Preliminary Info

These are notes taken from a portion of Math 554 (Complex Manifolds) at the University of Illinois Chicago in Fall 2022. The class was taught by Professor Izzet Coskun, and these notes are TeXed by Vignesh Jagathese (me!). There are still some minor errors in these notes, and by reading these and benefiting from them, all I ask in return is that you forward any errors to me.

These notes are for the class through around mid-October, where we followed Forester's book on Riemann Surfaces. We covered the theory of 1-dimensional complex analytic Manifolds (i.e. Riemann Surfaces) from a topological, differential, and algebraic perspective. Towards the end of these notes we prove that the category of Riemann Surfaces is equivalent to the category of projective algebraic curves, implying that one can study these purely algebreo-geometrically. The rest of the course covered these objects from that perspective, but unfortunately those notes are not as cleanly formatted, and have thus been lost to time.

If there any suggestions you would recommend or additions that you would like to add, please email me at vjagat2 [at] (My graduate school's website).

Contents

Chapter 1

Introduction

1.1 Some Preliminaries

This course will primarily focus on Complex Manifolds and Riemann Surfaces. We'll be using Forester for the more 'analytic' treatment first, then move on to ACGH for the more algebraic treatment.

1.2 Introduction to Riemann Surfaces

A ***Riemann Surface*** is a 1-dimensional complex analytic connected manifold. As we know from Math 549, a (Real) Manifold is:

- Hausdorff
- Second Countable
- A Topological Space
- Locally Homeomorphic to $\mathbb{R}^{2n}$ (i.e. $\exists$ some set of charts.)

A Manifold M is ***Complex Analytic*** if $\exists$ a chart (U_α, z_α) such that $M = \bigcup U_\alpha$ (i.e. U_α is an open cover of M) and $z_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^n$ is a homeomorphism to some open subset V_α, with complex analytic transition functions $f_{\alpha\beta} := z_\beta \circ z_\alpha^{-1} : \mathbb{C}^n \to \mathbb{C}^n$. In these notes you may see these referred to as "$\mathbb{C}$-folds".

1.2.1 Examples of Riemann Surfaces

(1) $\mathbb{C}$

(2) $\mathbb{CP}^1 = \mathbb{C} \cup \infty$. This is a $\mathbb{C}$-fold, as $\mathbb{CP}^1 \backslash \{\infty\} \cong \mathbb{C}$ and $\mathbb{CP}^1 \backslash \{0\} \cong \mathbb{C}$, both via stereographic projection. The transition map simply takes nonzero, finite elements to their reciprocal.

(3) Any domain (open, connected subset) of a Riemann Surface is also a Riemann Surface, such as the unit disk $\Delta := \{z \in \mathbb{C} \mid |z| < 1\}$

1.3 Morphisms

A continuous map of $\mathbb{C}$-folds $f : M \to N$ is said to be ***analytic*** (equivalently, ***holomorphic***) if for all local coordinates (U_α, z_α) on M and (V_β, w_β) on N, $f(U_\alpha) \cap V_\beta \neq \varnothing$ and $w_\beta \circ f \circ z_\alpha^{-1} : \mathbb{C}^n \to \mathbb{C}^n$ is a holomorphic function.

f is a ***biholomorphism*** if it is a holomorphic map with holomorphic inverse. If such an f were to exist, then we say that M and N are ***biholomorphic***. We say a map is ***conformal*** if it is holomorphic and bijective (equivalently, conformal maps are those that preserve angles).

A holomorphic map with codomain $\mathbb{C}$ is called a ***holomorphic function***. A holomorphic map with codomain $\mathbb{CP}^1$ that is not identically $f(x) = \infty$ is called ***meromorphic***.

For a $\mathbb{C}$-fold M, we let $\mathcal{H}(M)$ denote the $\mathbb{C}$-algebra of holomorphic functions on M, and $\mathcal{K}(M)$ denote the field of meromorphic functions on M. We will be interested in computing these when M is a Riemann surface.

Theorem 1.3.1. *Suppose M and N are $\mathbb{C}$-folds, with M compact. Then, either f is constant or f is surjective.*

Proof. We'll prove this when M and N are Riemann Surfaces, but the result holds in general. If f was non-constant, then f is an open map (via the open mapping theorem), implying that $f(M)$ is open. However, as M is compact, $f(M)$ is compact and closed. Therefore, $f(M)$ is open, closed, and compact, and as its codomain is connected, it follows that $f(M) = N$. In particular, f is surjective, and $f(M) = N$ is compact. $\square$

Corollary 1.3.1.1. *If M is a compact Riemann Surface, then $\mathcal{H}(M) = \mathbb{C}$.*

1.4 Uniformization Theorem

One major goal of this course is to classify all simply connected Riemann Surfaces up to biholomorphism.

Theorem 1.4.1. *Every simply connected Riemann Surface is biholomorphic to one of $\mathbb{C}, \mathbb{CP}^1, \Delta$. As expected, these three surfaces are distinct biholomorphically.*

While it is highly nontrivial that these are the only three simply connected Riemann surfaces, their distinctness is pretty simple. First, $\mathbb{CP}^1$ is the only compact space of the three, so it is biholomorphically distinct. Similarly, a simple Louiville's Theorem argument shows that any entire, holomorphic map from $\Delta \to \mathbb{C}$ is constant, so these cannot be biholomorphic.

1.5 Universal Covers

While not all Riemann Surfaces X are simply connected, they all have simply connected universal covers $\mathcal{U} \to X$. It's easy to check that universal covers of Riemann surfaces are also Riemann surfaces, so it follows that $\mathcal{U}$ is biholomorphic to one of $\mathbb{C}, \mathbb{CP}^1$, or Δ (usually Δ). In particular, we know that $\pi_1(X)$ acts on $\mathcal{U}$ via deck transformations, and that $X = \mathcal{U}/\pi_1(X)$. That means that, in general, if we know $\pi_1(X)$, we can determine X entirely from its universal cover, which can only be 1 of 3 things.

1.6 Computing Groups of Biholomorphisms

1.6.1 Biholomorphisms of $\mathbb{CP}^1$

Theorem 1.6.1. *Biholomorphisms on $\mathbb{CP}^1$ are precisely the möbius transformations, i.e. linear fractional transformations (LFTs). This implies that the group of biholomorphisms over $\mathbb{C}$ is isomorphic to $\mathrm{PGL}_2(\mathbb{C})$.*

Proof. It's already known that LFTs are biholomorphisms on $\mathbb{CP}^1$ (they are clearly holomorphic and their inverses are computable and still LFTs). Furthermore, $\forall p_1, p_2, p_3 \in \mathbb{CP}^1$, $\exists$ an LFT f such that $f(p_1) = 0, f(p_2) = \infty, f(p_3) = 1$. In particular, $f(z) = \frac{(p_3-p_2)(z-p_1)}{(p_3-p_1)(z-p_2)}$.

Therefore, if we were to pre and post-compose an arbitrary $\mathbb{CP}^1$-biholomorphism f with LFTs, then $f(0) = 0, f(\infty) = \infty$. $f(z)/z$ is analytic over all of $\mathbb{CP}^1$, and is bounded in a neighborhood of ∞, suggesting that $f(z)/z = c$, a constant. Thus, $f(z) = cz$, and f pre and post-composed is still an LFT, so we can conclude that all $\mathbb{CP}^1$-biholomorphisms are LFTs. □

1.6.2 Biholomorphisms of Δ

Theorem 1.6.2. *Biholomorphisms on Δ are LFTs of the form*

$$f(z) = \varepsilon \cdot \frac{z-\alpha}{1-\bar{\alpha}z} \qquad |\varepsilon| = 1, \alpha \in \Delta$$

This implies that the group of biholomorphisms over $\mathbb{C}$ is isomorphic to $\mathrm{PSL}_2(\mathbb{R})$.

To understand biholomorphisms of Δ, we must use Schwarz' Lemma.

Lemma 1.6.3. *Suppose $f : \Delta \to \Delta$ is a holomorphism. If $f(0) = 0$ then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. If either of those two inequalities is in fact an equality, then $f(z) = \varepsilon z$ for $|\varepsilon| = 1$.*

Proof. It's pretty easy to check that $z \mapsto \varepsilon \frac{z-a}{1-\bar{a}z}$ is a biholomorphism from $\Delta \to \Delta$. We just want to check that all biholomorphisms are of this form. Well, via pre/post-composing with LFTs, we can suppose that any biholomorphism f sends $0 \mapsto 0$. We apply Schwarz Lemma to see that $f(z) = cz$ for $|c| = 1$, so f is an LFT. By enforcing that f maps $\Delta \to \Delta$, f is then of the desired form. □

1.6.3 Bilholomorphisms of $\mathbb{C}$

Consider $f(z) = az + b$ for $a \neq 0$. We claim that such an f can be extended to $\mathbb{P}^1$, as the singularity at $\infty$is isolated. That is not enough though, as we need to verify that this singularity is non-essential. Well, this follows from the Casoratti-Weierstraß theorem.

Therefore, we can conclude that f has a simple pole at ∞. Note that, if we want f to act without fixed points, then $a = 1$. This is because, if there were a fixed point at z, then $az + b = z$, so if $a \neq 1$ f has a fixed point at $z = \frac{-b}{a-1}$, so it always has a fixed point.

Now let's look at group quotients by the action $z \mapsto z + 1$ (1 can be any $b \in \mathbb{R}$) on $\mathbb{C}$ (this will be used to get a variety of Riemann Surfaces with universal cover $\mathbb{C}$). This yields a cylinder $\mathbb{S}^1 \times \mathbb{R}$, which is a Riemann Surface, implying that a real cylinder has complex structure.

But what about the action of $z + \tau$ for $\tau \notin \mathbb{R}$. This yields a Torus, which is a Riemann Surface via this identification. Thus, one can put a complex structure on a Torus.

Now let Γ act on $\mathbb{C}$ properly and discontinuously. Let $V \subset \mathbb{C}$ be the smallest $\mathbb{R}$-vector space containing Γ. We have a couple cases:

- $\dim_{\mathbb{R}} V = 0$: In this case, $\Gamma = \mathrm{Id}$.
- $\dim_{\mathbb{R}} V = 1$: Then the smallest length element generates the group, implying that the group of deck transformations is $\mathbb{Z}$.
- $\dim_{\mathbb{R}} V = 2$: Yields a lattice in $\mathbb{C}$.

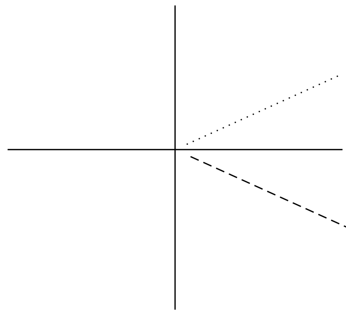
Over $\mathbb{CP}^1$, one can ask when biholomorphisms have fixed points. If there were a fixed point at z, then $\frac{az+b}{cz+d} = z$, suggesting that $az + b = cz^2 + dz$, so $cz^2 + (d - a)z - b = 0$. Thus we have the following cases:

- If $c \neq 0$, then this is a degree 2 polynomial, so it has 1 or 2 fixed points with multiplicity (the roots of the equation)
- If $c = 0$ but $d \neq a$, then it has 1 fixed point.
- If $c = 0$ and $d = a$, the LFT becomes $z + b/a$, which has a fixed point at ∞.

Thus, we always have fixed points.

