

Proof. We prove (1) first. Let P be an associated prime.

$$\begin{aligned} \dim(M) &= \dim(R/\text{Ann}_R(M)) \\ &= \sup(\dim(R/P) \mid P \in \text{Ass}_R(M)) \\ &\geq \inf(\dim(R/P) \mid P \in \text{Ass}_R(M)) \\ &\geq \delta(M) \end{aligned}$$

Since $\dim(M) = \delta(M)$, these are all equal, so $\dim(R/P_1) = \dim(R/P_2)$ for any two associated primes P_i . It follows that the sup of these dimensions is just this number, so $\dim(M) = \dim(R/P)$.

To prove (2), we use the homework problem to conclude that $\dim(M') = \dim(M) - r$ and $\delta(M') = \delta(M) - r$. Thus, it is clear that if the two quantities are equal for M , they are equal for M' , and vice versa. It follows that M is CM $\iff M'$ is.

Finally we prove (3). If $x_1, \dots, x_r \in P$ is an M -sequence, it follows that $x_1, \dots, x_r \in PR_P$ is an M_P sequence. Thus, $\delta(M_P) \geq \delta(P, M) = \dim(M_P)$. We still need to check that this

is, in fact, an equality. To check this, we induct on $\delta(P, M)$. When $\delta(P, M) = 0$; then P consists of zero divisors on M . Thus, $P \subset \bigcup_{Q \in \text{Ass}_R(M)} Q$, so by prime avoidance there exists a $Q \in \text{Ass}_R(M)$ such that $P \subset Q$. As $\dim(R/P) = \dim(M)$, it follows that P itself is an associated prime. Furthermore, P is a minimal element in $\text{Supp}(M)$, as the minimal elements of the support are precisely those that are minimal associated primes. By minimality, if we localize at P , all associated primes are minimal, so the minimal elements in the support are precisely the associated primes. Thus, $\text{Supp}_{R_P}(M_P) = \{PR_P\}$. It follows that $\dim_{R_P}(M_P) = 0$, so we have equality in the $\delta(P, M) = 0$ case.

We now assume that $\delta(P, M) > 0$. Now choose $x \in P$ to be M -regular, and let $M' = M/xM$. The dimension and depth both drop by 1 here, via the homework problem. By part (2), M' is CM, as we are killing a regular element. By induction, $\dim(M'_P) = \delta(P, M') = \delta(P, M) - 1$. From homework, killing a regular element drops the dimension by 1, so $\dim(M'_P) = \dim(M_P) - 1$, so it follows that $\dim(M_P) - 1 = \delta(P, M) - 1$, so $\dim(M_P) = \delta(P, M)$. The result follows. \square

26.1.1 More Examples

Using the theorem above, we can construct more (non) examples of CM modules.

- Suppose $R = k[[x, y, z]]$, $M = R/(x^2, xy)$. M is NOT CM, since M has an embedded prime (x, y) .
- $R = k[[x, y, z]]$, $M = R/(xy, xz)$. M is NOT CM, since there are two associated primes $P_1 = (x)$ and $P_2 = (y, z)$, where $\dim(R/P_1) = 2 \neq 1 = \dim(R/P_2)$. R/P . Thus both of these cannot equal $\dim(M)$, so M is not CM.

It's worth noting that the equi-dimensionality in (1) from the previous theorem is necessary, but not sufficient. For an example of this claim, consider the last non-example from the previous day's lecture. We have equi-dimensionality, but that module was not CM.

(3) tells us how to define Cohen-Macaulay for non-local rings. If, for a Noetherian ring R , R_P is Cohen-Macaulay for every $P \in \text{Spec}(R)$, then R is **Cohen-Macaulay**. (3) tells us that this definition is compatible with the definition given for Noetherian local rings, as being CM is preserved under localization in that case.

Theorem 26.1.2. *Suppose that (R, m) is CM and local. Choose a sequence $x_1, \dots, x_r \in m$. Then the following conditions are equivalent:*

- (1) x_1, \dots, x_r is an R -sequence (so, it is regular).
- (2) $\text{ht}(x_1, \dots, x_i) = i$ for any $1 \leq i \leq r$.
- (3) $\text{ht}(x_1, \dots, x_r) = r$.
- (4) x_1, \dots, x_r can be extended to a system of parameters.

This provides a nice way of checking whether rings are CM. If we choose one system of parameters of R and check that it is a regular sequence, then we can conclude via this theorem that R is CM.

Proof. We first check that (1) \Rightarrow (2). We have already seen that $\text{ht}(x_1, \dots, x_i) \leq i$. Since x_1, \dots, x_i is regular, $x_1 \notin P \forall P \in \text{Ass}(R)$. Thus, x_1 is not contained in any minimal prime. Thus, x_1 has positive height, so $\text{ht}(x_1) = 1$. Similarly for x_2 , x_2 is not contained in any minimal prime of (x_1) , so $\text{ht}(x_1, x_2) = 2$. Continuing this process, (2) follows.

(3) is simply a special case of (2), so we can move on to checking that (3) \Rightarrow (4). If $\dim(R) = r$, then we are done. Now let's assume that $\dim(R) > r$. Well, that means that m is NOT a minimal prime of (x_1, \dots, x_r) , as m has strictly larger height. Now, we make a similar argument to before. Pick $x_{r+1} \in m$, but not contained in any minimal prime of (x_1, \dots, x_r) (this is possible by prime avoidance). If we add x_{r+1} into the sequence, it's easy to see that $\text{height}(x_1, \dots, x_{r+1}) = r + 1$. This process continuous until we increase up to $\dim(R)$, where m would be a minimal prime, giving us an extension of (x_1, \dots, x_r) determining a system of parameters.

To conclude, we check that (4) \Rightarrow (1). It is sufficient to show that every system of parameters is a regular sequence, as part of a regular sequence is still a regular sequence. Say x_1, \dots, x_d is a system of parameters. x_1 is not contained in any minimal prime, and since R is CM, the minimal primes are just associated primes. It follows that x_1 is not contained in any associated primes. Thus, x_1 is NOT a zero divisor, so x_1 is a regular element in R .

Thus we can set $R' = R/(x_1)$. R is CM, so R' is CM by a previous lemma. x_1, \dots, x_d is a system of parameters, so the projection of x_2, \dots, x_d is a system of parameters in R' under the map $R \twoheadrightarrow R/(x_1)$. By induction, x_2, \dots, x_d is a regular sequence, so it follows that adding a regular element to it retains regularity. Thus, x_1, \dots, x_d is a regular sequence in R . \square

Here is an immediate consequence of this result:

Lemma 26.1.3. *If (R, m) is Noetherian and local, then R is CM \iff every system of parameters of R is a regular sequence, \iff one system of parameters of R is a regular sequence.*

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10/26/2022

27.1 Revisiting a Counter-Example

Following the Theorem from last class, we deduced the corollary that a Noetherian local ring is Cohen Macaulay \iff any system of parameters is an R -sequence. It follows then that every system of parameters must be an R -sequence. Recall the example from a couple classes ago, where $R = \frac{k[[x,y,u,v]]}{(x,y) \cap (u,v)} = \frac{k[[x,y,u,v]]}{(xu,xv,yu,yv)}$. We can pick the first regular element such that it avoids all the minimal primes.

For instance, let $x + u$ be the first element. This is nonzero, because if $x + u = 0$, $x = -u$, so $u \in (x, y)$, which is a contradiction. We can thus take an alleged system of parameters $x + u, y + v$. We'd like to verify that this is indeed such a system. By the corollary above, it is both necessary and sufficient that this is a regular sequence for R to be CM. First, observe that $\sqrt{(x + u, y + v)} = m \subset R$. It follows that $R/(x + u, y + v)$ has finite length. It follows that $\delta(R) \leq 2$, so $\dim(R) = \delta(R) \leq 2$. Now, consider the surjection $R \twoheadrightarrow R/(u, v) \cong k[x, y]$. It follows that R surjects to a ring of dimension 2. Thus, $\dim(R) \geq 2$. It follows that $\dim(R) = 2$, so $x + u, y + v$ is a system of parameters.

However, observe that this is NOT a regular sequence. $R/(x + u) = \frac{k[[x,y,u,v]]}{(xy,xv,yu,yv,x+u)}$. In this ring, $uv = v(x + u) - xv$, so $uv = 0$ in this ring. Similarly, $u(y + v) = uy + uv = 0$, but $u \neq 0$ in $R/(x + u)$. It follows that $y + v$ is a zero divisor in $R/(x + u)$, so it follows that $x + u, y + v$ is NOT a regular sequence, so R is NOT CM. This conclusion matches the conclusion we made previously, but we have come to this conclusion by using more elementary methods.

27.2 Catenary Rings

Theorem 27.2.1. *Suppose that (R, m) is CM Noetherian ring, and $I \subsetneq R$. Then $\text{ht}(I) = \delta(I, R)$, and $\text{ht}(I) + \dim(R/I) = \dim(R)$.*

Proof. If $\text{ht}(I) = h$, then choose $x_1, \dots, x_h \in I$ such that $\text{ht}(x_1, \dots, x_i) = i$ for any $1 \leq i \leq h$. It follows that x_1, \dots, x_h is a regular sequence in I , so $\delta(I, R) \geq h$. On the other hand, given any R -sequence $a_1, \dots, a_r \in I$, then $r = \text{ht}(a_1, \dots, a_r)$ by Krull's Height Theorem

and regularity. It follows that

$$r = \text{ht}(a_1, \dots, a_r) \leq \text{ht}(I) = h$$

It follows that $\delta(I, R) \leq h$, so we have equality. Now onto the second condition. Let S be the set of minimal primes of I . Then $\text{ht}(I) = \inf(\text{ht}(P) \mid P \in S)$. On the other hand, $\dim(R/I) = \sup(\dim(R/P) \mid P \in S)$. Thus, it suffices to show that $\text{ht}(P) + \dim(R/P) = \dim(R) \forall P \in S$, as if this holds, then infs and sups are hit. Thus, $\delta(P, R) = \text{ht}(P) = \dim(R_P) = h$.

Choose an R -sequence $x_1, \dots, x_n \in P$. As R is CM, $R/(x_1, \dots, x_h)$ is CM and of dimension $n - h$, for $h = \text{ht}(P)$. Since $\text{ht}(x_1, \dots, x_h) = h = \text{ht}(P)$, it follows that P is minimal over the ideal (x_1, \dots, x_h) . Thus, by CM-ness, $\dim(R/P) = \dim(R/(x_1, \dots, x_h)) = d - h$. Thus,

$$\text{ht}(P) + \dim(R/P) = h + d - h = d = \dim(R)$$

□

Now onto a definition. We say that any ring R is *Catenary* if

- Given any two prime ideals $P \subsetneq Q$, there is a saturated chain of primes between them.
- All such saturated chains between P and Q have the same length.

There are non-Catenary rings that are still Noetherian, due to Nagata. That being said, most typical examples of Noetherian rings satisfy the Catenary conditions.

Theorem 27.2.2. *Any homomorphic image of a CM local ring is Catenary.*

Proof. It suffices to show that CM local rings are Catenary, if S is the homomorphic image of R under some map θ , it follows that $R/\ker(\theta) = S$, so we can just pass these saturated chains into the quotient.

Thus, we assume that (R, m) is a CM local ring, and pick $P \subsetneq Q$ prime. R_Q is then CM, so $\text{ht}(PR_Q) + \dim(R_Q/PR_Q) = \dim(R_Q)$. It follows that $\text{ht}(PR_Q) = \text{ht}(P)$, and that $\dim(R_Q) = \text{ht}(Q)$. We can also see that $\dim(R_Q/PR_Q)$ is length of a saturated chain in R_Q/PR_Q , where primes in R_Q/PR_Q are precisely those that contain P and are contained in Q . Thus, $\dim(R_Q/PR_Q)$ is the length of a saturated chain between P and Q . This is precisely $\text{ht}_{R/P}(Q/P)$. It follows that $\text{ht}_{R/P}(Q/P) = \text{ht}(Q) - \text{ht}(P)$, so whatever saturated chain you pick, it will always be the same length. The Catenary condition follows. □

28.1 Unmixedness

We've shown that if M is CM, then $\dim(R/P) = \dim(M)$ for each prime $P \in \text{Ass}_R(M)$. In general, the converse is not true, but if we loosen our restrictions a bit, we get some form of the converse. This ends up being a useful characterization of CM modules.

Suppose that R is Noetherian and $I \subsetneq R$. Then I is *unmixed* if the height of all associated primes of R/I are the same. We say the *unmixedness theorem holds* for R if for every $r \geq 0$, every height r ideal I generated by r elements is unmixed.