1.7 Riemann Surfaces associated to multi-valued functions

Consider the Riemann Surfaces associated to $\sqrt{z}$. In polar coordinates, this is the map $Re^{i\theta} \mapsto \sqrt{R}e^{i\theta/2}$. This can take on multiple values depending on the range of θ. One way to force this to force a bound $0 < \theta < 2\pi$, but the issue is that this function can't even be extended to a continuous function. Thus, the answer is to make two separate copies of $\mathbb{C}$ then ***glue them***.

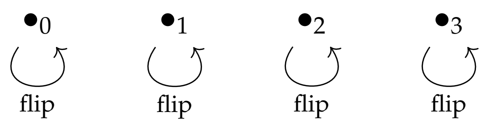


This is a bit easier to visualize over $\mathbb{CP}^1$. Viewing these rays from 0 to ∞, we slice a longitudinal half-circle from each, then glue the spheres along the open lines. This would then yield another sphere, allowing us to define a single-valued variant of a multivalued function over $\mathbb{CP}^1$ that retains sufficient information.

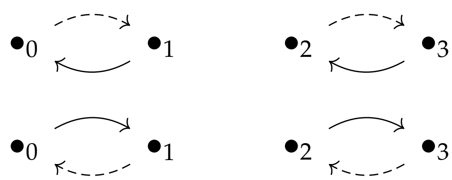
Generalizing a bit, consider the map $z \mapsto \sqrt{z(z-1)(z-2)(z-3)}$. We'd like to understand the behavior when rotating around the following points:

$$\bullet_0 \qquad \bullet_1 \qquad \bullet_2 \qquad \bullet_3$$

Rotating around each point individually flips orientation:



Suggesting that rotating around two of the points is a neutral action. This allows us to glue two copies of $\mathbb{C}$ as follows:



Which yields a Torus! In general, the Riemann Surface associated to the morphism $z \mapsto \sqrt{\prod_{i=1}^{2g+2}(z-a_i)}$ yields a g-torus. This gives a g-Torus a complex structure. It follows that all closed orientable surfaces have a complex structure (namely, as a quotient of $\mathbb{C}$ under the action of some multi-valued functions).

1.7.1 A Sidebar on Orientability

Theorem 1.7.1. *All $\mathbb{C}$-surfaces have a fixed orientation.*

Proof. Let $f(x,y) = u(x,y) + iv(x,y)$ be a holomorphic function viewed as a map $\mathbb{R}^2 \to \mathbb{R}^2$. Recall that this satisfies the Cauchy Riemann Equations $u_x = v_y$ and $u_y = -v_x$. Thus, we can compute the $\mathbb{R}$-Jacobian of f as follows:

$$\mathcal{J}_{\mathbb{R}}(f) = \det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = u_x v_y - u_y v_x = u_x^2 + u_y^2 > 0$$

Fixing an orientation for the Riemann Surfaces associated to the action of f. $\square$

But are all Riemann Surfaces determined by a morphism? Well, for this we have the following (very general) theorem:

Theorem 1.7.2. *Every compact Riemann Surface is an algebraic curve* $i : X \hookrightarrow \mathbb{P}^3$ *such that* $i(x)$ *is defined by a polynomial.*

Chapter 2

Holomorphic Maps

2.1 Local Behavior of Holomorphic Maps

Lemma 2.1.1. *$f : X \to Y$ is a holomorphic map of Riemann Surfaces, and assume $f(x) = y$, $x \in X, y \in Y$. Then $\exists k \geq 1$ and charts $\varphi : U \to V$ on X and $\psi : U' \to V'$ on Y such that:*

(a) $\varphi(x) = \psi(y) = 0$

(b) $f(U) \subset U'$

(c) For $F : \psi \circ f \circ \varphi^{-1} : V \to V'$, $F(z) = z^k$.

Proof. For $W, W' \subset \mathbb{C}$ such that $0 \in W, W'$, one can suppose that $f : W \to W'$ such that $f(0) = 0$. In some neighborhood of 0, f has a power series expansion $f = \sum a_n z^n$ such that $a_n = 0$ for $n > k$ and $a_k \neq 0$, so assume that the order of 0 is k (for the k fixed above). Then

$$f(z) = z^k(a_k + a_{k+1}z + \dots)$$

for $a_k \neq 0$. Let

$$g(z) = a_k + a_{k+1}z + \dots$$

In some small neighborhood, $g \neq 0$, so $\exists h$ such that $h(z)^k = g(z)$. Using $z' = zh$ as a new variable, we see that $F = (z')^k$.

As we know, nonconstant holomorphic maps are open, so (b) follows. □

As a corollary, we have that any nonconstant holomorphic map $f : X \to Y$ is open and discrete (i.e. the preimages of points are discrete).

If we choose $f_1, f_2 : X \to Y$ (holomorphic maps) and a sequence of distinct points $\{x_i\}$ such that $\lim x_i = x$, if $f_1(x_i) = f_2(x_i)$ for all i, then we have $f_1 = f_2$ (as the difference has non-isolated zeroes). As an example of this phenomenon, consider the punctured plane $\mathbb{C}^\times$, which has universal cover $\mathbb{C}$ with covering map $\exp : \mathbb{C} \to \mathbb{C}^\times$.

If X and Y are both compact, then non constant holomorphic maps between them are open and each point in the images has finitely many preimages.

2.2 Branch Points

$f : X \to Y$ is a nonconstant holomorphic map of Riemann Surfaces. $x \in X$ is a ***Branch Point*** if $\nexists V \ni x$ such that $f|_V$ is injective. In terms of the local description of holomorphic maps, it is sufficient to find the appropriate constant $k > 1$ such that $f \sim z^k$ at x.

Recall the following Theorem:

Theorem 2.2.1. ***(Riemann Removal Singularities Theorem)*** *Let $f : U \to \mathbb{C}$ be an analytic function in $U \backslash \{x\}$ (for $x \in U$) and bounded in a neighborhood of x. Then, f can be extended to be analytic at x.*

This suggests that if a Manifold has a complex structure at all but finitely many isolated points, then the complex structure can extend to the whole space, across the potential singularities.

Theorem 2.2.2. *$f : X \to Y$ is a nonconstant holomorphic map of Riemann Surfaces. f has no branch points $\iff$ f is an* **etale** *homomorphism (i.e. a local homeomorphism).*

Proof. Around each point $p \in X$, as there are no branch points, $\exists V \ni p$ such that $f|_V$ is injective. As f is an open map, $f(V)$ is open in Y, suggesting that $V \cong f(V)$ is a homeomorphism.

Conversely, if f is a local homeomorphism it is locally injective, so it cannot have a branch point. $\square$

Theorem 2.2.3. *Suppose Y is a Riemann Surface and X is a Hausdorff Topological Space. Suppose that $f : X \to Y$ is a local homeomorphism. Then, $\exists!$ $\mathbb{C}$-structure on X such that f is holomorphic and locally biholomorphic.*

Proof. The idea is to pull back charts on Y to charts on X by making the charts finer and finer until the fact that f is a local homeomorphism allows you to pull the charts back homeomorphically. $\square$

2.3 Liftings

We say that g is a lifting of f with respect to p if the following diagram commutes:

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Z \arrow[ur, dashed, "g"] \arrow[r, "f"] & X \end{tikzcd}$$

Theorem 2.3.1. *Suppose that X, Y are Hausdorff Topological spaces, p is a local homeomorphism, Z is a connected topological space, and we have the following diagram of maps:*

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Z \arrow[ur, dashed, shift left, "g_1"] \arrow[ur, dashed, shift right, "g_2"'] \arrow[r, "f"] & X \end{tikzcd}$$

If $g_1(z) = g_2(z)$ for some $z \in Z$, then $g_1 = g_2$.

Proof. Let $T = \{z \in Z \mid g_1(z) = g_2(z)\}$. We'd like to show that $T \neq Z$. We know by hypothesis that $T \neq \varnothing$, so it is sufficient to verify that T is both open and closed. T is closed because $T = (g_1, g_2)^{-1}(\Delta_{Y\times Y})$. The diagonal is closed because Y is Hausdorff, and as T is the inverse image of a closed set under a continuous map, it is itself closed.

We now conclude by checking that T is open. As p is a local homeomorphism, $\exists U \ni y, W \ni z \in T$ such that $g_i(W) \subset V$ for $i = 1, 2$, and $p(y) = f(z)$, and $p \circ g_i = f$. It follows that $g_1 = g_2$ agree on W, so we can conclude that T is also an open map, as any point in T has a open neighborhood contained in T. $\square$

Theorem 2.3.2. *Suppose X, Y, Z are Riemann Surfaces satisfying the following diagram:*

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Z \arrow[ur, "q"] \arrow[r, "f"] & X \end{tikzcd}$$

Suppose p is an unbranched holomorphic map and f is holomorphic. Then any continuous lifting of f (denoted above as q) is also holomorphic.

Proof. Choose $c \in Z$, $q(i) = b, f(i) = a$. As p is unbranched and holomorphic, it is a local homeomorphism. It follows that we can find neighborhoods $b \in V, a \in W$ such that $p : V \to W$ is a biholomorphism. Let $\varphi : W \to V$ be the holomorphic local inverse. We can find a neighborhood $U \ni c$ such that $q(U) = V$ by continuity of q. Thus, $f|_U = p \circ q|_U$ implies that $q|_U = \varphi \circ f|_W$. $\square$

Now back to covering maps. Suppose $f : X \to Y$ is a covering map. Then $\forall y \in Y$ $\exists U_y \ni y$ open such that $f^{-1}(U_y) = \coprod V_{y_i}$ where $f|_{V_{y_i}} : V_{y_i} \to U_y$ is a homeomorphism. Such coverings have the following curve lifting property:

Lemma 2.3.3. *A curve, or continuous map $[0, 1] \to X$ can be uniquely lifted (up to deck transformation) to Y such that the obvious diagram commutes.*

Proof. Lift locally, then conclude that the lift works globally via compactness of $[0, 1]$. $\square$

2.3.1 Some Topological Results

Theorem 2.3.4. *Suppose that X, Y are Hausdorff topological space and X is path connected. Let $p : Y \to X$ be a covering map. Then for any pair of points $x_0, x_1 \in X$, $p^{-1}(x_0)$ and $p^{-1}(x_1)$ have the same number of points.*

Proof. Take a path $u : [0, 1] \to X$ such that $u(0) = x_0, u(1) = x_1$. Take a lifting of u. By uniqueness of lifting, it turns out that this determines a bijection of fibers. $\square$

Theorem 2.3.5. *X, Y Hausdorff topological spaces, and $p : Y \to X$ a covering map, and Z is simply/path/locally connected with a continuous map $f : Z \to X$. Then for every $z_0 \in Z$ and $y_0 \in p^{-1}(f(z_0))$, $\exists!$ lifting $\widehat{f} : Z \to Y$ such that $\widehat{f}(z_0) = y_0$.*

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Z \arrow[ur, "\exists! \widehat{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

Proof. Choose $z \in Z$, and let u be a path $z_0 \to z$. Lift uniquely to a path in Y such that $\widehat{u}(0) = y_0$. Then we can define $\widehat{f}(z) = \widehat{u}(1)$. This is well defined as Z is simply connected, and homotopic paths lift uniquely. This map is continuous because p is a covering map (can verify continuity locally). $\square$

We say a continuous map between two locally compact sets is ***proper*** if the inverse image of every compact set is compact. It's pretty straightforward to see that proper maps are closed. For example, if Y is compact and Hausdorff, then $p : Y \to X$ being a continuous map of Hausdorff spaces immediately implies that p is proper. As a nonexample, take any morphism from a noncompact set to a point. This map is clearly not closed, so it isn't proper. Furthermore, the exponential map $\mathbb{C} \to \mathbb{C}^*$ is not proper.