Theorem 28.1.1. R is CM \iff the unmixedness theorem holds for R .

Proof. We prove the forward direction first, assuming that R is CM. Choose an ideal $I \subsetneq R$ such that $I = (x_1, \dots, x_r)$, and $\text{ht}(I) = r$ for some $r \geq 1$. Our goal is to show that $\text{ht}(P) = r$. Well, $\text{ht}(P) = \dim(R_P)$, where R_P is CM. As $P \in \text{Ann}(R/I)$, it follows that in R_P , x_1, \dots, x_r is a system of parameters, so $R_P/(x_1, \dots, x_r)R_P$ is $\dim(R_P) - r$ dimensional CM ring. As $P \in \text{Ass}(R/I)$, $R/P \hookrightarrow R/I$, so $R_P/PR_P \hookrightarrow R_P/IR_P$. As the maximal ideal includes into the ring, it follows that $\dim(R_P/IR_P) = 0$, so $\dim(R_P) - r = 0$, i.e. $\dim(R_P) = r$, and the result follows.

Now we prove the converse. Suppose that the unmixedness theorem holds. Then $\forall P \in \text{Spec}(R)$, R_P is CM. Let $\text{ht}(P) = r$. Thus we may choose $x_1, \dots, x_r \in P$ such that $\text{ht}(x_1, \dots, x_i) = i$ for any $1 \leq i \leq r$. Applying the unmixedness theorem, we know all the associated primes of $R/(x_1, \dots, x_i)$ are all height i and cannot contain x_{i+1} . Thus, x_{i+1} is $R/(x_1, \dots, x_i)$ -regular. As this is true for any i , it follows that x_1, \dots, x_r is a regular sequence, so $\delta(P, R) \geq r$. This holds on localizations too, so $\delta(P, R_P) \geq r$, but because $\text{ht}(P) = r$, $\delta(R_P) = \dim(R_P) = r$. It follows that R is CM. \square

Theorem 28.1.2. If R is CM, then so is $R[x]$.

This naturally inducts to $R[x_1, \dots, x_n]$.

Proof. It is sufficient to check that $R[x]_P$ is CM for every maximal ideal P of $R[x]$. Fix such an ideal P . Set $m = P \cap R$. It is easy to see that this is a maximal ideal of R . We have the inclusion $R_m[x] \hookrightarrow R[x]_P$, suggesting that $R[x]_P$ is a further localization of $R_m[x]$. Thus,

we can replace R with R_m , i.e we can assume that R is local. Let m be its unique maximal ideal.

Thus we have the following situation. Let (R, m) be a CM local ring. We want to show that $S = R[x]$ is also CM. Notice that $S/mS \cong (R/m)[x]$. R/m is a field, so $(R/m)[x]$ is a PID. It follows that P/mS is a principal ideal. Suppose that it is generated by f ; by maximality we know that f is a monic irreducible polynomial in S . Thus, let $P = mS + (f)$ in S . Choose a system of parameters a_1, \dots, a_d in R ; as R is CM, a_1, \dots, a_d is regular. This is equivalent to saying that the multiplication by a_{i+1} endomorphism on $R/(a_1, \dots, a_i)$ is an injection. Since S is flat over R , a_1, \dots, a_d is still a regular sequence in S (just tensor over the correspond koszul complex). Set $S' = S/(a_1, \dots, a_d)S = (R/(a_1, \dots, a_d))[x]$ (notice that $(a_1, \dots, a_d)S \subset mS$). The image of f is still a monic polynomial in S' . Thus, f is S' regular, and a_1, \dots, a_d, f is a S sequence. Since $a_1, \dots, a_d, f \in P$, $\delta(S_P) \geq d + 1$. Since $\dim(S) = d + 1 = \dim R + 1$, $\dim(S_P) \leq d + 1 \leq \delta(S_P)$, so $\delta(S_P) = \dim(S_P) = d + 1$, and the result follows. \square

29

10/31/2022

29.1 Some Homological Algebra

We began class by going over projective and injective resolutions. Projective modules are typically simpler to understand than injective modules, so we spend more time on the latter. While projective modules are direct summands of free modules, leading to convenient ways to verify projective dimension, we'd like a similar criterion for injective modules.

Lemma 29.1.1. E is an injective R -module $\iff \text{Ext}_R^1(R/I, E) = 0$ for all $I \subset R$.

Proof. The forward direction follows from the definition of injectivity (we just specialize to the case where "any module" is of the form R/I). The meat of the argument comes in the reverse case.

With that, onto the reverse case. Suppose that M is an R -module, and $L \subset M$. If we have an R -linear map $\varphi : L \rightarrow E$, we'd like to extend this to a morphism $M \rightarrow E$. This follows from the definition of injectivity as follows:

$$\begin{array}{ccccc}
 0 & \longrightarrow & L & \longrightarrow & M \\
 & & \downarrow & \swarrow \text{dashed} & \\
 & & E & &
 \end{array}$$

This diagram to commuting is equivalent to $\text{Hom}(-, E)$ being exact. As Hom is always left exact, right exactness is precisely checking the tail of the sequence $0 \rightarrow \text{Hom}(\text{coker}(L \rightarrow M), E) \rightarrow \text{Hom}(M, E) \rightarrow \text{Hom}(L, E)$ is surjective. This is surjective precisely when we can always pull back a morphism $\text{Hom}(L, E)$ to a map $\text{Hom}(M, E)$ such that it commutes with the map $L \rightarrow M$, which is just an inclusion. Thus,

$$0 \rightarrow \text{Hom}(\text{coker}(L \rightarrow M), E) \rightarrow \text{Hom}(M, E) \rightarrow \text{Hom}(L, E) \rightarrow 0$$

is a short exact sequence precisely \iff top diagram commutes. This implies it is sufficient just for Ext^1 to vanish to check injectivity for E . Furthermore, Applying Zorn's Lemma on the set

$$\mathcal{S} = \{(N, \psi) \mid L \subset N \subset M, \psi : N \rightarrow E, \psi|_L = \varphi\}$$

shows us that we only need to check Ext^1 vanishes on cyclic modules, which are of the form R/I . \square

Theorem 29.1.2. *The following are equivalent:*

- (1) $\text{InjDim}_R(M) \leq n$
- (2) $\text{Ext}_R^{n+1}(N, M) = 0$ for all R -modules N .
- (3) $\text{Ext}_R^{n+1}(R/I, M) = 0$ for all $I \subset R$.

Proof. It is sufficient to check that (3) \Rightarrow (1). Take an injective resolution of M of the form $0 \rightarrow M \rightarrow E^\bullet$, denoting

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow \dots \longrightarrow E^{n-1} \longrightarrow C^\bullet \longrightarrow 0$$

Where C^\bullet represents a complex that possibly continues. Shrink this to a short exact sequence

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow C^0 \longrightarrow 0$$

Now apply $\text{Hom}(R/I, -)$ and take a long exact sequence in homology.

$$\begin{array}{ccccccc} \dots & \longrightarrow & \underbrace{\text{Ext}_R^n(R/I, E^0)}_{=0} & \longrightarrow & \text{Ext}_R^n(R/I, C^0) & & \\ & & & \searrow & & & \\ & & \text{Ext}_R^{n+1}(R/I, M) & \longrightarrow & \underbrace{\text{Ext}_R^{n+1}(R/I, E^0)}_{=0} & \longrightarrow & \dots \end{array}$$

Implying via exactness that $\text{Ext}_R^{n+1}(R/I, M) \cong \text{Ext}_R^n(R/I, C^0)$. The same argument shows that $\text{Ext}_R^n(R/I, C^0) \cong \text{Ext}_R^{n-1}(R/I, C^1)$, so

$$\text{Ext}_R^{n+1}(R/I, M) \cong \dots \cong \text{Ext}_R^1(R/I, C^0)$$

where the first term vanishes by assumption. It follows that C^0 is injective. It follows that

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow \dots \longrightarrow E^{n-1} \longrightarrow C^0 \longrightarrow 0$$

Is an injective resolution, so $\text{InjDim}_R(M) \leq n$. \square

Lemma 29.1.3. *Suppose that R is Noetherian and M is an R -modules. Suppose further that N is a finitely generated R -module and $n > 0$. Then, if $\text{Ext}_R^n(R/P, M) = 0 \forall P \in \text{Supp}(N)$, then $\text{Ext}_R^n(N, M) = 0$.*

Proof. Consider a prime filtration of N

$$0 = N_0 \subsetneq N_1 \subsetneq \dots \subsetneq N_t = N$$

Where $N_{i+1}/N_i \cong R/P_i$. It follows from this construction that $P_i \in \text{Supp}(N)$. $N_1 \cong N_1/N_0 \cong R/P_1$. This gives us a short exact sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & N_2/N_1 \longrightarrow 0 \\
& & \Downarrow & & & & \Downarrow \\
& & R/P_1 & & & & R/P_2
\end{array}$$

Taking a long exact sequence of homology via the functor $\text{Hom}_R(-, M)$, we get an exact sequence

$$\cdots \longrightarrow \underbrace{\text{Ext}_R^n(R/P_2, M)}_{=0} \longrightarrow \text{Ext}_R^n(N_2, M) \longrightarrow \underbrace{\text{Ext}_R^n(R/P_1, M)}_{=0} \longrightarrow \cdots$$

Thus $\text{Ext}_R^n(N_2, M) = 0$. Repeating this process, we eventually see that $\text{Ext}_R^n(N_i, M) = 0$ for any $i \geq 1$, so $\text{Ext}_R^n(N, M) = 0$. \square

This result has the following useful corollary:

Lemma 29.1.4. *If R is Noetherian and M is an R module, then the following are equivalent:*

- (1) $\text{InjDim}_R(M) \leq n$
- (2) $\text{Ext}_R^{n+1}(R/P, M) = 0$ for any $P \in \text{Spec}(R)$.

We'd like to specialize this further in the local case.

Lemma 29.1.5. *let (R, m, k) is a Noetherian local ring. $P \subsetneq m$ and M is a finitely generated R -module. If $\text{Ext}_R^{n+1}(R/Q, M) = 0 \forall$ primes $Q \supsetneq P$, then $\text{Ext}_R^n(R/P, M) = 0$.*

Proof. Choose $x \in m \setminus P$. This gives us a short exact sequence

$$0 \longrightarrow R/P \xrightarrow{\cdot x} R/P \longrightarrow R/(P + (x)) \longrightarrow 0$$

taking a long exact sequence in homology via the functor $\text{Hom}_R(-, M)$, we get the sequence of maps

$$\cdots \longrightarrow \text{Ext}_R^n(R/P, M) \xrightarrow{\cdot x} \text{Ext}_R^n(R/P, M) \longrightarrow \text{Ext}_R^{n+1}(R/(P + (x)), M) \longrightarrow \cdots$$

Since $V(P + (x)) \subset \{Q \in \text{Spec}(R) \mid P \subsetneq Q\}$, it follows that $\text{Ext}_R^{n+1}(R/(P + (x)), M) = 0$. Thus, the multiplication map is surjective, so $\text{Ext}_R^n(R/P, M) = x \text{Ext}_R^n(R/P, M)$. By Nakayama's Lemma, $\text{Ext}_R^n(R/P, M) = 0$ (this is where the finitely generated hypothesis for M is needed). \square

30

11/02/2022

30.1 Injective Dimension

Last class, we proved that over a Noetherian ring, $\text{InjDim}(M) \leq n \iff \text{Ext}_R^{n+1}(R/P, M) = 0 \forall P \in \text{Spec}(R)$. In the case where (R, m) is also local, M is finitely generated, and $P \subsetneq m$ is a prime ideal, then $\text{Ext}_R^{j+1}(R/Q, M) = 0 \forall Q \supsetneq P \Rightarrow \text{Ext}_R^j(R/P, M) = 0$.

These results generalize as follows:

Theorem 30.1.1. *Suppose that (R, m, k) is a Noetherian local ring and M is finitely generated. Then,*

$$\text{InjDim}(M) = \sup \left(i \mid \text{Ext}_R^i(R/m, M) \neq 0 \right)$$

Proof. Let $t = \sup \left(i \mid \text{Ext}_R^i(R/m, M) \neq 0 \right)$. It's clear that $\text{InjDim}(M) \geq t$. It is thus sufficient to check that $\text{Ext}_R^{t+1}(R/P, M) = 0 \forall P \in \text{Spec}(R)$ (this is equivalent to checking that $\text{InjDim}(M) \leq t$ by the first previously stated result). We want to show that $\text{Ext}_R^{t+i}(R/m, M) = 0$ for $i \geq 1$. By the second such result, $\text{Ext}_R^{t+i-1}(R/P, M) = 0$ for any P with height $d - 1$. There is only one such prime ideal that properly contains P , namely m . Repeating this process for prime ideals of height $d - 2$ and so on, via the second stated result above, the theorem follows. \square

Theorem 30.1.2. *Suppose that M, N are R -modules for R a commutative ring. Fix $x \in R$. If x is R -regular, M -regular, and $xN = 0$, set $\bar{R} = R/(x)$, $\bar{M} = M/xM$. Then,*

- $\text{Hom}_R(N, M) = 0$
- $\text{Ext}_R^{i+1}(N, M) \cong \text{Ext}_{\bar{R}}^i(N, \bar{M})$ for all $i \geq 0$.

Proof. Since x is M -regular, the multiplication x map $M \rightarrow M$ is injective. Applying $\text{Hom}_R(N, -)$ to this map, we get an exact sequence $0 \rightarrow \text{Hom}_R(N, M) \rightarrow \text{Hom}_R(N, M)$. As $xN = 0$, the multiplication map has zero image. As it is also injective, we conclude that $\text{Hom}_R(N, M) = 0$.

For the second part, we'd like to show that $\{\text{Ext}_{\bar{R}}^i(-, \bar{M}) \mid i \geq 0\}$ is the collection of all derived functors of $\text{Hom}_{\bar{R}}(-, \bar{M})$. These derived functors necessarily must satisfy the following three properties:

- The $i = 0$ case must recover $\text{Hom}_{\bar{R}}(-, \bar{M})$.
- When $i \geq 1$, $\text{Ext}_{\bar{R}}^i(\bar{P}, \bar{M}) = 0$ for all projective \bar{R} -modules \bar{P} .
- A short exact sequences induce a long exact sequence in Ext^i .

Our goal is to show that $\{\text{Ext}_R^{i+1}(-, M) \mid i \geq 0\}$ also satisfies all 3 properties.

- When $i = 0$, we have the short exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot x} M \longrightarrow \bar{M} \longrightarrow 0$$

Applying $\text{Hom}_R(L, -)$ induces a long exact sequence in homology. By applying the first part, we see that most of the terms vanish, yielding

$$\text{Ext}_R^1(L, \bar{M}) \cong \text{Hom}(L, \bar{M}) \cong \text{Hom}_{\bar{R}}(L, \bar{M})$$

so the Hom functor is recovered.

- Let \bar{P} be a projective \bar{R} -module. Then $\text{Pdim}_R(\bar{P}) = 1$, so it follows that $\text{Ext}_R^{i+1}(\bar{P}, \bar{M}) = 0 \forall i \geq 1$.
- $\text{Ext}_R^{i+1}(-, M)$ already has this property.

It follows that $\text{Ext}_R^{i+1}(-, M) \simeq \text{Ext}_{\bar{R}}^i(-, \bar{M})$ as functors on the category of R -modules. \square

This result introduced the following neat corollary:

Lemma 30.1.3. *If (R, m) is Noetherian and local and M is finitely generated, and $x \in m$ is both R -regular and M -regular, then $\text{InjDim}_{\bar{R}}(\bar{M}) = \text{InjDim}_R(M) - 1$.*

30.2 Free Resolutions

Suppose that (R, m) is a Noetherian local ring and M is a finitely generated R -module. It is a corollary of Nakayama's Lemma that free, projective, and flat are all equivalent over (R, m) . An exact sequence

$$\dots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \dots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

is a *minimal free resolution* of M provided that:

- (1) each F_i is a free R -module of finite rank

$$(2) d_i(F_i) \subseteq mF_{i-1}$$

$$(3) F_0 \otimes_R R/m \cong M \otimes_R R/m$$

(1) and (2) are equivalent to saying that d_i can be represented as a matrix in which every entry is in m .

Now assume that $F_\bullet \rightarrow M \rightarrow 0$ is a minimal free resolution of M . In $F_\bullet \otimes_R R/m$, every differential is the zero map. Thus, $\text{Tor}_i^R(R/m, M) \cong F_i \otimes_R R/m$. Minimal free resolutions are also unique; if $F_\bullet \rightarrow M \rightarrow 0$ and $F'_\bullet \rightarrow M \rightarrow 0$ are both minimal free resolutions of M , then the identity map $\text{Id} : M \rightarrow M$ induces an isomorphism of complexes.

31

11/04/2022

31.1 More From Last Time

Using similar techniques to the previous class, we obtain the following result.

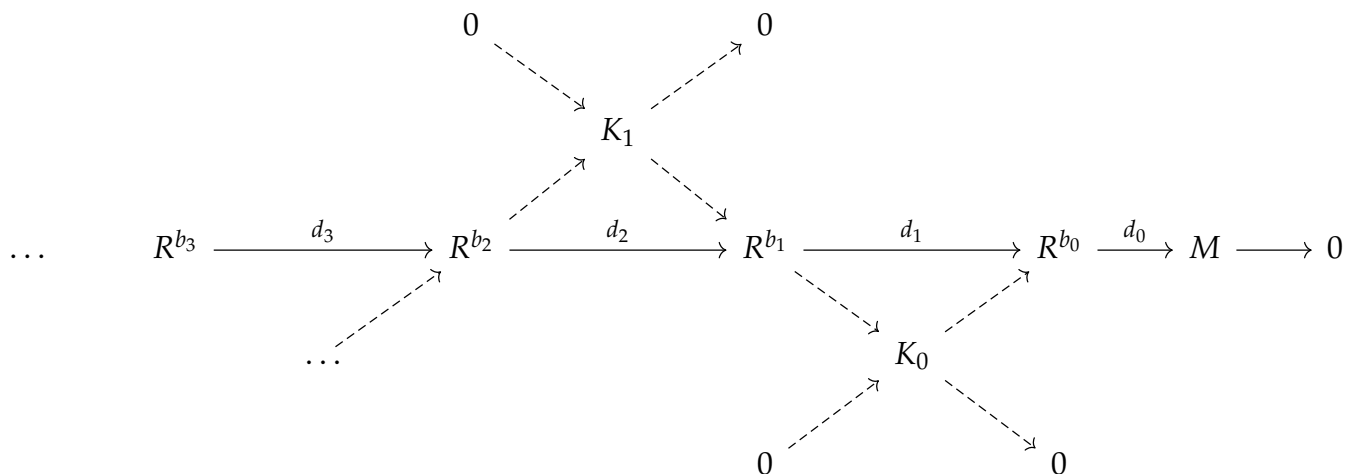
Theorem 31.1.1. *If R is a commutative ring and M is finitely generated, and $x \in R$ is both R -regular and M -regular and $xN = 0$, then*

$$\text{Ext}_R^i(M, N) = \text{Ext}_{\overline{R}}^i(\overline{M}, N)$$

$$\text{Tor}_i^R(M, N) = \text{Tor}_i^{\overline{R}}(\overline{M}, N)$$

31.2 Existence of Minimal Free Resolutions

Suppose that (R, m) is Noetherian local and M is a finitely generated R -module. Take a surjection onto generators $d_0 : R^{b_0} \rightarrow M \rightarrow 0$, where $b_0 = \dim(M/mM)$. This has a kernel K_0 , which itself has a surjection onto generators via $R^{b_1} \rightarrow K_0$ where $b_1 = \dim(K_0/mK_0)$. Repeating this process, this gives a resolution



With a bit of work, one can verify (via our choices of b_i) that this sequence is indeed minimal. This is because $K_0 = \ker(R^{b_0} \rightarrow M)$ and $R^{b_0}/mR^{b_0} \cong M/mM$, implying that $K_0 \subseteq mR^{b_0}$, so $\text{im}(d_1) \subseteq mR^{b_0}$. Similar logic can be applied to the following terms in the sequence.

For example, if we were to construct a free resolution this way for k where $R = k[[x, y]]$, $m = (x, y)$, $k = R/m$, then we in fact recover its Koszul complex! This actually holds over any regular local ring (R, m) where $m = (x_1, \dots, x_d)$, for $d = \dim(R)$. It follows that x_1, \dots, x_d is a regular sequence, so the Koszul complex on x_1, \dots, x_d provides a minimal free resolution for R/m . This implies that $\text{ProjDim}(R/m) \leq d$. Furthermore, $\text{InjDim}(M) = \sup \left(i \mid \text{Ext}_R^i(R/m, M) \neq 0 \right) \leq d$ as well, for any R -module M . Then we can conclude that $\text{Ext}_R^i(N, M) = 0 \forall i > d$, so $\text{ProjDim}_R(N) \leq d$ for any R -module N .

Lemma 31.2.1. *Suppose that (R, m, k) is Noetherian and local, and M is a finitely generated R -module. Let $F_\bullet \rightarrow M \rightarrow 0$ be a minimal free resolution. Then,*

- (1) $\dim_k \text{Tor}_i^R(M, k) = \text{rank}(F_i)$
- (2) $\text{ProjDim}_R(M) \leq \text{ProjDim}(k)$
- (3) *If $m \neq 0$ and $\text{Pdim}(M) = r < \infty$, then for each finitely generated R -module $N \neq 0$, $\text{Ext}_R^r(M, N) \neq 0$.*

Proof. Since each entry in $d_i : F_i \rightarrow F_{i-1}$ is in m , $d_i \otimes_R R/m$ is the 0 map. It follows that $\text{Tor}_i(M, k) \cong F_i \otimes_R R/m$. From this conclusion, (1) and (2) immediately follow.

To prove (3), Suppose that $F_r \neq 0$ but $F_{r+1} = 0$. This halts the resolution to $0 \rightarrow F_r \rightarrow F_{r-1} \rightarrow \dots$, where the first map is just d_r . Applying $\text{Hom}_R(-, N)$, we get an exact sequence

$$\text{Hom}_R(F_{r-1}, N) \rightarrow \text{Hom}_R(F_r, N) \rightarrow 0$$

via the map d_r^* . Implying that

$$\text{Ext}_R^r(M, N) \cong \frac{\text{Hom}_R(F_r, N)}{\text{im}(d_r^*)}$$

As we can associate the top cohomology to the cokernel of this mapping. It follows via minimality that the entries of d_r^* are contained in m . Thus, $\text{im}(d_r^*) \subset m \text{Hom}_R(F_r, N)$, so by Nakayama, $\text{im}(d_r^*) \neq \text{Hom}_R(F_r, N)$. It follows that $\text{Ext}_R^r(M, N)$ does not vanish. \square

31.3 Global Dimension

Theorem 31.3.1. *If R is a commutative ring and $n \geq 0$, the following are equivalent.*

- $\text{ProjDim}(M) \leq n \forall M$.
- $\text{ProjDim}(M) \leq n \forall M$ finitely generated.
- $\text{InjDim}(N) \leq n \forall N$.

- $\text{Ext}_R^{n+1}(M, N) = 0 \forall M, N$.

Proof. Exercise, but follows very easily from the previous results. \square

This lends us the following definition. The *Global Dimension* of R , denoted $\text{glDim}(R)$, is

$$\sup_{R\text{-Mod}} (\text{ProjDim}(M)) = \sup_{R\text{-Mod}} (\text{InjDim}(M))$$

When (R, m) is Noetherian and local, the global dimension is just $\text{ProjDim}_R(R/m)$.

Theorem 31.3.2. (Serre) *If (R, m) is Noetherian and local, then R is a regular local ring $\iff \text{glDim}(R) < \infty$.*

Proof. For the forward direction, let $m = (x_1, \dots, x_d)$. Taking the Koszul complex with respect to this sequence, which we know to be regular, yields a minimal free resolution of R/m . We know this to be of finite length, so it follows that $\text{glDim}(R)$ is finite. The reverse case is significantly harder and may be proved at another time. \square

This has the following corollary:

Lemma 31.3.3. *If (R, m) is a regular local ring, then R_P is regular for each prime ideal P .*

Proof. Since global dimension is finite, R/P has a finite projective resolution. Localizing each step of this resolution at P , we get a resolution of $\kappa(P)$. This shows that $\text{ProjDim}(\kappa(P)) < \infty$, so R_P has finite global dimension. \square

32

11/07/2022

32.1 Finishing Serre's Theorem

Last time, we gave a characterization of regular local rings (to be specific; we say that a Noetherian local ring (R, m) is regular $\iff \text{glDim}(R) < \infty$). We proved the forward case last time, and now we'll prove the reverse case.