Lemma 2.3.6. *Suppose X, Y are locally compact and $p : Y \to X$ is proper and discrete. Then,*

(1) $p^{-1}(x)$ is finite $\forall x$.

(2) if $x \in X$ and V a neighborhood of $p^{-1}(x)$, then $\exists U \ni x$ such that $p^{-1}(U) \subset V$.

Theorem 2.3.7. *Suppose that X, Y are locally compact and Hausdorff. Let $p : Y \to X$ be a proper local homeomorphism. Then, p is a covering map.*

Proof. Choose $x \in X$, $p^{-1}(x) = \{y_1, \ldots, y_n\}$ (finite by previous lemma). Choose $W_i \ni y_i$ such that $p|_{W_i}$ is a homeomorphism. Let $p(W_i) = U_i$, and define $U \subset U_1 \cap \cdots \cap U_n$ open and small enough such that $p^{=1}(U) = \coprod V_j$ such that $p|_{V_j} : V_j \cong U$. $\square$

2.4 Riemann-Hurwitz

The Riemann-Hurwitz Formula for non-constant holomorphic maps $f : Y \to X$ from a compact genus g Riemann Surface to a compact genus h Riemann Surface is given as follows:

$$2g - 2 = \deg(g)(2h - 2) + \deg(\text{Ram. Div.})$$

First, notice that $2g - 2$ is just $-\chi_{\text{Top}}$, where χ_{Top} is the topological Euler Characteristic. Thus, Riemann-Hurwitz is (in a sense) a purely topological statement.

Proof. Choose a triangulation of X such that the branch points are vertices of the triangles. We can lift this to a triangulation of Y. If $\deg(f) = d$, then each triangle on X lifts to d triangles on Y. If $\chi_{\text{Top}}(X) = \text{Faces + Edged + Vertices} = 2 - 2h$, each face on X lifts to d faces on Y, and each edge lifts to d edges. As for vertices, Each vertex has d lifts, but we are double counting ramification points so we subtract those out. It follows that

$$2 - 2g = \chi_{\text{Top}}(Y) = d\chi_{\text{Top}}(X) - \deg(\text{Ram. Div.}) = d(2 - 2h) - \deg(\text{Ram. Div.})$$

Multiply both sides by -1 to conclude. $\square$

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a map of degree d. It follows from this result that f has $2d-2$ ramifications. Also, it follows that the degree of the Ramification divisor is always even.

If f is etale (e.g. unramified), then we have the following possible cases when $\deg(\text{Ram.Div}) = 0$:

- $g = 0$, implying that $h = 0$ and thus $\deg(f) = 1$.
- $g = 1$, implying $h = 1$. In this case, $\deg(f)$ can be anything.
- If $g > 1$, either $g = h$ and $\deg(f) = 1$, or $g > h > 1$, and $\deg(f) > 1$ with $\deg(f)|(g-1)$.

Outside of etale maps, we also have a few cases (with less specificity).

- If $g = 0$ then $h = 0$.
- If $1 \leq g = h$, then either $\deg(f) = 1$ or $g = 1$. In both cases $\deg(\text{Ram.Div}) = 0$

Theorem 2.4.1. *Suppose X is a Riemann Surface, and $A \subset X$ is a closed discrete set of points. Let $X' = X \backslash A$, and define Y, Y' similarly. If $\pi' : Y' \to X'$ is a proper unbranched holomorphic map, then there exists a proper holomorphic map $\pi : Y \to X$ such that $\pi|_{Y \backslash \pi^{-1}(A)} = \pi'$.*

Proof. Proof by picture; not too hard. □

2.5 Local Monodromy

Let $f : Y \to X$ be a proper holomorphic map, and suppose that $p \in X$ is the image of a ramification map. If $\deg(f) = d$, then p has d pre-images $z^{k_1}, \ldots, z^{k_1}$. f is locally a covering map at this point, so we can see the actions between the pre-images *(****I don't fully get this)+*

Chapter 3

Differential Calculus on Holomorphic Forms

3.1 Sheaves of Holomorphic Functions

Let $U \subset \mathbb{C}^{(z)} = \mathbb{R}^2(x, y)$ be an open domain, and take $z = x + iy$. Let $\mathcal{E}(U)$ be the $\mathbb{C}$-algebra of C^∞ functions $f : U \to \mathbb{C}$. This naturally forms a sheaf on U. Now let X be a Riemann Surface, and $Y \subset X$ is open. We can similarly define

$$\mathcal{E}(Y) = \{f : Y \to \mathbb{C} \mid \text{on every chart, } Y \supseteq U \to V, (\widehat{f} : V \to \mathbb{C}) \in \mathcal{E}(V)\}$$

This too is a sheaf of C^∞ functions on a Riemann Surface.

3.2 Cotangent Spaces

We'd like to define the cotangent space at a point $a \in X$. We can take $\mathcal{O}(Y) \subset \mathcal{E}(Y)$ to be the sheaf of holomorphic functions on Y. both are in fact a sheaves of $\mathbb{C}$-algebras, so we can consider the maximal ideal m_a of functions $f \in \mathcal{E}(Y)$ that vanish at a. By maximality of m_a, we know that $T_a^1 := m_a / m_a^2$ is a vector space (in particular, it is the vector space of functions that vanish of order exactly 1 at a). We define this to be the cotangent space at a.

We can let $d_a f := (f - f(a))$ (modulo m_a^2) be a representative of f. Furthermore, for $a \in X$ for X a Riemann Surface, with coordinate system (U, z) in a neighborhood of a, then $d_a x, d_a y$ form a basis of T_a^1, but so do $d_a z, d_a \overline{z}$. Using this basis, we can write

$$d_a f = \frac{\partial f}{\partial x}(a) d_a x + \frac{\partial f}{\partial y}(a) d_a y$$

Where, via the Cauchy Riemann Equations,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}$$

$$\frac{1}{i}\frac{\partial}{\partial y} = \frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}$$

Thus,

$$d_a f = \frac{\partial f}{\partial z}(a) d_a z + \frac{\partial f}{\partial \overline{z}}(a) d_a \overline{z}$$

Now let's take two different coordinate systems around a, (U, z) and (U', z'), where $\frac{\partial z}{\partial z'} = c$. This implies that $\frac{\partial \overline{z}}{\partial \overline{z}'} = \overline{c}$. where $\frac{\partial \overline{z}}{\partial z} = \frac{\partial z}{\partial \overline{z}} = 0$. Thus, $d_a z' = c d_a z$ and $d_a \overline{z}' = \overline{c} d_a \overline{z}$. Thus, it follows that $T_a(1)$ (the one dimensional subspace spanned by $d_a z, d_a \overline{z}$, are defined independently of the choice of coordinates for f. We define

$$T_a^{1,0} = \mathbb{C} d_a z, \qquad\qquad T_a^{0,1} = \mathbb{C} d_a \overline{z}$$

It is clear that T^1 is also a sheaf, called the ***cotangent sheaf***.

3.3 Differential Forms

We define a ***differential*** 1***-form*** to be a section of T^1, which is of the form $\omega : Y \to \bigcup_{a \in Y} T_a^1$. A differential form is of type $(1,0)$ (resp. type $(0,1)$) if $\omega(a) \in T_a^{1,0}$ $\forall a \in Y$ (resp. $\omega(a) \in T_a^{0,1}$ $\forall a \in Y$). We say the form is differentiable (resp. holomorphic) if for every chart (U, z), $\omega = f dz + g d\overline{z}$ for $f, g \in \mathcal{E}(U \cap Y)$ (resp. $\omega = f dz$ for $f \in \mathcal{O}(U \cap Y)$). Let $\mathcal{E}^1(U)$ be the differentiable 1-forms on U; this shows that $\mathcal{E}^1$ is a sheaf. We have two subsheaves $\mathcal{E}^{1,0}, \mathcal{E}^{0,1}$ of differentiable 1 forms of type $(1,0)$ and $(0,1)$ respectively. We also define $\Omega^1(U)$ to be the the space of holomorphic functions, implying that Ω^1 is a sheaf of holomorphic 1 forms, which is a subsheaf of $\mathcal{E}^{1,0}$.

3.3.1 Residues of 1-forms

Suppose that Y is a Riemann Surface, and $a \in Y$. Let ω be a holomorphic 1-form o $Y \backslash a$, with (U, z) local coordinates around a such that $z(a) = 0$. We can write $\omega = f dz$. where $f = \sum_{n \in \mathbb{Z}} c_n z^n$, Define the ***Residue*** $\operatorname{Res}_a \omega = c_{-1}$.

We say a is a ***removable singularity*** if $c_n = 0$ for $n < 0$, and that a is a ***pole of order*** $k > 0$ if $c_n = 0$ for $n < -k$ and $c_{-k} \neq 0$. Finally, we say a is an ***essential singularity*** if there are infinitely many $c_n \neq 0$ for $n < 0$.

Suppose Y is a Riemann surface, and ω a 1-form on Y. Choose coordinates (U, z) around $a \in Y$ such that $z(a) = 0$. As ω is a holomorphic 1-form, $\omega = f dz$ for some holomorphic f. Take a Laurent series expansion $f = \sum_{n \in \mathbb{Z}} c_n z^n$, and define the residue to be $\operatorname{Res}_a \omega = c_{-1}$. This is known to be independent of the chosen chart (U, z). To see this, suppose that g is holomorphic on $V \backslash \{0\}$, for $V = f(U)$. This gives an expansion $g = \sum_{n \in \mathbb{Z}} b_n z^n = \sum_{n=-\infty}^{-1} b_n z^n + \sum_{n=0}^{\infty} b_n z^n$. Thus,

$$dg = \sum_{n=-\infty}^{-1} n b_n z^{n-1} dz + \sum_{n=0}^{\infty} n b_n z^{n-1} dz$$

In particular, this shows that the coefficient of $z^{-1}dz$ in dg is 0.c If g is holomorphic with a zero of order 1 at 0, so $\mathrm{Res}_0(g^{-1}dg) = 1$. For $g(v) = z$, we have a Taylor expansion $g(v) = \sum_{n=1}^{\infty} b_n v^n$ (note that $b_0 = 0$). Notice that $(\sum_{n=1}^{\infty} b_n v^n)^m$ has no negative terms for positive m. and is of the form $v^m h$ for h a holomorphic function for negative m.

$$\omega = f dz = \sum_{n\in\mathbb{Z}} c_n (g(v))^n = \sum_{n\in\mathbb{Z}} c_n \left(\sum_{m=1}^{\infty} b_m v^m\right)^n$$

It follows (somehow?) that $\omega = dg + c_{-1}\frac{dv}{v}$. This shows that our conclusions are independent of chart (somehow?).

We say a 1 form is ***meromorphic*** on Y if ω is holomorphic on $Y' \subset Y$, and $Y \backslash Y'$ is a set of isolated points and ω has poles at each of them. Let $M^1(Y)$ be the set of meromorphic 1-forms on Y, called the set of abelian differentials.

3.3.2 2 Forms

Define $T_a^2 := \bigwedge^2 T_a^1$. If $d_a x, d_a y$ is a basis of T_1^a, it follows that $d_a x \wedge d_a y$ is a basis of T_a^2. As $d_a z, d_a \bar{z}$ is also a basis of T_1^a, it translates into the basis

$$dz \wedge d\bar{z} = (dx + idy) \wedge (dx - idy) = -idx \wedge dy + idy \wedge dx = -2idx \wedge dy$$

So $dz \wedge d\bar{z}$ is also a basis of T_a^2, with multiplying by $2i$ being the appropriate change of basis. We say $\omega : Y \to \cup_a T_a^2$ is a 2-***form*** provided that $\omega(a) \in T_a^2$. A 2-form is differentiable if for every chart (U, z), $\omega = f dz \wedge d\bar{z}$ with $f \in \mathcal{E}(Y \cap U)$. It follows that $\omega_1, \omega_2 \in \mathcal{E}^1(Y) \Rightarrow \omega_1 \wedge \omega_2 \in \mathcal{E}^2(Y)$.

3.3.3 Exterior Differentials

We want to define the maps $d, d', d'' : \mathcal{E}^1(U) \to \mathcal{E}^2(U)$. Let $\omega = \sum f_k dg_k = f_1 dz + f_2 d\bar{z}$. Define

$$d\omega = \sum df_k \wedge dg_k$$
$$d'\omega = \sum d'f_k \wedge dg_k$$
$$d''\omega = \sum d''f_k \wedge dg_k$$

A very long computational slog shows that this definition is independent of representative $\omega = \sum f_k dg_k$. This can be written (in terms of f, g) as

$$\omega = \sum \left(\frac{\partial f_k}{\partial x}\frac{\partial g_k}{\partial y} - \frac{\partial f_k}{\partial y}\frac{\partial g_k}{\partial x}\right)$$

Notice that $dd = d'd' = d''d'' = 0$ (this is pretty easy to check. Also, as like when we took differentials of functions, we have that $d\omega = d'\omega + d''\omega$ and $d(f\omega) = df \wedge \omega + fd\omega$ (and similarly for d', d'').

3.4 Harmonic Maps

For any 2-form ω, it can be written locally as $\sum f_k \wedge dg_k$. Recall the definitions of d, d', d'', and that $d = d' + d''$. Furthermore, recall that differentials satisfy the Leibniz rule: $d(f\omega) = df \wedge \omega + f d\omega$, where f is a holomorphic map. From these rules we have

$$0 = dd = (d' + d'')(d' + d'') = d'd' + d'd'' + d''d' + d''d'' = d'd'' + d''d'$$

Thus,

$$d'd''f = -d''d'f$$

Where

$$d'd''f = \frac{\partial^2 f}{\partial z \partial \overline{z}} dz \wedge d\overline{z} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right)\right) + \frac{1}{2i}\frac{\partial}{\partial y} = \frac{1}{2i}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx \wedge dy = \frac{\Delta f dx \wedge dy}{2i}$$

We say that f is ***harmonic*** if $d'd''f = 0$ (equivalently, if the Laplacian Δf vanishes).

3.5 De Rham Cohomology

A differential form ω is called ***closed*** if $d\omega = 0$. It is called exact if $\omega = df$ for some f. Define the first De Rham Cohomology, denoted $\mathrm{H}^1(Y)$ to be the quotient of the $\mathbb{C}$-module of closed forms by the $\mathbb{C}$-submodule of exact forms.

Theorem 3.5.1. *Suppose $Y \subset X$, each are Riemann Surfaces. Then, every holomorphic 1-form is closed. Furthermore, every closed 1-form $\omega \in \mathcal{E}^{1,0}(Y)$ is holomorphic.*

Proof. Choose ω to be a differential 1-form of type $(1,0)$. Thus, $\omega = f dz$, so

$$d\omega = df \wedge dz = \left(\frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \overline{z}} d\overline{z}\right) \wedge dz = -\frac{\partial f}{\partial z} dz \wedge d\overline{z}$$

This is equal to 0 $\iff \frac{\partial f}{\partial z} = 0$, so we're done. □

As a corollary, we have

Lemma 3.5.2. *If u is harmonic, then $d'u$ is a holomorphic 1-form.*

Proof.

$$dd'u = (d'' + d')d'u = d''d'u = 0$$

Thus $d'u$ is a closed 1 form, so it is holomorphic. □

3.5.1 Pullbacks

Let $F : X \to Y$ be a holomorphic map. This gives us a map $F^* : \mathcal{E}(U) \to \mathcal{E}(F^{-1}(U))$ locally on $U \subset Y$ open. via the map $f \mapsto f \circ F$. More generally, for a 1-form $\omega \in \mathcal{E}^1(U)$, we can write $\omega = \sum f_j \wedge dg_j$ locally, and define $F^*(\omega) = \sum F^* f_j \wedge dF^* g_j$. For a two form $\eta = \sum f_j dg_j \wedge dh_j$, define $F^*(\omega) = \sum F^* f_j dF^* g_j \wedge dF^* h_j$. One can verify that $F^*(d\omega) = dF^*(\omega)$. Similar statements hold if you replace d with d' or d''.

Lemma 3.5.3. *If f is harmonic on Y, then $F^* f = f \circ F$ is harmonic on X.*

3.5.2 Integration

Choose $\omega = \sum f_k dx_k + g_k dy_k \in \mathcal{E}^1(X)$, and $c : [0,1] \to X$ a piecewise continuously differentiable curve. As $[0,1]$ is compact, it follows that $c([0,1])$ can be covered by finitely many charts $(U_1, z_1), \ldots, (U_n, z_n)$. Define a partition of $0 = t_1 < t_2 < \cdots < t_n = 1$ such that $[t_i, t_{i+1}] \subset U_i$. Thus we can define

$$\int_c \omega := \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left(f_k(c(t)) \frac{dx_k(c(t))}{dt} + g_k(x(t)) \frac{dy_k(c(t))}{dt} \right) dt$$

Theorem 3.5.4. ***(Independence of parameterization and charts)*** *If $F \in \mathcal{E}(X)$, then $\int_C dF = F(c(1)) - F(c(0))$.*

Proof. Recall that $dF = \sum \frac{\partial F}{\partial x_k} dx_k + \frac{\partial F}{\partial y_k} dy_k$. Thus,

$$\begin{aligned}
\int_c dF &= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left(\frac{\partial F}{\partial x_k}(c(t)) \frac{dx_k(c(t))}{dt} + \frac{\partial F}{\partial y_k}(x(t)) \frac{dy_k(c(t))}{dt} \right) dt \\
&= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{d}{dt} F(c(t)) dt \\
&= \sum_{k=1}^{n} F(c(t_k)) - F(c(t_{k-1})) \\
&= F(c(1)) - F(c(0))
\end{aligned}$$

□

We say that for $\omega \in \mathcal{E}^1(X)$, a function F is a ***primitive*** of ω if $\omega = dF$. If F and G are primitives of ω, then $dF = dG = \omega$, i.e. $d(F - G) = 0$. Thus, $0 = \int_c d(F - G) = (F - G)(c(1)) - (F - G)(c(0))$, implying that $F - G$ is a constant. Thus, primitives must only differ by a constant at most.

We now show that such primitives exist for any closed form, at least locally. Take $\mathcal{U} = \{z \in \mathbb{C} | |z| < r\}$ and $\omega = f dx + g dy \in \mathcal{E}^1(U)$ closed. As

$$0 = d\omega = df \wedge dx + dg \wedge dy = \left(-\frac{\partial f}{\partial y} + \frac{\partial g}{\partial x} \right) dx \wedge dy$$

Thus ω is closed $\iff$ $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$. Define

$$F(x, y) = \int_0^1 (f(tx, ty)x + g(tx, ty)y) dt$$

Then

$$\frac{d}{dt} f(tx, ty) = \frac{\partial f}{\partial x}(tx, ty)x + \frac{\partial f}{\partial y}(tx, ty)y$$

Implying that

$$\begin{aligned}
\frac{\partial F(x,y)}{\partial x} &= \int_0^1 \left(f(tx,ty) + tx\frac{\partial f}{\partial x}(tx,ty) + ty\frac{\partial g}{\partial x}(tx,ty) \right) dt \\
&= \int_0^1 \left(f(tx,ty) + t\frac{d}{dt}f(tx,ty) \right) dt \\
&= \frac{\int_0^1 d\left(tf(tx,ty)\right) dt}{dt} \\
&= tf(tx,ty)\Big|_0^1 \\
&= f(x,y)
\end{aligned}$$

and similarly,

$$\frac{\partial F(x,y)}{\partial y} = \int_0^1 \left(tx\frac{\partial f}{\partial y}(tx,ty) + g(tx,ty) + ty\frac{\partial g}{\partial y}(tx,ty) \right) dt = f(x,y)$$

Note that if ω is a holomorphic 1-form, then the existence of a local primitive is easy. If $\omega = fdz$ for $f = \sum_{n=0}^{\infty} c_n z^n$, then primitive would just be $F(z) = \sum_{n=0}^{\infty} \frac{c_n z^{n+1}}{n+1}$ as expected.

3.6 Sheaves of Primitives

Theorem 3.6.1. *Suppose X is a Riemann Surface, and $\omega \in \mathcal{E}^1(X)$ is a closed differential form. Then, $\exists$ a covering map $p : \widetilde{X} \to X$ with $\widetilde{X}$ connected and a primitive $F \in \mathcal{E}(\widetilde{X})$ such that $dF = p^*\omega$.*

To prove this, we ought to use the language of sheaves. For each $U \subset X$ open, assign

$$\mathscr{F}(U) = \{f \in \mathcal{E}(U) \mid df = \omega \text{ on } U\}$$

This is naturally a sheaf, as if there are f, g such that $df = dg = \omega$, then we know that $f - g$ is constant (i.e. this sheaf satisfies the identity theorem). If $\mathscr{F}$ is a sheaf, we call $\mathscr{F}_x$ the ***stalk*** at a point x, corresponding to the germs at a point.

$$\mathscr{F}_x = \{(U \ni x, f) \mid f \in \mathcal{F}(U)\}/\sim$$

Where $(U, f) \sim (V, g) \iff \exists W \subset U \cap V$ such that $f|_W = g|_W$. We can take these stalks and glue them back into a general ***Espace étalé*** defined as follows:

$$|\mathscr{F}| = \bigcup_{x \in X} \mathscr{F}_x$$

With a natural map $p : |\mathscr{F}| \to X$, and $|\mathscr{F}|$ has a topology defined on it via the basis $\{[U, f \mid f \in \mathscr{F}(U)]\}$ If $\mathscr{F}$ satisfies the identity theorem, then $|\mathscr{F}|$ is a Hausdorff topological space. Note that all this is true for any sheaf $\mathscr{F}$, but now we return to the case where

$\mathscr{F}$ is the sheaf of local primitives on ω. In this case, $p : |\mathscr{F}| \to X$ is a covering map. This is because $\forall x \in X, \exists U \ni x$ open such that ω has a primitive on U, denoted by f. Thus, we have an open set $[U, f]$ such that $df = \omega$. Furthermore, $p^{-1}(U) = \bigcup_{c\in\mathbb{C}}[U, f + c]$. So we can take a connected component of $|\mathscr{F}|$ called $\widetilde{X}$, and restrict p to still get a covering map $p : \widetilde{X} \to X$, where $p^*\omega$ has a primitive. As a corollary of this theorem, we have the following:

Lemma 3.6.2. *Suppose that X is a Riemann Surface and $\pi : \widetilde{X} \to X$ the universal covering. For $\omega \in \mathcal{E}^1(X)$ a closed 1-form, $\exists f \in \mathcal{E}(\widetilde{X})$ such that $df = \pi^{-1}\omega$. In other words, on any simply connected Riemann Surface, every closed 1-form is exact.*

3.7 Independence of Path

Theorem 3.7.1. *Suppose X is a Riemann Surface, $\pi : \widetilde{X} \to X$ is the universal cover, and $\omega \in \mathcal{E}^1(X)$ is a closed differential 1-form. Let F be a primitive of $\pi^*\omega$. If $c : [0, 1] \to X$ is a curve, take a lifting $\widehat{c} : [0, 1] \to \widetilde{X}$. Then*

$$\int_c \omega = F(\widehat{c}(1)) - F(\widehat{c}(0))$$

and

$$\int_{\widehat{c}} \pi^*\omega = \int_{\pi\circ\widehat{c}} \omega$$

These are both independent of choice of F and choice of lifting $\widehat{c}$.