Proof. Set $g = \text{glDim}(R)$, and $e = \dim_{R/m}(m/m^2)$. We say that e is the *embedded dimension* of R . Let's induct on e , which is necessarily finite. When $e = 0$, $m = 0$, so R is a field (and thus clearly regular). Now, assume that $e > 0$ (so $m \neq 0$). We claim that $m \notin \text{Ass}(R)$. If it were, then $m = \text{Ann}_R(a)$ for some a . Since g is finite, R/m admits a finite free resolution $F_\bullet \rightarrow R/m \rightarrow 0$ of length g . Minimality implies that $F_g \subset mF_{g-1}$, so $aF_g \subset amF_{g-1} = 0$, since $am = 0$. This is a contradiction.

Now use prime avoidance to pick $x \in m$, but not in m^2 nor in any associate prime of R . It follows that x is regular (and m -regular). Set $\bar{R} = R/(x)$, $\bar{m} = m/(x)$, $m' = m/xm$. It's worth noting that $m \otimes_R \bar{R} = m'$, so we'd like to derive facts about m' that would imply things about \bar{m} . From the last couple lectures, we know that $\text{Ext}_R^i(m, N) \cong \text{Ext}_{\bar{R}}^i(m', N)$, but $\text{Ext}_R^i(m, N) = 0$ for all $i > g$, so $\text{Ext}_{\bar{R}}^i(m', N) = 0$ for all $i > g - 1$. It follows that $\text{ProjDim}_{\bar{R}}(m') < \infty$. We want to use this to show that $\text{ProjDim}_{\bar{R}}(\bar{R}/\bar{m}) < \infty$ as well. Consider the short exact sequence $0 \rightarrow \bar{m} \rightarrow \bar{R} \rightarrow \bar{R}/\bar{m} \rightarrow 0$. As $\text{Ext}_{\bar{R}}^i(\bar{R}, N)$ vanishes, it follows that

$$\text{Ext}_{\bar{R}}^i(\bar{R}/\bar{m}, N) \cong \text{Ext}_{\bar{R}}^{i-1}(\bar{m}, N)$$

So it is sufficient to check that $\text{ProjDim}_{\bar{R}}(\bar{m}) < \infty$. Well, it is clear that $m' \twoheadrightarrow \bar{m}$, as $xm \subsetneq (x)$. We also get a natural inclusion $\bar{m} \hookrightarrow m'$, and the composition $\bar{m} \hookrightarrow m' \twoheadrightarrow \bar{m}$ is the identity. It follows via the splitting lemma that $m' \cong \bar{m} \oplus C$ for some C . Thus, we have that

$$\text{Ext}_{\bar{R}}^i(m', N) = 0 \Rightarrow \text{Ext}_{\bar{R}}^i(\bar{m}, N) = 0$$

Thus, $\text{ProjDim}_{\bar{R}}(\bar{m}) \leq \text{ProjDim}_{\bar{R}}(m') < \infty$, so \bar{m} , and thus \bar{R}/\bar{m} , has finite projective dimension. Inductively, we know \bar{R} to be regular, so \bar{m} is generated by $\dim \bar{R}$ elements. Thus, m can be generated by $\dim(\bar{R}) + 1 = \dim(R)$ elements. It follows that R is regular. \square

From this, we know (via a lemma from last class) that this theorem implies that being regular is a local property. This leads to the following definition: A Noetherian ring R is **Regular** if R_P is regular for all $P \in \text{Spec}(R)$ (equivalently, R_m is regular for every $m \in \text{MaxSpec}(R)$).

Theorem 32.1.1. *If R is regular, then so is $R[x]$. (and therefore, so is $R[x_1, \dots, x_n]$)*

Proof. Let m be a maximal ideal of $R[x]$, and set $P = m \cap R$, which is known to be prime. $R[x]_m$ is a further localization of $R_P[x]$, a polynomial ring over a local ring. It follows that we can assume that R is local to begin with. Once we prove that $R_P[x]$ is regular, then $R[x]_m$ is regular, so it is sufficient that $R_P[x]$ is regular.

Well, $R_P[x]/PR_P[x] \cong \kappa(P)[x]$, and since $\kappa(P)$ is a field, it follows that $\kappa(P)[x]$ is a PID. Thus, the image of m is a principal ideal (f) . We may assume that f is monic. Now, pick a preimage F of f in $R_P[x]$. It follows that $m = (PR_P[x], F)$. Well, PR_P can be generated by $\dim(R_P)$ elements by regularity, so we need only one more generator. Thus, m can be generated by $\dim(R_P) + 1$ elements, but $\dim(R_P[x]) = \dim(R_P) + 1$, so $(R_P[x])_m$ is regular, so $R_P[x]$ is regular, so $R[x]$ is regular. \square

It's worth noting that the converse of this statement is clearly true, as x is a regular element over $R[x]$, and $R[x]/(x) = R$.

32.2 Completion

Let R be a commutative ring, $0 \neq I \subsetneq R$ is an ideal, and M is an R -module. We'd like to define an ***I-adic topology*** on M . Assign $\{I^n M\}_{n \geq 0}$ to be a system of open neighborhoods for $0 \in M$. $\forall z \in M$, $\{z + I^n M\}_{n \geq 0}$ is a system of open neighborhoods for z . This topology is $T_1 \iff \bigcap_{n \geq 0} I^n M = 0$ (i.e. we can separate two points with open neighborhoods). If (R, \mathfrak{m}) is Noetherian local and if M is finitely generated, we know that this criterion holds.

Given this topology, we say that a sequence $(x_i) \subset M$ is **Cauchy** in this I -adic topology if $\forall m \geq 0, \exists n_0$ such that $(x_{n+1} - x_n) \in I^m M \forall n \geq n_0$. M is ***I-adically complete*** if every Cauchy sequence has a unique limit in M .

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33.1 More on Completion

R is any commutative ring, I is a non-trivial ideal, and M is an R -module. Last time, we defined the I -adic topology, and discussed Cauchy Sequences in this topology. With this in mind, we consider the collection of all Cauchy Sequences. Define a relation as follows: say $(z_n) \sim (z'_n)$ if $\forall m > 0, \exists n_0$ such that $z_n - z'_n \in I^m M \ \forall n \geq n_0$. Now define $\mathfrak{C}_I(M)$ to be the set of all Cauchy Sequences on M with respect to the I -adic topology, modulo this equivalence. This is clearly also an R -module, and can be viewed as the *completion* of M under the I -adic topology.

We can construct $\mathfrak{C}_I(M)$ in an alternative way as follows. Say that (z_n) is a Cauchy Sequence. In $M/I^m M$, the images under the projection $M \rightarrow M/I^m M$ of z_{n+1} and z_n are the same, for $n \gg 0$. Thus, $\forall m, (z_n)$ determined a unique element in $M/I^m M$. Call this element x_m . It follows that (z_n) determines a sequence (x_m) such that

$$M/IM \longleftarrow M/I^2M \longleftarrow \dots \longleftarrow M/I^m M \longleftarrow M/I^{m+1}M$$

$$x_1 \longleftarrow x_2 \longleftarrow \dots \longleftarrow x_m \longleftarrow x_{m+1}$$

This assignment determines an assignment $\mathfrak{C}_I(M) \rightarrow \varprojlim M/I^m M$

Theorem 33.1.1. *Under the above identification,*

$$\mathfrak{C}_I(M) \cong \varprojlim_m M/I^m M$$

Proof. Surjectivity follows from applying the logic above backwards; take $(x_m) \in \varprojlim M/I^m M$, and choose a preimage $z_m \in M$ of x_m under the canonical projection. $z_{m+1} - z_m \in I^m M$, since they have the same image in $M/I^m M$, so it follows that (z_m) is a Cauchy sequence whose image under this identification is (x_m) .

It is thus sufficient to check injectivity. Suppose that (z_n) maps to the 0 sequence under this identification. That is, $\forall m$ the image of z_n under the canonical projection is 0 in $M/I^m M$, so it follows that $z_n \in I^m M$. Thus, $(z_n) \sim (0)$. \square

Lemma 33.1.2. *R is I -adically complete, then $I \subset \mathcal{J}(R)$.*

Proof. As we proved some time ago, $I \subset \mathcal{J}(R) \iff 1 + a$ is unit for any $a \in I$. The sequence $1 - a + a^2 - a^3 + \dots$ exists in R because R is I -adically complete, and it is the inverse of $1 + a$. \square

We let \widehat{R}_I denote the **Completion** of R with respect to I . If the context is clear, we can omit the subscript and just write \widehat{R} .

Lemma 33.1.3. (Hensel's Lemma) *Suppose that (R, \mathfrak{m}) is a Noetherian local ring, and that R is \mathfrak{m} -adically complete. Now assume that $F(x) \in R[x]$ is a monic polynomial, and let $f(x)$ denote its projection to $(R/\mathfrak{m})[x]$. If $f(x)$ can be factored as $g(x)h(x)$, where both g, h are monic, then $\exists G, H \in R[x]$ such that $F = GH$ and $H \mapsto h, G \mapsto g$ under $R \twoheadrightarrow R/\mathfrak{m}$.*

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34.1 More Properties of Completion

Let's focus on the case where R is Noetherian and M is a finitely generated R -module.

Theorem 34.1.1. *Suppose R is Noetherian and M is a finitely generated R -module. Further assume that $I \subset R$ is an ideal, and $N \subset M$ is an R -submodule of M . This gives us the short exact sequence*

$$0 \longrightarrow \varprojlim N/(I^m M \cap N) \longrightarrow \varprojlim M/I^m M \longrightarrow \varprojlim M/(N + I^m M) \longrightarrow 0$$

Proof. We already have a short exact sequence

$$0 \longrightarrow N/(I^m M \cap N) \longrightarrow M/I^m M \longrightarrow M/(N + I^m M) \longrightarrow 0$$

for any m . Taking an inverse limit is left exact, so it sufficient to check that, in this case, the map $\varprojlim M/I^m M \longrightarrow \varprojlim M/(N + I^m M)$ is surjective. Given any $(x_m) \in \varprojlim M/(N + I^m M)$. We'd like to find a point in the fiber denoted (y_m) , and we do by inducting on m . Pick $y_1 \in M$ such that $y_1 \mapsto x_1$ and $y'_2 \in M$ such that $y'_2 \mapsto x_2$, under the map $M/I^k M \longrightarrow M/(N + I^k M)$ for $k = 1, 2$ respectively. It follows that $y'_2 - y_1 \in N + IM$. Thus, $y'_2 - y_1 = a_1 + b_1$ such that $a_1 \in N, b_1 \in IM$. Set $y_2 = y'_2 - a_1$. It follows that $y_2 \mapsto x_2$ as well, as it differs from y'_2 by an element of the quotient. Thus, $y_2 - y_1 = y'_2 - a_1 - y_1 = b_1 \in IM$.

Now onto the inductive step. Assume we've constructed y_1, \dots, y_j . Pick y'_{j+1} such that $y'_{j+1} \mapsto x_{j+1}$ under the map $M/I^{j+1} M \longrightarrow M/(N + I^{j+1} M)$. Thus, $y'_{j+1} - y_j \in N + I^j M$, so $y'_{j+1} - y_j = a_j + b_j$ for $a_j \in N, b_j \in I^j M$, and set $y_{j+1} = y'_{j+1} - a_j$. Via similar logic to last time, we can conclude that x_{j+1} has a natural fiber. To conclude, we'd like to use this to imply surjectivity in the inverse limit. To do this however, is not immediate, as finite convergence is not sufficient to conclude the case over the limit. To do this, we need The *Mittag-Leffler Condition*, which states that $\forall j \in \mathbb{Z}_{>0}, \exists C$ such that $\text{im}(A_n \rightarrow A_j) =$

$\text{im}(A_{n+1} \rightarrow A_j)$ for any $n \geq c$, given a directed system (A_n) . This means we only need to check finitely many cases, so we can conclude once we show this.

This result is an application of Artin-Rees. It's an easy consequence of Artin-Rees that the I -adic topology on N coincides with the subspace topology induced from the I -adic topology on M . This implies that $\varprojlim N/(I^m \cap N) \cong \varprojlim N/I^m N$. \square

Via this application of Artin-Rees, we can conclude the following theorem.