Proof. It's clear that this is independent of choice of primitive, as primitives differ by a constant. As for independence of lifting, suppose we had two liftings $\widehat{c}$ and $\widetilde{c}$, associated via a deck transformation $\nu(\widehat{c}) = \widetilde{c}$. $\pi : \widetilde{X} \to X$ is a Galois cover, and $\nu \circ \pi = \pi$. It follows that $\nu^*\pi^*\omega = \pi^*\omega$, so ν^F is also a primitive of $\pi^*\omega$. Following independence of these choices, the actual results above follow from previous work. $\square$

Theorem 3.7.2. *If X is a Riemann Surface and $\omega \in \mathcal{E}^1(U)$ is a closed differential form, then integrals are independent of path, i.e. if $u \simeq v$ on X are curves $a \to b$ (or alternatively, u, v are two closed freely homotopic curves), then $\int_u \omega = \int_v \omega$.*

Proof. Pull back to the universal cover, lifting u, v initial points to the same initial point in $\widetilde{X}$. It follows via the continuous lifting property that the lifts $\widehat{u}, \widehat{v}$ have the same end point, and are homotopic (as $\widetilde{X}$ is simply connected). Thus,

$$\int_u \omega = \int_{\widehat{u}} \pi^*\omega = F(\widetilde{b}) - F(\widetilde{a}) = \int_{\widehat{v}} \pi^*\omega = \int_v \omega$$

For the freely homotopic case, as $v \sim h^{-1} \circ u \circ h$ for some h, we have

$$\int_v \omega = \int_{h^{-1}\circ u\circ h} \omega = -\int_h \omega + \int_u \omega + \int_h \omega = \int_u \omega$$

$\square$

3.8 Periods of a Riemann Surface

Let X be a Riemann Surface, $\omega \in \mathcal{E}^1(X)$ a closed differential 1-form. For $\sigma \in \pi_1(X)$, define an assignment $\sigma \to \int_\sigma \omega$. One gets a homomorphism $\pi_1(X) \to \mathbb{C}$ associated to a given closed 1-form ω. We call this the ***period homomorphism*** associated to ω.

For example, for $X = \mathbb{C}^*, \pi_1(\mathbb{C}^*) = \mathbb{Z}$. $\omega = dz/z$ is a holomorphic 1-form on $\mathbb{C}^\times$, so it is closed. As $\pi_1(CC^*)$ is generated by a 1-time loop around the missing zero (denoted σ), we conclude that $\int_\sigma \omega = 2\pi i$. In general, $\int_\ell \omega = 2\pi i k$ for $k \in \mathbb{Z}$ correspond to the positively negative count of counterclockwise rotations around the center. This implies that the period map is $k \mapsto 2\pi i k$.

What are the meromorphic functions on the torus? The Torus (which is always of the form $\mathbb{C}/\Gamma$ for some lattice Γ) has $\mathbb{C}$ as a universal cover. Thus, we can take a doubly period meromorphic functions on $\mathbb{C}$ and push it down to the torus quotient, and that yields a meromorphic function on $\mathbb{C}/\Gamma$. In particular, this is the only way to construct meromorphic functions.

Recall that not all toruses are created equal (see Homework), and have a complex structure that is completely dependent on the lattice one is quotienting out by (up to scaling).

3.9 Summands of Automorphy

Let $G = \mathrm{Deck}(\widetilde{X}/X) \cong \pi_1(X)$ be the group of deck transformations on a covering space. For any $f : \widetilde{X} \to \mathbb{C}$, σ acts on f by $\sigma \cdot f := f \circ \sigma^{-1}$. It is clear that

$$\sigma \cdot (f + g) = \sigma \cdot f + \sigma \cdot g$$

$$\sigma \cdot (fg) = (\sigma \cdot f)(\sigma \cdot g)$$
$$\sigma \cdot (\tau \cdot f) = (\sigma\tau) \cdot f$$

A function $f : \widetilde{X} \to C$ is ***additively automorphic*** with constant summands of automorphy if $f - \sigma f = a_\sigma$ where a_σ is a constant for any $\sigma \in G$. Notice that if this is the case,

$$a_{\sigma\tau} = f - \sigma\tau f = f - \sigma f + \sigma f - \sigma\tau f = a_\sigma + a_\tau$$

Following from the fact that $\sigma f - \sigma\tau f = \sigma a_\tau = a_\tau$. It follows then that $f - \sigma^{-1} f = -a_\sigma$. Thus, using the word additive is apt. If f is a function invariant under deck transformations, then $f = \sigma f \ \forall \sigma \in G$, so $a_\sigma = 0$ for any such $\sigma \in G$.

Theorem 3.9.1. *Suppose that X is a Riemann Surface with universal cover $\widetilde{X}$.*

(1) $\omega \in \mathcal{E}^1(X)$ is a closed differential 1 form, and let F be a primitive of $\pi^*\omega$. Then, F is additively automorphic with constant summand of automorphy, where the summands of automorphy are the periods.

(2) *Conversely, suppose that $F \in \mathcal{E}(\widetilde{X})$ is additively automorphic. Then, $\exists! \omega \in \mathcal{E}^1(X)$ that is closed such that $dF = \pi^*\omega$.*

Proof. As we've seen before, deck transformations send primitives to other primitives, and primitives differ by a constant. Thus, $F - \sigma F$ is constant, so F is additively automorphic. Choose a path $\nu : [0,1] \to \widetilde{X}$ such that $\nu(0) = y_0, \nu(1) = \sigma(y_0)$. Let $\mu = \pi(\nu)$. Because we like to confuse ourselves, let $\sigma(y_0) = z_0, y_0 = \sigma^{-1}(z_0)$. Then,

$$\int_\mu \omega = F(\nu(1)) - F(\nu(0)) = F(z_0) = F(\sigma^{-1}(z_0)) = F - \sigma F = a_\sigma$$

Which is precisely the period. Now we want to prove (2). Suppose that F has constant summands of automorphy $a_\sigma \in \mathbb{C}$. Then

$$\sigma^* dF = d\sigma^* F = d\left(F + a_\sigma\right) = dF$$

So dF is invariant under the action of the deck group. Since $\pi : \widetilde{X} \to X$ is locally biholomorphic, $\exists \omega \in \mathcal{E}^1(X)$ such that $dF = \pi^*\omega$ such that it is closed and uniquely determined. $\square$

Theorem 3.9.2. *Suppose that X is a Riemann Surface and $\omega \in \mathcal{E}^1(X)$ is a closed differential 1-form. Then ω has a primitive $\iff$ all the periods of ω vanish.*

Proof. If $\omega = dF$, then integrating ω over a closed loop yields 0, so it has no period. Conversely, if $\pi : \widetilde{X} \to X$ is the universal cover, then $\pi^*\omega$ has a primitive F. As the periods vanish, $a_\sigma = 0$, so $F = \sigma F$, so $\exists f \in \mathcal{E}(X)$ such that $F = \pi^* f$. It follows that $\omega = df$ by pulling everything back.

$$\pi^* df = d\pi^* f = dF = \pi^*\omega \Rightarrow df = \omega$$

$\square$

In practice, if ω is a closed 1-form on a Riemann Surface, we can explicitly compute a primitive by integrating ω along a path, i.e for any $x \in X$ we take a path C from some x_0 to x, then define $f(x) = \int_C \omega$. This (at first glance) seems dependent on the choice of C, but if we define $f(x)$ and $\widetilde{f}(x)$ with respect to different choices of C, we see that $f(x) - \widetilde{f}(x) = \int_{\text{closed loop with 0 period}} \omega = 0$.

Theorem 3.9.3. *Let X be a compact Riemann Surface, and $\omega_1, \omega_2 \in \Omega(X)$. Then $\omega_1 = \omega_2 \iff$ they define the same period homomorphisms.*

Proof. The forward direction is trivial so we just need to prove the reverse case. $\omega_1 - \omega_2$ has 0 period (via the hypothesis) and is closed (all holomorphic 1 forms are closed), so it has a primitive $df = \omega_1 - \omega_2$. As X is compact, we have no nonconstant holomorphic maps, so f is constant, so $df = 0$. It follows that $\omega_1 = \omega_2$. $\square$

Chapter 4

Cohomology

4.1 Cech Cohomology

Let $\mathcal{U} = \bigcup U_i$ be an open cover of X and $\mathscr{F}$ a sheaf on X. Define $\mathcal{U}_{i_0,\ldots,i_n} = U_{i_0} \cap \cdots \cap U_{i_n}$. Define the ***Cech Groups*** as

$$\mathcal{C}^n(\mathcal{U}, \mathscr{F}) = \prod_{i_1 < \cdots < i_n} \mathscr{F}(\mathcal{U}_{i_0,\ldots,i_n})$$

We use this to define the ***Cech Complex***

$$\mathcal{C}^0 \to \mathcal{C}^1 \to \ldots$$

Where the transition maps are as follows:

$$\delta_0(f_i) = (f_j - f_i) \in \mathcal{U}_{i,j}$$

$$\delta_1(f_{ij}) = f_{jk} - f_{ik} + f_{ij} \in \mathcal{U}_{i,j,k}$$

And so on. Notice that $\ker(\delta_0) = \mathscr{F}(X)$, i.e. $\mathcal{C}^0$ can be associated to the global sections of the sheaf. One can take the cohomology of this sequence to obtain $\mathrm{H}^i(\mathcal{U}, \mathscr{F}) = \ker(\delta_i)/\operatorname{im}(\delta_{i+1}) =: \mathrm{Z}^i(\mathcal{U}, \mathscr{F})/B^i(\mathcal{U}, \mathscr{F})$. At first glance, it seems that these groups are dependent on the choice of covering $\mathcal{U}$ of X. An open covering $\bigcup V_k$ is called ***finer*** than $\bigcup U_i$ if $\forall k \exists i$ such that $V_k \subset U_i$. If $\mathcal{V}$ is a finer covering than $\mathcal{U}$, this gives us a map $\mathrm{H}^i(\mathcal{U}, \mathscr{F}) \to \mathrm{H}^i(\mathcal{V}, \mathscr{F})$, descending to a map on cocycles $\mathrm{Z}^i(\mathcal{U}, \mathscr{F}) \to \mathrm{Z}^i(\mathcal{V}, \mathscr{F})$ sending an element $f_{j_0,\ldots,j_i} \in \mathcal{U}_{j_0,\ldots,j_i}$ to itself contained in $V_{j_0,\ldots,j_i}$. Thus, we can define the ***Sheaf Cohomology*** groups to be

$$\mathrm{H}^i(X, \mathscr{F}) := \varinjlim \mathrm{H}^i(\mathcal{U}, \mathscr{F})$$

Theorem 4.1.1. *Suppose X is a Riemann Surface. Then $\mathrm{H}^1(X, \mathcal{E}) = 0$, and furthermore, $\mathrm{H}^1(X, \mathcal{E}^1) = \mathrm{H}^1(X, \mathcal{E}^{1,0}) = \mathrm{H}^1(X, \mathcal{E}^2) = 0$.*