Theorem 34.1.2. *Over a Noetherian ring, completion is exact on finitely generated R -modules.*

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35.1 Completion via Base Change

Theorem 35.1.1. *If R is Noetherian and M is a finitely generated R -module, then $\widehat{M}_I \cong M \otimes_R \widehat{R}_I$.*

Proof. This is clear when M is free. as M is finitely generated over a Noetherian ring, it admits a finite presentation

$$R^{\oplus b} \longrightarrow R^{\oplus a} \longrightarrow M \longrightarrow 0$$

tensoring by \widehat{R}_I is right exact, so we have a right exact sequence

$$R^{\oplus b} \otimes_R \widehat{R}_I \longrightarrow R^{\oplus a} \otimes_R \widehat{R}_I \longrightarrow M \otimes_R \widehat{R}_I \longrightarrow 0$$

Further note that completion is right exact. As such the theorem holds over finitely generated free modules, we get the following isomorphisms:

$$\begin{array}{ccccccc} R^{\oplus b} \otimes_R \widehat{R}_I & \longrightarrow & R^{\oplus a} \otimes_R \widehat{R}_I & \longrightarrow & M \otimes_R \widehat{R}_I & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \widehat{R}_I^{\oplus b} & \longrightarrow & \widehat{R}_I^{\oplus a} & \longrightarrow & \widehat{M}_I & \longrightarrow & 0 \end{array}$$

By exactness, we can conclude that the rightmost map is an isomorphism as well. □

Therefore, exactness of completion is equivalent to the following theorem:

Theorem 35.1.2. *If R is Noetherian, \widehat{R}_I is a flat R -module.*

The standard completion map $R \rightarrow \widehat{R}_I$ has kernel $\bigcap_{m \geq 1} I^m$. If $I \subset \mathcal{J}(R)$, we know that this intersection vanishes. Thus, the standard completion map is simply an inclusion map in this case. This case occurs regularly, like for instance when R is local and all ideals are contained in the Jacobson Radical.

Lemma 35.1.3. *The I -adic topology on $M/I^m M$ is discrete.*

Proof. This is clear from the fact that open neighborhoods of $z \in M$ are $z + I^j(M/I^m M)$ for $j \geq m$, implying that z is an open point. Discreteness follows from the fact that all points are generic. \square

A natural corollary is that $(\widehat{M/I^m M})_I = M/I^m M$. Over the case of a Noetherian ring, notice that

$$R/I^m = (\widehat{R/I^m})_I \cong R/I^m \otimes_R \widehat{R}_I \cong \widehat{R}_I/I^m \widehat{R}_I$$

In the case where (R, \mathfrak{m}) is Noetherian and local, we see that $R/\mathfrak{m} = (\widehat{R/\mathfrak{m}})_{\mathfrak{m}} \cong \widehat{R}_{\mathfrak{m}}/\mathfrak{m}\widehat{R}_{\mathfrak{m}}$. Thus, the completion with respect to the unique maximal ideal of a local ring (R, \mathfrak{m}) still yields a noetherian local ring. $(\widehat{R}_{\mathfrak{m}}, \mathfrak{m}\widehat{R}_{\mathfrak{m}})$.

Theorem 35.1.4. *If R is Noetherian, $I, J \subset R$, and M is a finitely generated R -module, then*

$$\begin{aligned} (\widehat{JM})_I &= J\widehat{M}_I \\ (\widehat{M/JM})_I &\cong \widehat{M}_I/J\widehat{M}_I \end{aligned}$$

Proof. Since R is Noetherian, J is finitely generated. Let $J = (g_1, \dots, g_n)$. Now consider the map $M^{\oplus n} \rightarrow M$ given by $(m_1, \dots, m_n) \mapsto \sum g_i m_i \in JM$. It follows that we have a right exact sequence

$$M^{\oplus n} \xrightarrow{[g_1 \dots g_n]} M \longrightarrow M/JM \longrightarrow 0$$

Completing with respect to I yields

$$\widehat{M}_I^{\oplus n} \xrightarrow{\beta} \widehat{M}_I \xrightarrow{\alpha} (\widehat{M/JM})_I \longrightarrow 0$$

Where $\text{im}(\beta) = J\widehat{M}_I$ and $\ker(\alpha) = (\widehat{JM})_I$. By exactness these are equal, so we have the first part of the theorem. We can see the equality explicitly by taking the short exact sequence

$$0 \longrightarrow JM \longrightarrow M \longrightarrow M/JM \longrightarrow 0$$

And complete with respect to I , we get the sequence

$$0 \longrightarrow (\widehat{JM})_I \longrightarrow \widehat{M}_I \longrightarrow (\widehat{M/JM})_I \longrightarrow 0$$

This shows us that $J\widehat{M}_I = (\widehat{JM})_I$, and

$$(\widehat{M/JM})_I \cong \widehat{M}_I/(\widehat{JM})_I \cong \widehat{M}_I/J\widehat{M}_I$$

\square

Lemma 35.1.5. *If $R[x_1, \dots, x_n]$ is a polynomial ring over R and $I = (x_1, \dots, x_n)$, then*

$$R[\widehat{x_1, \dots, x_n}]_I \cong R[[x_1, \dots, x_n]]$$

This is clear, but we can consider the following generalization:

Theorem 35.1.6. *If R is Noetherian and $I = (g_1, \dots, g_n)$, then*

$$\widehat{R}_I = R[[x_1, \dots, x_n]] / (x_1 - g_1, \dots, x_n - g_n)$$

Proof. Set $S = R[x_1, \dots, x_n]$ and $H = (x_1, \dots, x_n)$. Set $J = (x_1 - g_1, \dots, x_n - g_n) \subset S$, implying that $R \cong S/J$. Viewing R as an S -module, then H -adic topology on R coincides with the I -adic topology on R . Therefore,

$$\widehat{R}_I \cong (\widehat{R/J})_H \cong \widehat{S}_H / j\widehat{S}_H \cong R[[x_1, \dots, x_n]] / (x_1 - g_1, \dots, x_n - g_n)$$

□

Theorem 35.1.7. *Let (R, \mathfrak{m}) be a Noetherian local ring. Then R is CM $\iff \widehat{R}_{\mathfrak{m}}$ is.*

Proof. Recall that R is CM $\iff \delta(R, \mathfrak{m}) = \dim(R)$. We have an explicit description of this depth, mainly that

$$\delta(R, \mathfrak{m}) = \inf \left(i \mid \text{Ext}_R^i(R/\mathfrak{m}, R) \neq 0 \right)$$

Since R/\mathfrak{m} is finitely generated,

$$\text{Ext}_R^i(R/\mathfrak{m}, R) \otimes_R \widehat{R}_{\mathfrak{m}} \cong \text{Ext}_R^i(\widehat{R}_{\mathfrak{m}}/\mathfrak{m}\widehat{R}_{\mathfrak{m}}, \widehat{R}_{\mathfrak{m}})$$

Thus, $\dim(R) = \dim(\widehat{R})$ and $(R, \mathfrak{m}) \rightarrow (\widehat{R}_{\mathfrak{m}}, \mathfrak{m}\widehat{R}_{\mathfrak{m}})$ is a flat and local map. Thus, \widehat{R} is faithfully flat, so $\text{Ext}_R^i(R/\mathfrak{m}, R)$ vanishes if and only if $\text{Ext}_R^i(\widehat{R}_{\mathfrak{m}}/\mathfrak{m}\widehat{R}_{\mathfrak{m}}, \widehat{R}_{\mathfrak{m}})$ does. Thus, $\delta(R, \mathfrak{m}) = \delta(\widehat{R}_{\mathfrak{m}}, \mathfrak{m}\widehat{R}_{\mathfrak{m}})$, and since these both have the same dimension, it follows that depth equals dimension of the completion if and only if the depth equals dimension of the base ring. □

Such a result holds for regular rings as well.

Theorem 35.1.8. *Let (R, \mathfrak{m}) be a Noetherian and local ring. Then R is regular $\iff \widehat{R}_{\mathfrak{m}}$ is.*

Proof. For the forward case, $\widehat{R}_{\mathfrak{m}} \cong R[[x_1, \dots, x_d]] / (x_1 - a_1, \dots, x_d - a_d)$. We know that regularity is preserved as we take polynomial rings and take quotients, so $R[[x_1, \dots, x_d]] / (x_1 - a_1, \dots, x_d - a_d)$ is regular if and only if R is.

Conversely, $\text{glDim}(R) = \text{ProjDim}(R/\mathfrak{m}) = \sup \left(i \mid \text{Tor}_i^R(R/\mathfrak{m}, R/\mathfrak{m}) \neq 0 \right)$. However, the Tor functor commutes with Tensor, so

$$\text{Tor}_i^R(R/\mathfrak{m}, R/\mathfrak{m}) \neq 0 \iff \text{Tor}_i^R(R/\mathfrak{m}, R/\mathfrak{m}) \otimes_R \widehat{R}_{\mathfrak{m}} \neq 0 \iff \text{Tor}_i^R(\widehat{R}_{\mathfrak{m}}/\mathfrak{m}\widehat{R}_{\mathfrak{m}}, \widehat{R}_{\mathfrak{m}}/\mathfrak{m}\widehat{R}_{\mathfrak{m}}) \neq 0$$

Since $\text{ProjDim}(\widehat{R}_{\mathfrak{m}}) < \infty$, it follows that $\text{Tor}_i^R(R/\mathfrak{m}, R/\mathfrak{m}) = 0$ for some $i \gg 0$. Thus, $\text{glDim}(R) < \infty$, so R is regular local. □

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36.1 Essential Extensions

We'd like an analogue of a "minimal free resolution" in the injective case. For R any commutative ring and N a submodule of M (both R -modules), $N \hookrightarrow M$ is called an *essential extension* if $L \cap N \neq 0$ for every nonzero submodule $L \subset M$.

36.1.1 Examples

- Let $R = k[x], M = \text{Frac}(R)$. Then $R \subset M$ is an essential extension. It is sufficient to check the essential extension condition when L is cyclic (these are the "smallest" submodules). Take any $0 \neq m \in M$. We want to show $R \cap R \cdot m \neq 0$ where $m = \frac{f}{g}$ and $g \neq 0$ and $f, g \in R$. Well, $m \neq 0$, so $f \neq 0$. Well, $f = gm \in R$, so $f \in R \cap Rm$, so $R \cap Rm$ is nonzero.
- Take R as before and set $N = R/(x) \cong k$. We would like to show that $K = \text{frac}(R)/R$ is an essential extension of k . $M = k(x)/k[x]$ can be considered explicitly. $k[x]$ comprises all nonnegative degree polynomials, so M consists entirely of purely negative degree polynomials. Thus we can write

$$M = k(x)/k[x] \cong \bigoplus_{i \geq 1} k \cdot \frac{1}{x^i}$$

This has a natural R -module structure as follows:

$$x^j \cdot \frac{1}{x^i} = \begin{cases} \frac{1}{x^{i-j}} & i > j \\ 0 & i \leq j \end{cases}$$

We can thus define an R -linear map $N \rightarrow M$ sending $1 \mapsto \frac{1}{x}$. These are both killed by x , so this extends nicely to an R -module homomorphism with trivial kernel. Another way to see this is to consider the map $k[x] \rightarrow M$ sending $1 \mapsto \frac{1}{x}$, and see that the kernel of this map is precisely (x) , implying that $k[x]/(x) \hookrightarrow M$ by the first isomorphism theorem. $N = k[x]/(x)$, which allows us to conclude the result.

Regardless, this shows that we have an extension, but we need to check that it is an essential extension. Note that every element of M is killed by a suitably large power of x . Thus, each non-zero submodule of M has a nonzero element that is killed by x . Since $R \cdot \frac{1}{x} = N$, it follows that we can choose i_z for any nonzero element z such that $x^{i_z}z = 0$ and $x^{i_z-1}z \neq 0$. It follows that $x^{i_z-1}z$ is killed by x , so $x^{i_z-1}z \in N$, implying that N is essential.

This last case gives rise to a general construction. If R is a commutative ring and M is an R -module, the **Injective Hull** of M is an essential extension of M that is also an injective R -module. This is normally denoted by $E_R(M)$, or $E(M)$ if the base ring is clear. Both of the examples above are examples of injective hulls, as they are both divisible modules over the PID $k[x]$, which implies that they are injective.

Theorem 36.1.1. *Suppose that R is a commutative ring. Then E is an injective R -module $\iff E$ has no proper essential extensions.*

Proof. We prove the forward direction direct. Suppose that E is injective and $E \hookrightarrow E'$ is an essential extension. Since E is injective, this injection splits, and $E' \cong E \oplus E''$ for some submodule $E'' \subset E'$. Since $E \cap E'' = 0$ and E'' is a submodule of E' , it follows that $E'' = 0$. Thus, $E' \cong E$.

For the reverse case, assume that E has no proper essential extensions. Given any $N \subset M$ with an R -linear morphism $\varphi : N \rightarrow E$, we need to extend this morphism to a morphism $M \rightarrow E$ (this is the definition of injectivity). In other words, we'd like to construct the following morphism:

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{i} & M \\ & & \downarrow \varphi & \swarrow \bar{\varphi} & \\ & & E & & \end{array}$$

Take the pushout diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{i} & M \\ & & \downarrow \varphi & & \downarrow f \\ & & E & \xrightarrow{g} & P \end{array}$$

Where

$$P = \frac{M \oplus E}{\{i(n) - \varphi(n) \mid n \in N\}}$$

We claim that since i is injective, g is also injective. To see this, choose $e \in \ker(g)$. If $g(e) = 0$, then $(0, e) = (i(n), -\varphi(n))$ for some $n \in N$. Thus, $i(n) = 0$, so $n = 0$ by injectivity of i . Thus, $-\varphi(n) = -\varphi(0) = 0$, so $e = 0$.

Now consider all submodules $L \subset P$ such that $L \cap E = 0$. Ordering these by inclusion, by Zorn's lemma there exists a maximal element L among all such submodules. As E is disjoint from L , may view E as a submodule of P/L . By maximality of L , P/L is an essential extension of E . Since E has no proper essential extensions, $E = P/L$. Since

$E \cap L = 0$, it follows that $P = E \oplus L$. Let $\pi : P \rightarrow E$ be the projection to the first coordinate. This gives a section

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{i} & M \\ & & \downarrow \varphi & \xrightarrow{\pi} & \downarrow f \\ & & E & \xrightarrow{g} & P \end{array}$$

Composing π, f yields a morphism $\pi \circ f : M \rightarrow E$, which by commutativity extends φ . □

Theorem 36.1.2. *If M embeds into an injective module I , then a maximal essential extension of M in I is an injective hull of M .*

Proof. Set \mathcal{S} to be the set of all essential extensions of M in I . Ordering this set by inclusion, we use Zorn's lemma to construct a maximal essential extension of M in I , denoted E . It is sufficient to check that E is an injective module, and we can do so using the previous theorem by showing that E has no proper essential extensions. Note that we are not only showing that E has no proper essential extensions contained in I , but that E has no proper essential extensions in general.

Assume that $E \hookrightarrow E'$ is an essential extension. Since I is injective, The morphism $E \hookrightarrow I$ extends to a morphism $\varphi : E' \rightarrow I$, where $\varphi|_E$ is the inclusion map $E \hookrightarrow I$. We'd like to show that φ is injective, If $\ker(\varphi) \neq 0$, then $\ker(\varphi) \cap E \neq 0$, but that would mean that $\varphi|_E$ has nontrivial kernel, which is a contradiction. Thus, we can conclude that φ is injective. We may thus view E' as a submodule of I . Since $M \rightarrow E, E \rightarrow E'$ are essential extensions, $M \rightarrow E'$ is an essential extension. By maximality, $E = E'$. □

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11/18/2022

37.1 Uniqueness of Injective Hulls

Suppose that $M \hookrightarrow I$, for I an injective R -module. Last time, we proved that any maximal essential extension of M in I is an injective hull of M . We'd now like to show that injective hulls are, in fact, unique.

Theorem 37.1.1. *Assume E and E' are injective hulls of M . Then, they are isomorphic as R -modules.*

Proof. $M \hookrightarrow E$ and $M \hookrightarrow E'$. Since E is an injective module, we can extend $M \hookrightarrow E'$ into a morphism $E \rightarrow E'$. As $M \hookrightarrow E$ and $M \hookrightarrow E'$, they both have vanishing kernels. Thus, $E \rightarrow E'$ also has vanishing kernel, so it too is an injective map. Since E is an injective module, this morphism splits, i.e. $\exists E'' \subset E'$ such that $E' \cong E \oplus E''$.

Notice that $E'' \cap E = 0$, but $M \subset E$. Thus, $E'' \cap M = 0$. However, $M \rightarrow E'$ is an essential extension, so $E'' = 0$. Thus, $E' \cong E$. \square

37.2 Minimal Injective Resolutions

We'd like to use an injective hull to construct a minimal injective resolution of a module. We should probably define this formally. Let $0 \rightarrow M \rightarrow I^\bullet$ be an injective resolution of M . It is called a *minimal injective resolution* if $\forall j$, I^{j+1} is the injective hull of $\text{im}(d^j) \subset I^{j+1}$.

To construct these, first take the injective hull of M , denoted $E(M)$, then take the cokernel of the map $M \rightarrow E(M)$, which is $E(M)/M$, then take the injective hull of that, and so forth. This yields the following resolution:

$$\begin{array}{ccccccc}
& & & & 0 & & \\
& & & & \uparrow & & \\
& & & & \text{---} & & \\
& & & & E(C_0)/C_0 := C_1 & & \\
& & & & \uparrow & \searrow & \\
0 & \longrightarrow & M & \longrightarrow & E(M) & \longrightarrow & E(C_0) & \longrightarrow & E(C_1) & \longrightarrow & \dots \\
& & & & \downarrow & & \uparrow & & & & \\
& & & & E(M)/M := C_0 & & & & & & \\
& & & & \downarrow & & & & & & \\
& & & & 0 & & & & & &
\end{array}$$

Lemma 37.2.1. *Any two minimal injective resolutions are isomorphic.*

One neat fact that we'll need about injective hulls is that taking them commutes with localizations.

Theorem 37.2.2. *Suppose R is Noetherian, $W \subset R$ is a multiplicative set, and M is an R -module. Then,*

$$W^{-1}E_R(M) \cong E_{W^{-1}R}W^{-1}M$$

Proof. First, let's prove that $W^{-1}E_R(M)$ is injective over $W^{-1}R$. Recall that an R -module N is injective $\iff \text{Ext}_R^1(R/I, N) = 0$ for any ideal $I \subset R$. We want to show that $\text{Ext}_{W^{-1}R}^1(W^{-1}R/JW^{-1}R, W^{-1}E_R(M)) = 0$, and conclude by uniqueness. R/I is a finitely generated module by Noetherianity of R , so it is also finitely presented. Well,

$$\text{Ext}_{W^{-1}R}^1(W^{-1}R/JW^{-1}R, W^{-1}E_R(M)) \cong W^{-1}\text{Ext}_R^1(R/I, E_R(M)) = 0$$

Because $\text{Ext}_R^1(R/I, E_R(M)) = 0$, as $E_R(M)$ is injective. Thus, it is sufficient to verify that $W^{-1}M \hookrightarrow W^{-1}E_R(M)$ is an essential extension. It is clearly that this is an extension, i.e. an injective map, so we just need to verify that this is essential. Thus, we need to show that $\forall z \neq 0$ contained in $W^{-1}E_R(M)$, $W^{-1}Rz \cap W^{-1}M \neq 0$. Write $z = \frac{x}{w}$ for $x \in E_R(M), w \in W$. We may replace z by x and assume that $z \in E_R(M)$. Consider $\mathcal{S} = \{\text{Ann}(wz) \mid w \in W\}$. R is Noetherian, so there exists a maximal element. Denote this by $\text{Ann}(uz)$, for $u \in W$. We may replace z by uz , so we can assume that $\text{Ann}(z)$ is a maximal element in \mathcal{S} .

Since $z \neq 0$, $M \hookrightarrow E_R(M)$ is an essential extension, since $Rz \cap M \neq 0$. Well, $Rz \cap M \subset Rz \cong R/\text{Ann}(z)$, so $Rz \cap M$ is a submodule of $R/\text{Ann}(z)$. $R/\text{Ann}(z)$ is a ring, so $Rz \cap M$ is necessarily an ideal. That is, $Rz \cap M \cong Iz$ for $I \subset R$ is an ideal of R that contained $\text{Ann}(z)$. As R is Noetherian, we can let $I = (g_1, \dots, g_n)$. If $g_i z = 0$ in $W^{-1}E_R(M)$, then there exists some $w_i \in W$ such that $w_i g_i z = 0$. Set $w = \prod w_i$. Then, $w \cdot g_i z = 0 \forall i$.

$\text{Ann}(z) \subset \text{Ann}(wz)$, but by construction, $\text{Ann}(z)$ is a maximal element within \mathcal{S} , so $\text{Ann}(z) = \text{Ann}(wz)$. Thus, because $w g_i z = 0 \iff g_i(wz) = 0$, $g_i \in \text{Ann}(wz) = \text{Ann}(z)$, so $g_i z = 0$, so $Iz = 0$. Thus, $Rz \cap M = Iz = 0$, which is a contradiction. Thus, $\exists g_i$ such that $g_i \neq 0$ in $W^{-1}E_R(M)$. Thus, $0 \neq g_i z \in W^{-1}E_R(M) \cap W^{-1}M$. \square

This has the following corollary:

Lemma 37.2.3. *Suppose R is Noetherian and $P \in \text{Spec}(R)$. Then,*

$$E_R(R/P)_P \cong E_{R_P}(\kappa(P))$$

37.3 Structure Theorem of Injective Modules

Such a structure theory is comprised of multiple parts, which we can prove one by one.

Theorem 37.3.1. (Structure of indecomposable injective R -modules) *Let R be a Noetherian ring and $P, Q \in \text{Spec}(R)$.*

- (1) $E(R/P)$ is indecomposable (i.e. it cannot be written as a direct sum of submodules)
- (2) If E is an indecomposable nonzero injective R -module, then $\exists P \in \text{Spec}(R)$ such that $E \cong E(R/P)$.
- (3) Given any element $x \in R \setminus P$, the multiplication by x map $E(R/P) \rightarrow E(R/P)$ is an isomorphism. (Note: This gives $E(R/P)$ a natural R_P -module structure, implying that $E(R/P)_P = E(R/P)$.)

Proof. We prove (1) first. Given any two submodules $E_1, E_2 \subset E(R/P)$, it is sufficient to check that $E_1 \cap E_2 \neq \emptyset$. Since $R/P \hookrightarrow E(R/P)$ is an essential extension, $J_1 := E_1 \cap R/P \neq 0$, $J_2 := E_2 \cap R/P \neq 0$, and both are contained in R/P . Because R/P is an integral domain, $J_1 J_2 \neq 0$ over R/P . However, $J_1 J_2 \subset J_1 \cap J_2$, so the J_i have nonzero intersection. It follows that $E_1 \cap E_2 \neq 0$.

To prove (2), we notice that since $E \neq 0$, $\text{Ass}(E) \neq \emptyset$. Thus, choose $P \in \text{Ass}(E)$. This gives us an injection $R/P \hookrightarrow E$. Thus, $E(R/P) \hookrightarrow E$. As $E(R/P)$ is an injective R -module, this injection splits, yielding $E \cong E(R/P) \oplus E'$. However, E is indecomposable to begin with. Thus, $E' = 0$, so $E \cong E(R/P)$.

To prove (3), notice that $m_x : R/P \rightarrow R/P$ is injective, as R/P is an integral domain. This extends to $E(R/P) \rightarrow E(R/P)$, which is also injective. Thus, $E(R/P)$ is isomorphic to its image. Since $\text{im}(E(m_x)) \hookrightarrow E(R/P)$, we get a splitting $E(R/P) \cong \text{im}(E(m_x)) \oplus E'$. However, $E(R/P)$ is indecomposable, so $E' = 0$ and this map is indeed a surjection. It follows that $E(m_x)$ is an isomorphism. \square

11/28/2022

38.1 Computing Bass Numbers

Recall that any minimal free resolution over $(R, \mathfrak{m}, \mathfrak{K})$, of the form $F_\bullet \rightarrow M \rightarrow 0$ tensored with \mathfrak{K} yields the resolution $F_\bullet \otimes_R \mathfrak{K}$ where all the differential maps are 0 maps.

Theorem 38.1.1. *Suppose R is Noetherian and M is an R -module. Let $0 \rightarrow M \rightarrow E^\bullet$ be an injective module (i.e. if $d^i : E^i \rightarrow E^{i+1}$, then E^{i+1} is the injective hull of $\text{im}(d^i)$) where $E^i = \bigoplus_{P \in \text{Spec}(R)} E(R/P)^{\mu_i(M,P)}$. $\mu_i(M, P)$ is the i th Bass Number of M with respect to P . Then, $\mu_i(M, P) = \dim_{\kappa(P)} \text{Ext}_{R_P}^i(\kappa(P), M_P)$.*

It turns out, this is true because hitting this resolution with $\text{Hom}_{R_P}(\kappa(P), -)$ sets all differentials to zero maps, similarly to how $\mathfrak{K} \otimes_R -$ acts on minimal free resolutions.