Proof. Let $\mathcal{U} = \bigcup U_i$ be an open cover. Take a ***partition of unity*** $\{\psi_i\}$ such that $\operatorname{Supp} \psi_i \subset U_i$, every point $x \in X$ has a neighborhood meets only finitely many of the supports $\operatorname{Supp} \psi_i$, and $\sum \psi_i \equiv 1$. We want to show that $\mathrm{H}^1(X, \mathcal{E}) = 0$. Take a 1-cycle f_{ij} such that

$f_{jk} - f_{ik} + f_{ij} = 0$. We'd like to show that f_{ij} is a coboundary. $f_{ij} \in U_i \cap U_j$, so consider $\psi_j f_{ij}$ on $U_i \cap U_j$ and extend by 0 to all of U_i. Set $g_i = \sum_j \psi_j f_{ij}$ on U_i (this is OK to do because $\psi_j f_{ij} = 0$ for all but finitely many J).

$$\delta_0(g_i) = g_j - g_i = \sum_k \psi_k(f_{jk}) \sum_k \psi_k(f_{ik}) = \sum_k \psi_k(f_{jk} - f_{ik}) = \sum_k \psi_k(-f_{ij}) = -f_{ij}$$

It follows that f_{ij} is a coboundary, so $\overline{f_{ij}} = 0 \in \mathrm{H}^1(X, \mathcal{E})$. A similar proof follows for the second part of the claim. $\square$

Theorem 4.1.2. *X is a simply connected Riemann Surface. Then $\mathrm{H}^1(X, \mathbb{C}) = \mathrm{H}^1(X, \mathbb{Z}) = 0$, where $\mathbb{C}, \mathbb{Z}$ correspond to the constant sheaves on X.*

Proof. $\mathrm{Z}^1(\mathcal{U}, \mathbb{C}) \subset \mathrm{Z}^1(\mathcal{U}, \mathcal{E})$. We just saw that $\mathrm{H}^1(\mathcal{U}, \mathcal{E}) = 0$. Choose $c_{ij} \in \mathrm{Z}^1(\mathcal{U}, \mathbb{C})$ where $c_{ij} = f_i - f_j$. Then, $dc_{ij} = df_i - df_j$, but $dc_{ij} = 0$, so $df_i = df_j$ on all U_{ij}. Thus $\exists \omega \in \mathcal{E}^1(X)$ such that $\omega|_{U_i} = df_i$. Thus, $d\omega = 0$, so it is closed. Since X is simply connected, we can conclude that ω is then exact. Thus, $df = \omega$ for some f. Now set $c_i = f_i - f$. Then $dc_i = df_i - df = 0$ on U_i. Thus, these c_i are all locally constant. So $c_{ij} = f_i - f_j = c_i - c_j$, so $c_{ij} \in B^1(\mathcal{U}, \mathscr{F})$.

Similarly, choose $(a_{jk}) \in \mathrm{Z}^1(\mathcal{U}, \mathbb{Z})$. $a_{jk} = c_j - c_k$ on U_{jk} where $c_j, c_k \in \mathcal{C}^0(\mathcal{U}, \mathbb{C})$. Notice that $e^{2\pi i c_j} = e^{2\pi i c_k}$ and X is connected. Then $\exists b \in \mathbb{C}^\times$ such that $b = e^{2\pi i c_j}\ \forall j$. Choose c such that $e^{2\pi i c} = b$. $a_j = c_j - c$. Well, $e^{2\pi i a_j} = 1$, so $a_j \in \mathbb{Z}$. Now take $a_j - a_k = c_j - c - c_k + c = c_j - c_k = a_{jk}$. It follows that $\mathrm{H}^1(\mathcal{U}, \mathbb{Z}) = 0$. $\square$

Theorem 4.1.3. ***(Leray)*** *$\mathscr{F}$ is a sheaf of abelian groups, and $\mathcal{U} = \bigcup U_i$ is an open cover such that $\mathrm{H}^1(U_i, \mathscr{F}) = 0$ for each i. Then $\mathrm{H}^1(X, \mathscr{F}) = \mathrm{H}^1(\mathcal{U}, \mathscr{F})$.*

We call these types of coverings ***Leray Coverings***. The upshot of this is that we only need to find a cover that has locally trivial cohomology to compute general cohomology, rather than relying on computing a whole direct limit of complexes.

Proof. If you take a refinement $\mathcal{V}$, we get a map $\mathrm{H}^1(\mathcal{U}, \mathscr{F}) \hookrightarrow \mathrm{H}^1(\mathcal{V}, \mathscr{F})$ (this is true in general). In this specific case, however, we'd like to show that this map is also surjective. Choose $(f_{\alpha\beta}) \in \mathrm{Z}^1(\mathcal{U}, \mathscr{F})$. By hypothesis, $\mathrm{H}^1(U_i \cap U_j, \mathscr{F}) = 0$. Viewing τ as the inclusion map of indices on $\mathcal{U}$ to $\mathcal{V}$, it follows that $F_{\tau\alpha,\tau\beta} \in Z(\mathcal{U}, \mathscr{F})$, and $F_{\tau\alpha,\tau\beta} - f_{\alpha\beta} = 0$ on $V_{\alpha\beta}$. $F_{ij} = g_{j\alpha} - g_{i\alpha}$ on $U_i \cap U_j \cap V_\alpha$, and similarly $f_{\alpha\beta} = g_{i\alpha} - g_{i\beta}$ Then

$$F_{\tau\alpha,\tau\beta} - f_{\alpha\beta} = g_{\tau\beta,\alpha} - g_{\tau\alpha,\alpha} - (g_{i\alpha} - g_{i\beta})$$

Taking $i = \tau\beta$, we get that

$$F_{\tau\alpha,\tau\beta} - f_{\alpha\beta} = g_{\tau\beta,\beta} - g_{\tau\alpha,\alpha}$$

$\square$

We use this to compute $\mathrm{H}^1(\mathbb{C}^\times, \mathbb{Z})$. Take $U_1 = \mathbb{C} \backslash \{x \in \mathbb{R} \mid x \geq 0\}$ and $U_2 = \mathbb{C} \backslash \{x \in \mathbb{R} \mid x \leq 0\}$ It follows that these are both simply connected (so they have trivial homology) and they cover $\mathbb{C}^\times$. We can use this covering, via the previous theorem, to compute the sheaf cohomology. It follows that $\mathbb{Z}(U_1 \cap U_2) = \mathbb{Z} \oplus \mathbb{Z}$, as we have two connected components. Everything is thus a cycle, so the boundaries are precisely the points of the form (a, a), so it follows that

$$\mathrm{H}^1(\mathbb{C}^\times, \mathbb{Z}) = Z^1(\mathbb{C}^\times, \mathbb{Z})/B^1(\mathbb{C}^\times, \mathbb{Z}) = (\mathbb{Z} \oplus \mathbb{Z})/\Delta = \mathbb{Z}$$

4.2 L^2-Norms

Theorem 4.2.1. *If X is a compact Riemann Surface, then $\mathrm{H}^1(X, \mathcal{O})$ is finite dimensional.*

Let $D \subset \mathbb{C}$ be an open set, and $f \in \mathcal{O}(D)$. Define the L^2***-norm*** to be

$$\|f\|_{\mathcal{L}(D)} := \sqrt{\int_D |f|^2 dxdy}$$

Let $\mathcal{L}^2(D, \mathcal{O})$ be the vector space of functions f on D bounded on this norm. If f is bounded, then $\|f\|_{\mathcal{L}^2(D)} \leq \|f\|_D \mathrm{Vol}(D)$, for $\|f\|_D = \sup_{x \in D} f(x)$. Thus $\mathcal{L}^2(D, \mathcal{O})$ comes with an inner product

$$\langle f, g \rangle = \iint_D f\overline{g} dxdy$$

Where $|f\overline{g}| \leq \frac{1}{2}(|f|^2 + |g|^2)$. Notice that in an open ball $B_R(a)$, we have an orthonormal basis $\{(z-a)^n \mid n \in \mathbb{N}\}$. This is orthonormal because

$$\langle (z-a)^n, (z-a)^m \rangle = \iint (z-a)^n (\overline{z} - \overline{a})^n dxdy = \int_0^{2\pi} \int_0^R r^n e^{in\theta} r^m e^{-im\theta} r dr d\theta$$

If $n = m$, this integral is 0. If not, then we get

$$2\pi \int_0^R r^{2n+1} = \frac{2\pi R^{2n+2}}{2n+2}$$

Observe that

$$\|z - a\|_{\mathcal{L}^2(B_R(a))} = \sqrt{\frac{\pi}{n+1}} R^{n+1}$$

Therefore, for any $f \in \mathcal{L}^2(B_R(a), \mathcal{O})$, we have a local power series $f(z) = \sum_{n=0}^\infty a_n (z-a)^n$, implying that

$$\|f\|^2_{\mathcal{L}^2(B_R(a))} = \sum_{n=0}^\infty |a_n| \frac{R^{2n+2}}{n+1}$$

Theorem 4.2.2. *choose $D \subset \mathbb{C}$ open, and define $D_r = \{z \in \mathbb{C} \mid B_r(z) \subset D\}$. $\forall f \in \mathcal{L}^2(D, \mathcal{O})$,*

$$\|f\|_{D_r} \leq \frac{1}{r\sqrt{\pi}} \|f\|_{\mathcal{L}^2(D)}$$

Proof. Write $f(z) = \sum_{n=0}^\infty c_n (z-a)^n$ for $a \in D_r$.

$$|f(a)| = |c_0| \leq \frac{1}{r\sqrt{\pi}} \|f\|_{\mathcal{L}^2(B_r(a)} \leq \frac{1}{r\sqrt{\pi}} \|f\|_{\mathcal{L}^2(D)}$$

□

In particular, $L^2(D,\mathcal{O})$ is a Hilbert space. If I take a Cauchy sequence $(f_n) \subset L^2(D,\mathcal{O})$,then f_n converges uniformly on a compact subset of D to a holomorphic f.

Lemma 4.2.3. *Choose $D' \subset D \subset \mathbb{C}$ such that D is open in $\mathbb{C}$, and D' is open in D with compact closure which is also contained in D. Given $\varepsilon > 0, \exists$ a closed vector subspace $A \subset \mathcal{L}^2(D,\mathcal{O})$ of finite codimension such that*

$$\|f\|_{\mathcal{L}^2(D')} \leq \varepsilon \|f\|_{\mathcal{L}^2(D)}$$

$\forall f \in A$.