Proof. Consider $0 \rightarrow M_P \rightarrow E_P$. Then

$$E(R/Q)_P = \begin{cases} 0 & Q \not\subseteq P \\ E(R/Q) & \text{otherwise} \end{cases}$$

It follows that, upon localizing, each indecomposable of E_P^\bullet corresponds to a prime $Q \subseteq P$. Thus, we can just assume that we are over R_P to begin with, i.e. we can assume that $(R, \mathfrak{m}, \mathfrak{K})$ is actually local. Applying $\text{Hom}_R(\mathfrak{K}, -)$ to E^\bullet , this computes Ext . Well, $\text{Hom}_R(\mathfrak{K}, E^i) = \{z \in E^i \mid \mathfrak{m}z = 0\}$. E_i is an essential extension of $\text{im}(d^{i-1})$, so $\text{Hom}_R(\mathfrak{K}, E^i) \cap \text{im}(d^{i-1}) \neq \emptyset$. In particular, it follows that $Rz \cap \text{im}(d^{i-1}) \neq \emptyset \forall z \in \text{Hom}_R(\mathfrak{K}, E^i)$. Thus, $Rz \subset \text{im}(d^{i-1})$, so $\text{Hom}_R(\mathfrak{K}, E^i) \subset \text{im}(d^{i-1})$. By exactness, it follows that $d^i(\text{Hom}_R(\mathfrak{K}, E^i)) = 0$, so the corresponding map d_*^i on $\text{Hom}_R(\mathfrak{K}, E^i) \rightarrow \text{Hom}_R(\mathfrak{K}, E^{i+1})$ is the zero map for each i . It follows that

$$\text{Ext}_R^i(\mathfrak{K}, M) \cong \text{Hom}_R(\mathfrak{K}, E^i)$$

Now write $E^i = \bigoplus_{P \in \text{Spec}(R)} E(R/P)^{\mu_i(M,P)}$. If $P \not\subseteq \mathfrak{m}$, then $\forall x \in \mathfrak{m} \setminus P$, the multiplication by x map is an automorphism on $E(R/P)$. Thus, $\text{Hom}_R(\mathfrak{K}, E(R/P)) = 0$. Therefore,

$$\text{Hom}_R(\mathfrak{K}, E^i) = \text{Hom}_R\left(\mathfrak{K}, E(\mathfrak{K})^{\mu_i(M,\mathfrak{m})}\right) \cong \mathfrak{K}^{\mu_i(M,\mathfrak{m})}$$

Completing the proof. □

38.2 Gorenstein Rings

Theorem 38.2.1. *Suppose that $(R, \mathfrak{m}, \mathfrak{K})$ is a Noetherian local ring, and $d = \dim(R)$. Then, the following are equivalent:*

- (1) $\text{InjDim}_R(R) < \infty$
- (2) $\text{InjDim}_R(R) = d$
- (3) $\text{Ext}_R^i(\mathfrak{K}, R) = \begin{cases} \mathfrak{K} & i = d \\ 0 & i \neq d \end{cases}$
- (4) $\text{Ext}_R^i(\mathfrak{K}, R) = 0$ for some $i > d$.

If a Noetherian local ring $(R, \mathfrak{m}, \mathfrak{K})$ satisfies any of these condition, then R is called a **Gorenstein Ring**. For instance, regular local rings have finite global dimension, so they have finite injective dimension, and are thus Gorenstein. The 3rd condition above tells us that $\inf \{i \mid \text{Ext}_R^i(\mathfrak{K}, R) \neq 0\} = d$, which is precisely the depth of the ring. Thus, depth is equal to dimension, so Gorenstein rings are Cohen-Macaulay.

Proof. First we check that (1) \Rightarrow (2). As $(R, \mathfrak{m}, \mathfrak{K})$ is Noetherian local, let $\dim(R/P) = \delta$. Then, $\text{Ext}_R^{i+\delta}(\mathfrak{K}, M) = 0$, so $\text{Ext}_{R_P}^i(\kappa(P), M_P) = 0$. Set $t = \text{InjDim}(R)$. Pick a minimal prime P of R such that $\dim(R/P) = d$. Since $\text{Hom}_{R_P}(\kappa(P), R_P) \neq 0$, $\text{Ext}_R^d(\mathfrak{K}, R) \neq 0$. Thus, $t \geq d$. Induct on t ; when $t = 0$, then $d = 0$, so $t = d = 0$. Now assume that $t > 0$. Then, $\exists P \in \text{Spec}(R)$ such that $\text{Ext}_R^t(R/P, R) \neq 0$. If $P \neq \mathfrak{m}$, then pick $x \in \mathfrak{m} \setminus P$ and consider the short exact sequence

$$0 \longrightarrow R/P \xrightarrow{\cdot x} R/P \longrightarrow R/(P + (x)) \longrightarrow 0$$

Computing the long exact sequence over $\text{Ext}_R^t(-, R)$, we get

$$\dots \longrightarrow \text{Ext}_R^t(R/P, R) \xrightarrow{\cdot x} \text{Ext}_R^t(R/P, R) \longrightarrow \underbrace{\text{Ext}_R^{t+1}(R/(P + (x)), R)}_{=0} \longrightarrow \dots$$

Thus, $\text{Ext}_R^t(R/P, R) = x \text{Ext}_R^t(R/P, R)$. Since $\text{Ext}_R^t(R/P, R)$ is finitely generated, by Nakayama's lemma, $\text{Ext}_R^t(R/P, R) = 0$, a contradiction. Thus, $P = \mathfrak{m}$.

Now we check what happens if \mathfrak{m} is an associated prime. If this is true, then $\mathfrak{K} \hookrightarrow R$, giving us a short exact sequence $0 \rightarrow \mathfrak{K} \rightarrow R \rightarrow \text{coker} \rightarrow 0$. This induces a long exact sequence on Ext as follows:

$$\dots \longrightarrow \underbrace{\text{Ext}_R^t(R, R)}_{=0} \longrightarrow \text{Ext}_R^t(\mathfrak{K}, R) \longrightarrow \underbrace{\text{Ext}_R^{t+1}(\text{coker}, R)}_{=0} \longrightarrow \dots$$

Thus, $\text{Ext}_R^t(\mathfrak{K}, R) = 0$, a contradiction. It follows that \mathfrak{m} is not an associated prime, so R has positive depth. Pick $r \in \mathfrak{m}$ to be an R -regular element, and let $\bar{R} = R/(r)$. Then $\text{InjDim}_R(\bar{R}) = \text{InjDim}_R(R) - 1$. We've proved that $\text{Ext}_{\bar{R}}^i(N, \bar{R}) = \text{Ext}_R^{i+1}(N, R)$ for each \bar{R} -module N where $\text{Hom}_R(N, R) = 0$. By induction, it follows that $t - 1 = \text{InjDim}(\bar{R}) = \dim(\bar{R}) = d - 1$, so $t = d$.

Now onto (2) \Rightarrow (3). When $d = 0$, $\mathfrak{m} \in \text{Ass}(R)$, so $\mathfrak{K} \hookrightarrow R$. It follows from (2) that R is injective, as it has 0 injective dimension. It follows that $R = \text{Hom}_R(R, R) \twoheadrightarrow \text{Hom}_R(\mathfrak{K}, R)$, as $\text{Hom}_R(-, R)$ is exact. Thus, $\text{Hom}_R(\mathfrak{K}, R)$ is a cyclic module (generated by 1 element). Since it is also nonzero, it follows that $\text{Hom}_R(\mathfrak{K}, R) \cong \mathfrak{K}$, since $\text{Hom}_R(\mathfrak{K}, R)$ is killed by \mathfrak{m} and is a quotient of R . Since R has 0 injective dimension, it follows that $\text{Ext}^i(\mathfrak{K}, R) = 0$ for $i > 0$, giving us the desired result in the dimension 0 case.

Now let's induct on dimension, and suppose that $d > 0$. Via similar logic to the (1) \Rightarrow (2) case, it follows that $\delta(R) > 0$ as well. Pick $r \in \mathfrak{m}$ that is R -regular, and set $\bar{R} = R/(r)$. $\dim(\bar{R}) = \dim(R) - 1$ and $\text{InjDim}(\bar{R}) = \text{InjDim}(R) - 1$. By induction,

$$\text{Ext}_R^i(\mathfrak{K}, \bar{R}) = \begin{cases} \mathfrak{K} & i = d - 1 \\ 0 & i \neq d - 1 \end{cases}$$

Because $\text{Ext}_R^{i+1}(\mathfrak{K}, R) = \text{Ext}_{\bar{R}}^i(\mathfrak{K}, \bar{R})$. The result follows. Since it is clear that (3) \Rightarrow (4) is trivial, it suffices to prove (4) \Rightarrow (1). We solve this via inducting on $\dim(R)$. If $d = \dim(R) = 0$, then $\text{Spec}(R) = \{\mathfrak{m}\}$, so it follows that $\text{Ext}_R^i(\mathfrak{K}, R) = \text{Ext}_R^i(R/\mathfrak{m}, R) = 0$. Thus, $\text{InjDim}(R) \leq i - 1$. Now, assume that $d > 0$, for each prime ideal $P \neq \mathfrak{m}$, set $\delta = \dim(R/P)$, which is precisely the height of the ideal \mathfrak{m}/P . It follows that if $\text{Ext}_R^i(R/\mathfrak{m}, R) = 0$, then $\text{Ext}_{R_P}^{i-\delta}((R/P)_P, R_P) = 0$. On the other hand, $\dim(R/P) = \text{ht}(P) \leq d - \delta$ [because $\text{ht}(P) + \dim(R/P) \leq \dim(R)$]. Thus $\delta < i$, so by the inductive hypothesis, $\text{InjDim}(R_P) < i - \delta < \infty$. Thus, for any finitely generated R -module M ,

$$\text{Ext}_R^i(M, R)_P = \text{Ext}_{R_P}^i(M_P, R_P) = 0$$

It follows that $\text{Supp}_R(\text{Ext}_R^i(M, R)) \subset \{\mathfrak{m}\}$. Since $\text{Ext}_R^i(M, R)$ is finitely generated, its support is precisely $V(\text{Ann}(\text{Ext}_R^i(M, R)))$. It follows that $\mathfrak{m} \subset \sqrt{\text{Ann}(\text{Ext}_R^i(M, R))}$, so in particular, since R is Noetherian, a finite power of \mathfrak{m} is contained in the annihilator. Thus, $\text{Ext}_R^i(M, R)$ is an Artinian module, and thus has finite length.

To conclude, it is sufficient to check that $\text{Ext}_R^i(R/P, R) = 0 \forall P \in \text{Spec}(R)$. Well, suppose not. Then, $\exists P \in \text{Spec}(R)$ such that $\text{Ext}_R^i(R/P, R) \neq 0$. As R is Noetherian, we can assume that P was chosen such that P is maximal among all primes where $\text{Ext}_R^i(R/P, R) \neq 0$. This P is not \mathfrak{m} , so choose $x \in \mathfrak{m} \setminus P$. Take the short exact sequence

$$0 \longrightarrow R/P \xrightarrow{\cdot x} R/P \longrightarrow R/(P + (x)) \longrightarrow 0$$

Now consider a prime filtration of $R/(P + (x))$.

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_t = R/(P + (x))$$

Where $M_{j+1}/M_j \cong R/P_j$. Each prime P_j appearing in this filtration properly contains P . Thus for every such P_j , $\text{Ext}_R^i(R/P_j, R) = 0$, by maximality of P . Consider the short exact sequence

$$0 \longrightarrow (M_1 =)R/P_1 \longrightarrow M_2 \longrightarrow R/P_2 \longrightarrow 0$$

It follows that from this that $\text{Ext}_R^i(M_2, R) = 0$. From here we repeat this process again on the sequence

$$0 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow R/P_3 \longrightarrow 0$$

To conclude that $\text{Ext}_R^i(M_3, R) = 0$, and so on. Repeating this process finitely many times allows us to conclude that $\text{Ext}_R^i(R/(P + (x)), R) = 0$. It follows that if we take the long exact sequence in Ext of the first short exact sequence we took, we see that

$$\dots \longrightarrow \underbrace{\text{Ext}_R^i(R/(P + (x)), R)}_{=0} \longrightarrow \text{Ext}_R^i(R/P, R) \xrightarrow{\cdot x} \text{Ext}_R^i(R/P, R) \longrightarrow \dots$$

Thus, the multiplication by x map, denoted φ , on $\text{Ext}_R^i(R/P, R)$ is injective by exactness. Therefore, $\text{Ext}_R^i(R/P, R) \cong \text{im}(\varphi)$, so they both have the same length. As we've proved that $\text{Ext}_R^i(R/P, R)$ has finite length, it follows that φ is indeed surjective. Thus, $\text{Ext}_R^i(R/P, R) = x \text{Ext}_R^i(R/P, R)$, so by Nakayama's Lemma, $\text{Ext}_R^i(R/P, R) = 0$. \square

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12/02/2022

39.1 Uniqueness of Canonical Modules

Theorem 39.1.1. $(R, \mathfrak{m}, \mathfrak{K})$ is a Cohen Macaulay ring with canonical module C .