Proof. $\overline{D}'$ is compact and contained in D, by hypothesis. Thus $\exists r > 0$ and points $a_1, \ldots, a_k \in D$ such that

- $B_r(a_i) \subset D$
- $D' \subset B_{r/2}(a_i)$

for each i. Choose n such that $2^{-n-1}k < \varepsilon$. The set of functions $f \in \mathcal{L}^2(D,\mathcal{O})$ that vanish to order n at each a_i has codimension $\leq kn$. Set A to be that set of all such functions. Now suppose that $f \in A$. Then $f = \sum_{i=n}^{\infty} c_i (z - a_j)^i$ around each a_j. Then

$$\|f\|^2_{\mathcal{L}(B_\rho(a_j),\mathcal{O})} \leq \sum_{\ell} \frac{\pi \rho^{2\ell+2}}{\ell+1} |c_\ell|^2$$

By the previous theorem. $\rho = r/2$, This inequality becomes

$$\|f\|^2_{\mathcal{L}(B_{r/2}(a_j),\mathcal{O})} \leq 2^{-n-1} \|f\|_{\mathcal{L}^2(B_r(a),\mathcal{O})}$$

Thus,

$$\|f\|_{\mathcal{L}^2(D')} \leq k \sum_j \|f\|_{\mathcal{L}^2(B_{r/2}(a_j),\mathcal{O})} \leq k2^{-n-1} \|f\|_{\mathcal{L}^2(D,\mathcal{O})} < \varepsilon \|f\|_{\mathcal{L}^2(D)}$$

□

Let X be a compact Riemann Surface, and choose finitely many charts $\mathcal{U} = \{(U_i^*, z_i)\}$ such that $z_i(U_i^*)$ is a desk centered at 0. We want to put $\mathcal{L}^2$ norms on $\mathcal{C}^0, \mathcal{C}^1$. Say $U_i \subset U_i^*$ are open sets such that $\bigcup U_i = X$. For $\eta = (f_i) \in \mathcal{C}^0(\mathcal{U},\mathcal{O})$, define

$$\|\eta\|^2_{\mathcal{L}^2(\mathcal{U})} := \sum_i \|f_i\|^2_{\mathcal{L}^2(U_i)}$$

For $\zeta = (f_{ij}) \in \mathcal{C}^1(\mathcal{U},\mathcal{O})$, define

$$\|\zeta\|^2_{\mathcal{L}^2(\mathcal{U})} := \sum_{i,j} \|f_{ij}\|^2_{\mathcal{L}^2(U_i \cap U_j)}$$

the cochains with finite norm are a vector space, denoted via taking the subset $\mathcal{C}^q_{\mathcal{L}^2}(\mathcal{U},\mathcal{O}) \subset \mathcal{C}^q(\mathcal{U},\mathcal{O})$. Now suppose we have $V_i \subset U_i$ such that the compact closure of V_i is contained in U_i. In terms of notation we label this as taking $\mathcal{V} \ll \mathcal{U}$ and $V_i \Subset U_i$. Choose $\zeta \in \mathcal{C}^q(\mathcal{U},\mathcal{O})$. It follows that $\|\zeta\|_{\mathcal{L}^2(\mathcal{U})} < \infty$, so $\exists A$, a closed subspace of $Z_{\mathcal{L}^2(\mathcal{U},\mathcal{O})}$ of finite codimension such that $\|\zeta\|_{\mathcal{L}^2(\mathcal{V})} \leq \varepsilon \|\zeta\|_{\mathcal{L}^2(\mathcal{U})}$ $\forall \zeta \in A$ and fixed $\varepsilon > 0$.

Lemma 4.2.4. *Suppose that X is a compact Riemann Surface and $\mathcal{U}^*$ is a finite family of charts, where $\mathcal{W} \ll \mathcal{V} \ll \mathcal{U} \ll \mathcal{U}^*$. Then $\exists c > 0$ such that $\forall \zeta \in Z^1_{\mathcal{L}^2}(V, \mathcal{O})$ $\exists \alpha \in Z^1_{\mathcal{L}^2}(\mathcal{U}, \mathcal{O}), \beta \in \mathcal{O}^0_{\mathcal{L}^2}(\mathcal{W}, \mathcal{O})$ such that*

$$\alpha = \zeta + \delta^0 \beta \text{ and } \max\left(\|\alpha\|_{\mathcal{L}^2(\mathcal{U})}, \|\beta\|_{\mathcal{L}^2(\mathcal{W})}\right) \leq c\|\zeta\|_{\mathcal{L}^2(\mathcal{V})}$$

Proof. $\zeta = (f_{ij}) \in Z^1_{\mathcal{L}^2}(\mathcal{V}, \mathcal{O})$. $\mathcal{O} \subset \mathcal{E}$, so $\exists g_i \in \mathcal{C}^0(\mathcal{V}, \mathcal{E})$ such that $f_{ij} = g_i - g_j$ on V_{ij}. Thus $d'' f_{ij} = 0 \Rightarrow d'' g_j = d'' g_i$ on V_{ij}. We now have $\omega \in \mathcal{E}^{0,1}(\mathcal{V})$ with the property that $\omega|_{V_i} = d'' g_i$. As $\mathcal{W} \ll \mathcal{V}$, $\exists \psi \in \mathcal{E}(X)$ such that the support of ψ is contained in elements of the $\mathcal{V}$ cover, and if we restrict ψ to the elements of the $\mathcal{W}$ cover, ... (will finish next time). ☐

4.3 Finite Dimensionality of $\mathrm{H}^1(X, \mathcal{O})$

We're working towards proving that, for X is a Compact Riemann Surface. Then, $\mathrm{H}^1(X, \mathcal{O})$ is finite dimensional. To prove this, we need the lemma from last time, which we need to finish proving.

Proof. Let $\zeta = (f_{ij}) \in Z^1_{\mathcal{L}^2}(\mathcal{V}, \mathcal{O})$. As $\mathcal{O} \subset \mathcal{E}$, $\exists g_i \in \mathcal{C}^0(\mathcal{U}, \mathcal{E})$ such that $f_{ij} = g_i - g_j$ on $V_i \cap V_j$. Thus, as $d'' f_{ij} = 0$, we have that $d'' g_i = d'' g_j$ on $V_i \cap V_j$. Choose $\omega \in \mathcal{E}^{0,1}\left(\bigcup V_i\right)$ such that $\omega|_{V_i} = d'' g_i$. As $\bigcup W_i \Subset \bigcup V_i$, $\exists \psi \in \mathcal{E}(X)$ such that ψ is supported over $\mathcal{V}$ and $\psi|_{\bigcup W_i} = 1$, so $\psi_{\mathcal{W}} \in \mathcal{E}^{0,1}(X)$. Dolbeault's Lemma says that $\exists h_i \in \mathcal{E}(U_i^*)$ such that $d'' h_i = \psi_{\mathcal{W}}$ on U_i^*, and $d'' h_i = d'' h_j$ on $U_i^* \cap U_j^*$. Thus for $F_{ij} = h_j - h_i \in \mathcal{O}(U_i^* \cap U_j^*)$. Let $\alpha = (F_{ij})_{\mathcal{U}^*}$. As $\mathcal{U} \Subset \mathcal{U}^*$, $\alpha \in Z^1_{\mathcal{L}^2}(\mathcal{U}, \mathcal{O})$ as desired. On W_i, $d'' h_i = \psi_{\mathcal{W}} = d'' g_i$. Thus $h_i - g_i$ is holomorphic on W_i and bounded. Thus let $\beta = \left((h_i - g_i)|_{W_i}\right) \in \mathcal{C}^0_{\mathcal{L}^2}(\mathcal{W}, \mathcal{O})$. Therefore,

$$F_{ij} - f_{ij} = (h_j - g_j) - (h_i - g_i) = \delta\beta$$

on $W_i \cap W_j$. ☐

Lemma 4.3.1. *$\exists S \subset Z^1(\mathcal{U}, \mathcal{O})$ such that S is finite dimensional and $\forall \zeta \in Z^1(\mathcal{U}, \mathcal{O})$, $\exists \sigma \in S$ and $\eta \in \mathcal{C}^0(\mathcal{W}, \mathcal{O})$ with $\sigma = \zeta + \delta\eta$. In other words, the restriction map from $\mathrm{H}^1(\mathcal{U}, \mathcal{O}) \twoheadrightarrow \mathrm{H}^1(\mathcal{W}, \mathcal{O})$ has finite image.*

Proof. $\exists$ a finite codimension subspace $A \subset Z^1(\mathcal{U}, \mathcal{O})$ such that $\|\zeta\|_{\mathcal{L}^2(\mathcal{V})} < \varepsilon\|\zeta\|_{\mathcal{L}^2(\mathcal{U})}$ (which we proved last time) for any ε. In this case, let $\varepsilon = \frac{1}{2c}$. Letting S be $A^\perp$, we want to show that S is finite dimensional.

Let $\psi \in Z^1(\mathcal{U}, \mathcal{O})$ and $\mathcal{V} \ll \mathcal{U}$. Then $\|\psi\|_{\mathcal{L}^2(\mathcal{U})} = M < \infty$. Then $\exists \psi_0 \in Z^1_{\mathcal{L}^2(\mathcal{U})}(\mathcal{O})$ and $\eta_0 \in \mathcal{C}^0_{\mathcal{L}^2}(\mathcal{W}, \mathcal{O})$ such that $\psi_0 = \psi + \delta\eta_0$ on $\mathcal{W}$ with $\|\psi_0\|_{\mathcal{L}^2(\mathcal{U})} \leq cM$ and $\|\eta_0\|_{\mathcal{L}^2(\mathcal{W})} \leq cM$. Now write $\psi_0 = \xi_0 + \sigma_0$ where $\xi_0 \in A, \sigma_0 \in S$. Construct a sequence $\psi_v \in Z^1_{\mathcal{L}^2(\mathcal{U})}(\mathcal{O}), \eta_v \in \mathcal{C}^0_{\mathcal{L}^2}(\mathcal{W})$ such that $\psi_v = \xi_v + \sigma_v$, where again $\xi_v \in A, \sigma_v \in S$. It follows that

- $\psi_v = \xi_v + \delta\eta_v$

- $\psi_v = \xi_v + \sigma_v$
- $\|\psi_v\|_{\mathcal{L}^2(\mathcal{U})} \leq 2^{-v}cM$ and $\|\eta_v\|_{\mathcal{L}^2(\mathcal{W})} \leq 2^{-v}cM$

Therefore,

$$\|\xi_v\|_{\mathcal{L}^2(\mathcal{V})} \leq \varepsilon\|\psi_v\|_{\mathcal{L}^2(\mathcal{U})} \leq 2^{-v}\varepsilon cM = 2^{-v-1}M$$

Therefore, $\exists \zeta_{v+1} \in Z^1_{\mathcal{L}^2}(\mathcal{U}, \mathcal{O})$ and $\eta_{v+1} \in C^0_{\mathcal{L}^2(\mathcal{W})}(\mathcal{O})$ such that $\psi_{v+1} = \xi_v + \delta\eta_{v+1}$ on $\mathcal{W}$. Hence,

$$\max\left(\|\psi_{v+1}\|_{\mathcal{L}^2(\mathcal{U})}, \|\eta_{v+1}\|_{\mathcal{L}^2(\mathcal{W})}\right) \leq 2^{-v-1}cM$$

Taking the orthogonal decomposition, $\psi_{v+1} = \xi_{v+1} + \sigma_{v+1}$. Letting $\sigma = \sum \sigma_v, \eta = \sum \eta_v$, it follows that $\sigma = \zeta + \delta\eta$. □

From this, we conclude the following theorem:

Theorem 4.3.2. *X is a Riemann Surface, $Y_1 \Subset Y_2 \subset X$ open subsets. The restriction homomorphism $\mathrm{H}^1(Y_2, \mathcal{O}) \twoheadrightarrow \mathrm{H}^1(Y_1, \mathcal{O})$ has finite dimensional image.*

It follows from this that, if X were also compact, we can set $Y_1 = Y_2 = X$ and we get that $\mathrm{H}^1(X, \mathcal{O})$ is finite dimensional, as desired. We now prove the above theorem.

Proof. (above theorem): Take

$$Y_1 \subset W_i = Y' \Subset \bigcup U_i = Y'' = Y_2$$

$Z_i(W_i), Z_i(U_i)$ are disks, and by the previous result we know that $\mathrm{H}^1(\mathcal{U}, \mathcal{O}) \twoheadrightarrow \mathrm{H}^1(\mathcal{W}, \mathcal{O})$ has finite dimensional image, Well, $\mathrm{H}^1(U_i, \mathcal{O}) = \mathrm{H}^1(W_i, \mathcal{O}) = 0$, so these are Leray coverings, and we have the morphisms

$$\mathrm{H}^1(Y_2, \mathcal{O}) \to \mathrm{H}^1(\mathcal{U}, \mathcal{O}) \to \mathrm{H}^1(\mathcal{V}, \mathcal{O}) \to \mathrm{H}^1(Y_1, \mathcal{O})$$

□

Via this finite dimensionality, we can define the ***genus*** of X to be $g(X) := \dim \mathrm{H}^1(X, \mathcal{O})$. This is equivalent to counting the "number of holes" of X, viewing X as a g-torus. We also (will) know that $g(X) = \dim \mathrm{H}^0(X, \Omega_X)$ (this equality holds in general, and it is called Serre Duality). In particular, we know that $g(\mathbb{P}^1) = 0$.