- (1) If N is a CM R -module (i.e. $\delta(N) = \dim(N) = t$) then $\text{Ext}^j(N, C)$ is Cohen Macaulay of dimension t if $j = d - t$, and 0 otherwise.
- (2) If $\delta(N) = d$, then

$$\text{Hom}_R(N, C) \otimes_R R/(x_1, \dots, x_d) \cong \text{Hom}_{R/(x_1, \dots, x_d)}(N/(x_1, \dots, x_d)N, C/(x_1, \dots, x_d)C)$$

for every maximal R -sequence x_1, \dots, x_d .

Proof. We prove statement (1) first. Recall that over a Noetherian local ring, if $M, N \neq 0$ and finitely generated with depths n, t respectively, then $\text{Ext}^i(N, M) = 0$ for any $i < n - t$. It follows that $\text{Ext}^j(N, C) = 0$ for $j < d - t$. For the remaining case (where $j > d - t$) we now induct on t . When $t = 0$, $\exists n$ such that $\mathfrak{m}^n N = 0$. Thus, N has a composition series

$$0 = N_0 \subsetneq N_1 \subsetneq \dots \subsetneq N_\ell = N$$

Where $N_{i+1}/N_i \cong \mathfrak{K}$. Since $\text{Ext}_R^j(\mathfrak{K}, C) = 0$ for $j > d$, it follows that $\text{Ext}^j(N, C) = 0$ for $j > d = d - 0$. Thus, Ext vanishes for any $i \neq d - t$.

Now onto the inductive step. Assume $t > 0$. Choose $x \in \mathfrak{M}$ that is N -regular and take the short exact sequence

$$0 \longrightarrow N \xrightarrow{\cdot x} N \longrightarrow \overline{N} = N/xN \longrightarrow 0$$

Taking a long exact sequence in Ext gives the sequence

$$\dots \longrightarrow \text{Ext}_R^j(N, C) \xrightarrow{\cdot x} \text{Ext}_R^j(N, C) \longrightarrow \text{Ext}_R^{j+1}(\overline{N}, C) \longrightarrow \dots$$

$\delta(\overline{N}) = t - 1 = \dim(\overline{N})$, so by induction $\text{Ext}_R^{j+1}(\overline{N}, C)$ vanishes when $j + 1 > d - t + 1$ (so when $j > d - t$). It follows that $\text{Ext}_R^j(N, C) = x \text{Ext}_R^j(N, C)$, so $\text{Ext}_R^j(N, C) = 0$ by Nakayama's Lemma.

Thus all that remains is the case where $i = d - t$. To compute this we induct on t again. When $t = 0$, as before $\mathfrak{m}^n \cdot N = 0$ for some n , so $\mathfrak{m}^n \cdot \text{Ext}_R^{d-t}(N, C) = 0$. Therefore, $\text{Ext}_R^{d-t}(N, C)$ is Cohen Macaulay of dimension 0 as desired. For the inductive step, suppose that $t > 0$, and choose $x \in \mathfrak{m}$ to be N -regular as before. Taking the short exact sequence

$$0 \longrightarrow N \xrightarrow{\cdot x} N \longrightarrow \bar{N} = N/xN \longrightarrow 0$$

again and looking at the long exact sequence in Ext, we get

$$\underbrace{\text{Ext}_R^{d-t}(\bar{N}, C)}_{=0} \longrightarrow \text{Ext}_R^{d-t}(N, C) \xrightarrow{\cdot x} \text{Ext}_R^{d-t}(N, C) \longrightarrow \text{Ext}_R^{d-t+1}(\bar{N}, C) \longrightarrow \dots$$

As $\dim(N) = t - 1$, $\text{Ext}_R^j(\bar{N}, C) = 0$ for $j \neq d - (t - 1) = d - t + 1$, thus x is regular on $\text{Ext}_R^{d-t}(N, C)$ and we have the isomorphism

$$\text{Ext}_R^{d-t+1}(\bar{N}, C) = \frac{\text{Ext}_R^{d-t}(N, C)}{x \text{Ext}_R^{d-t}(N, C)}$$

By induction,

$$\dim \left(\text{Ext}_R^{d-t+1}(\bar{N}, C) \right) = \delta \left(\text{Ext}_R^{d-t+1}(\bar{N}, C) \right) = t - 1$$

Thus we can pick a regular sequence $y_1, \dots, y_{t-1} \in \mathfrak{m}$ on $\text{Ext}_R^{d-t+1}(\bar{N}, C)$, suggesting that x, y_1, \dots, y_{t-1} is a regular sequence on $\text{Ext}_R^{d-t}(N, C)$. Thus,

$$\delta \left(\text{Ext}_R^{d-t}(N, C) \right) = t = \dim \left(\text{Ext}_R^{d-t}(N, C) \right)$$

Proving part (1). For part (2), we use (1) to conclude that $\text{Hom}_R(N, C)$ is CM of dimension d , and $\text{Ext}_R^j(N, C) = 0$ for $j > 0$. If x is R -regular, then

$$0 \longrightarrow R \xrightarrow{\cdot x} R \longrightarrow \bar{R} = R/xR \longrightarrow 0$$

Plugged into the functor $\text{Hom}_R(-, C)$ gives us

$$0 \longrightarrow \underbrace{\text{Hom}_R(\bar{R}, C)}_{=0} \longrightarrow C \xrightarrow{\cdot x} C \longrightarrow 0$$

It follows that x is regular over C as well. It follows that every R -sequence x_1, \dots, x_d is also a C -sequence. Therefore, we can act on the short exact sequence

$$0 \longrightarrow C \xrightarrow{\cdot x_1} C \longrightarrow \bar{C} = C/x_1C \longrightarrow 0$$

Via the functor $\text{Hom}_R(N, -)$ to get

$$0 \longrightarrow \text{Hom}_R(N, C) \xrightarrow{\cdot x_1} \text{Hom}_R(N, C) \longrightarrow \text{Hom}_R(N, \bar{C}) \longrightarrow \underbrace{\text{Ext}_R^1(N, C)}_{=0}$$

It follows that

$$\mathrm{Hom}(N, C) \otimes_R R/(x_1) \cong \mathrm{Hom}_R(N, C)/x_1 \mathrm{Hom}_R(N, C) \cong \mathrm{Hom}(N, \bar{C}) \cong \mathrm{Hom}_{\bar{R}}(\bar{N}, \bar{C})$$

So Hom commutes with quotients by regular elements, and (2) is proved. \square

Lemma 39.1.2. *Suppose that (R, \mathfrak{m}) is Noetherian and local, where $f : M \rightarrow N$ is an R -linear morphism of finitely generated R -modules. If $(x_1, \dots, x_d) = \underline{x}$ is an N -sequence, then if $f \otimes_R R/(\underline{x})$ is an isomorphism, then so is f .*

Proof. We can conclude surjectivity immediately by Nakayama's Lemma, so it suffices to verify injectivity. Set $K = \ker(f)$. It is sufficient to verify this in the case where $d = 1$, i.e. $\underline{x} = x$. Taking the short exact sequence $0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0$ and tensoring by $R/(x)$, because $\mathrm{Tor}_1(N, R/(x)) = 0$, by vanishing we see that $K/xK = 0$, so $K = xK$. By Nakayama's Lemma, $K = 0$, so f is injective. \square

Theorem 39.1.3. *Suppose $(R, \mathfrak{m}, \mathfrak{K})$ is CM, C, C' are canonical modules, and $\underline{x} = (x_1, \dots, x_d)$ is a maximal R -sequence. Let $\bar{R} = R/(\underline{x})$, $\bar{C} = C/(\underline{x})C$, and $\bar{C}' = C'/(\underline{x})C'$. Then,*

$$(1) C/\underline{x}C \cong E_{\bar{R}}(\mathfrak{K})$$

$$(2) C \cong C'$$

$$(3) \mathrm{Hom}_R(C, C') \cong R$$

Proof. As $\mathrm{InjDim}_R(C) = d$, it follows that $\mathrm{InjDim}_{\bar{R}}\bar{C} = 0$. Thus,

$$\mathrm{Hom}_{\bar{R}}(\mathfrak{K}, \bar{C}) \cong \mathrm{Ext}_{\bar{R}}^d(\mathfrak{K}, \bar{C}) \cong \mathfrak{K}$$

Thus, \bar{C} is an essential extension of \mathfrak{K} . As $\{\mathfrak{m}\} = \mathrm{Ass}_{\bar{R}}(\bar{C})$, it follows that $\mathfrak{K} \hookrightarrow \bar{C}$, as \mathfrak{m} is an associated prime. Take any $0 \leq L \subseteq \bar{C}$. Thus, $\mathrm{Hom}_R(\mathfrak{K}, L) \neq 0$. Since Hom is left exact and $L \hookrightarrow \bar{C}$, we see that

$$\mathrm{Hom}_R(\mathfrak{K}, L) \hookrightarrow \mathrm{Hom}(\mathfrak{K}, \bar{C}) = \mathfrak{K}$$

Since $\mathrm{Hom}_R(\mathfrak{K}, L) \neq 0$ and is contained in \mathfrak{K} , it follows that $\mathrm{Hom}_R(\mathfrak{K}, L) = \mathfrak{K}$. Therefore, we have minimality and $\bar{C} = E_{\bar{R}}(\mathfrak{K})$.

For part (2), we know that $\mathrm{Hom}_R(C, C') \otimes_R \bar{R} = \mathrm{Hom}_{\bar{R}}(\bar{C}, \bar{C}')$ via the previous theorem. From part (1), we see that this is the same thing as $\mathrm{Hom}_{\bar{R}}(E_{\bar{R}}(\mathfrak{K}), E_{\bar{R}}(\mathfrak{K})) \cong \bar{R}$. Therefore, we see that $\mathrm{Hom}_{\bar{R}}(\bar{C}, \bar{C}')$ is cyclic, so by Nakayama's Lemma, $\mathrm{Hom}_R(C, C')$ is cyclic as well. Picking a generator $\varphi \in \mathrm{Hom}_R(C, C')$, we see via the isomorphism $\mathrm{Hom}_{\bar{R}}(E_{\bar{R}}(\mathfrak{K}), E_{\bar{R}}(\mathfrak{K})) \cong \bar{R}$ that $\bar{\varphi}$ can be assigned to $\bar{1}$, so $\bar{\varphi}$ is an isomorphism. Since $\bar{\varphi}$ is an isomorphism, it follows that φ is an isomorphism, so $C \cong C'$. This also tells us part (3). \square

Therefore, Canonical modules are unique. As for existence, we have the following result:

Theorem 39.1.4. *If (R, \mathfrak{m}) is CM and Noetherian, R admits a Canonical Module $\iff R$ is a homomorphic image of a Gorenstein ring.*

Proof. We prove the reverse case first. If $A \twoheadrightarrow R$ where A is a Gorenstein ring, it follows that

$$\text{Ext}_A^{\dim(A)-\dim(R)}(R, A)$$

is a canonical module. We can see this by killing a regular sequence in A to make $\dim(A) = \dim(R)$. In that case, we reduce to $\text{Hom}_A(R, A)$. Then kill a maximal regular sequence to reduce to the Artinian case. In the Artinian case, A is an injective module over itself, so it follows that $\text{Hom}_A(R, A)$ is an injective R -module via tensor-Hom adjunction.

The forward case is much harder, and uses a technique called Nagata's Idealization. The idea is that, if you are given an ring R and R -module M , there exists a new ring, denoted $R * M$, where M is naturally an ideal of $R * M$. Points in $R * M$ are of the form (r, m) , and we define multiplication by

$$(r_1, m_1) \cdot (r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$$

With addition being defined the normal way. This is clearly a ring. Thus, if R is a CM Noetherian ring, and C is the canonical module, one can verify that $R * C$ is Gorenstein, where $R * C \twoheadrightarrow R$ by forgetting the second coordinate. \square

This far generalizes to any Noetherian ring:

Theorem 39.1.5. *A Noetherian ring R admits a dualizing complex $\iff R$ is a homomorphic image of a Gorenstein ring.*