Theorem 4.3.3. *Suppose X is a Riemann Surface and $Y \Subset X$ is a (compactly supported) open set. $\forall p \in Y$, $\exists$ a meromorphic function $f \in \mathcal{M}(Y)$ such that f has a pole at p and is holomorphic in $Y \setminus \{p\}$.*

This has the following corollary:

Lemma 4.3.4. *If X is compact and $a_1, \ldots, a_n$ are distinct points on X and $c_1, \ldots, c_n$ are distinct points on $\mathbb{C}$, then $\exists$ a meromorphic function $f \in \mathcal{M}(X)$ such that $f(a_i) = c_i$.*

Chapter 5

Divisors and Duality

5.1 Interpolation

5.1.1 On $\mathbb{P}^1$

Choose $x_1, \ldots, x_n \in PP^1$, $c_1, \ldots, c_n \in \mathbb{C}$. Then $\exists$ a polynomial f such that $f(x_i) = c_i$ for any $i \leq n$. Let

$$q_i(x) = \frac{(x - x_2)\ldots(x - x_n)}{(x_1 - x_2)\ldots(x_1 - x_n)}$$

And set

$$f(x) = \sum_{i=1}^{n} c_i q_i(x)$$

5.1.2 On Any Compact Riemann Surface

As before, choose $a_1, \ldots, a_n \in X$ (X is a compact Riemann surface) and $c_1, \ldots, c_n \in \mathbb{C}$. We'd like to show $\exists f \in \mathcal{M}(X)$ such that $f(a_i) = c_i\ \forall i$. Can find f_{ij} with a pole at a_i and holomorphic on $X \backslash a_i$. Now choose $\lambda_{ij} \in \mathbb{C}^\times$ such that $f_{ij}(a_k) \neq f_{ij}(a_j) - \lambda_{ij}$ for any k. Now set

$$g_{ij} = \frac{f_{ij} - f_{ij}(a_j)}{f_{ij}(a_j) - \lambda_{ij}}$$

This is 0 at a_j and holomorphic elsewhere. Set $h_i = \prod_{j \neq i} g_{ij}$. This is zero at all a_k except for with $k = i$ and is holomorphic at a_i. It follows that $\sum_{i=1}^{n} a_i h_i$ is the function we are looking for.

5.2 Riemann Roch and Serre Duality

A ***divisor*** on a Riemann Surface X is of the form $D = \sum_{i=1}^{n} a_i p_i$ for $p_i \in X$, $a_i \in \mathbb{Z}$ (these are just formal sums, and must be finite). Set $\deg(D) := \sum_{i=1}^{n} a_i$ to be the ***degree*** of D. A divisor D is ***principal*** if $\exists f \in \mathcal{M}(X)$ such that $D = \operatorname{div}(f)$, where

$$\operatorname{div}(f) := \sum_{p \in X} \operatorname{ord}_p(f) p$$

Where the order is the degree of the pole or zero at p. There are only finitely many zeroes and poles for f as $f \in \mathcal{M}(X)$, so this sum is finite and it is indeed a divisor. It's pretty easy to check that $\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$ for any $f, g \in \mathcal{M}(X)$. Divisors also have a partial ordering. We say that $D_1 \geq D_2 \iff D_1 - D_2 = \sum n_i p_i$ with each $n_i \geq 0$. In particular, a divisor D is called ***effective*** if $D \geq 0$.

Given a divisor D on X, we can naturally assign a sheaf $\mathcal{O}(D)$ to D such that

$$\mathcal{O}(D)(U) = \{f \in \mathcal{M}(U) \mid \operatorname{div}(f)|_U + D|_U \geq 0\}$$

Explicitly, this is just a sheaf of meromorphic functions that bound the possible number of zeroes/poles of f. If D has a pole of order n at p, then f **must** have a zero at p of order at least n, lest the corresponding coefficient be negative. Conversely, if D has a zero of order m at q, then f is allowed to have a pole, but only of order at most m, at q (otherwise the corresponding coefficient would be < 0 again).

We say that divisors D_1 and D_2 are ***linearly equivalent*** (denoted $D_1 \sim D_2$) if $D_1 - D_2 = \operatorname{div}(f)$ for some $f \in \mathcal{M}(X)$. This is clearly an equivalence relation. It follows that $\mathrm{H}^0(\mathcal{O}(D))$ (the global sections of $\mathcal{O}(D)$) is precisely the set of divisors linearly equivalent to D. Furthermore, it follows that $D \sim D'$ implies that $\mathcal{O}(D) \simeq \mathcal{O}(D')$. This is because, if $D - D' = \operatorname{div}(g)$ for some $g \in \mathcal{M}(X)$, it follows that $\mathcal{O}(D)(U) \cong \mathcal{O}(D')(U)$ via the isomorphism $f \mapsto gf$ (this has natural inverse $f \mapsto f/g$).

Let $h^0(D) = \dim \mathrm{H}^0(\mathcal{O}(D))$. This is independent of the linear equivalence class of D.

Theorem 5.2.1. ***(Riemann Roch)*** *Let X be a compact Riemann Surface of genus g. $g = \dim \mathrm{H}^1(\mathcal{O})$, and*

$$h^0(D) - h^1(D) = \deg(D) - g + 1$$

We can associate a divisor to a meromorphic 1-form $\omega = f dz$ as follows:

$$\operatorname{div}(\omega) := \sum_{p \in X} \operatorname{ord}_p(\omega) p$$

It's pretty clear divisors associated to any two meromorphic 1-forms are linearly equivalent, as if $\omega_1 = f\omega_2$, then $\operatorname{div}(\omega_1) - \operatorname{div}(\omega_2) = \operatorname{div}(f)$. Thus, we can associate a divisor to a meromorphic function on X, and this is canonically defined over X. We let K_X denote this divisor, and we call it the ***canonical divisor***.

When X is a compact Riemann Surface, we have the following short exact sequence of sheaves

$$0 \to \Omega \to \mathcal{E}^{1,0} \to \mathcal{E}^2 \to 0$$

Where Ω is the sheaf associated to the kernel of the surjective morphism $d : \mathcal{E}^{1,0} \twoheadrightarrow \mathcal{E}^2$. As $\mathrm{H}^1(X, \mathcal{E}^{1,0}) = 0$, it follows that

$$\mathrm{H}^1(X, \Omega) = \frac{\mathcal{E}^2(X)}{d\mathcal{E}^{1,0}(X)}$$

By taking a long exact sequence in cohomology. We can write down the definition of this sheaf Ω on divisors (denoted $\Omega(D)$) as follows:

$$\Omega(D)(U) := \{\omega \text{ a meromorphic one form} \mid \operatorname{div}(\omega) + D \geq 0\}$$

This associates a natural linear pairing $\Omega(-D) \times \mathcal{O}(D) \to \Omega$ sending $(\omega, f) \mapsto f\omega$. This induces a pairing on cohomology:

$$\mathrm{H}^0(X, \Omega(-D)) \times \mathrm{H}^1(X, \mathcal{O}(D)) \to \mathrm{H}^1(X, \Omega) \to \mathbb{C}$$

Where $(\omega, f) \mapsto f\omega \mapsto \operatorname{Res} f\omega$, for $\operatorname{Res}\omega = \frac{1}{2\pi i}\iint_X \xi$, where ξ is the representative of the corresponding class of $\mathcal{E}^2(X)$ corresponding to $\omega \in \mathrm{H}^1(X, \Omega)$.

Theorem 5.2.2. ***(Serre Duality)*** *The pairing above is a perfect (non-degenerate, billinear) pairing.*

$$\mathrm{H}^0(\mathcal{O}(K_X - D)) \cong \mathrm{H}^1(\mathcal{O}(D))^*$$

in particular,

$$h^0(K_X - D) = h^1(D)$$

Thus we can rewrite the conclusion of Riemann Roch as:

$$h^0(D) - h^0(K_X - D) = \deg(D) - g + 1$$

5.3 Proof of Riemann Roch

Proof. (Riemann-Roch) This is clearly true for $D = 0$. Now onto the general case. Given any divisor D and a point p, there exists an injection $\mathcal{O}(D) \hookrightarrow \mathcal{O}(D+p)$, with natural kernel $\mathcal{O}_p$, where $\mathcal{O}_p$ is the ***skyscraper sheaf*** at p. This is defined by

$$\mathcal{O}_p(U) = \begin{cases} \mathbb{C} & p \in U \\ 0 & \text{otherwise} \end{cases}$$

It follows that $\mathcal{O}_p(X) = \mathbb{C}$, as $p \in X$, so $\mathrm{H}^0(\mathcal{O}_p) = \mathbb{C}$, with higher cohomology vanishing. This gives us a short exact sequence

$$0 \longrightarrow \mathcal{O}_(D) \longrightarrow \mathcal{O}(D+p) \longrightarrow \mathcal{O}_p \longrightarrow 0$$

Giving us a long exact sequence in cohomology:

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^0(\mathcal{O}(D)) \arrow[r] & \mathrm{H}^0(\mathcal{O}(D+p)) \arrow[r] & \mathrm{H}^0(\mathcal{O}_p) \arrow[dll] \\
 & \mathrm{H}^1(\mathcal{O}(D)) \arrow[r] & \mathrm{H}^1(\mathcal{O}(D+p)) \arrow[r] & \mathrm{H}^1(\mathcal{O}_p)
\end{tikzcd}
$$

Thus, $\chi(\mathcal{O}(D)) + 1 = \chi(\mathcal{O}(D+p))$. But, $\chi(\mathcal{O}(D)) = \deg(D) - g + 1, \chi(\mathcal{O}(D+p)) = \deg(D+p) - g + 1$. It follows that we have two cases:

- $h^0(D+p) = h^0(D)$ and $h^1(D+p) = h^1(D) - 1$
- $h^0(D+p) = h^0(D) + 1$ and $h^1(D+p) = h^1(D)$

□

Here are some neat applications:

(1) If $D = 0$, this gives us that $h^0(K_X) = h^1(\mathcal{O}) = g$, so $h^0(K_X) = g$. This gives us another definition of genus (the number of global sections of the canonical divisor)

(2) If $D = K$,

$$\underbrace{h^0(K)}_{g} - \underbrace{h^0(\mathcal{O})}_{1} = \deg(K) - g + 1$$

So $\deg(K) = 2g - 2$.

5.4 Maps to $\mathbb{P}^n$

Let X be a compact Riemann Surface of genus g. We'd like to construct a map fron X to projective space. First, take a basis $s_0, \ldots, s_n$ of $\mathrm{H}^0(\mathcal{O}_X(D))$. We can construct a map $X \to \mathbb{P}^n$ by mapping $x \mapsto [s_0(x), \ldots, s_n(x)]$. This is well defined precisely when $s_i(x) \neq 0$ for at least one i. If all s_i vanish at some $p \in X$, we say that p is a ***basepoint*** of $\mathcal{O}_X(D)$. We'd like to find D such that $\mathcal{O}_X(D)$ is ***basepoint free***. p is not a basepoint if and only if $\mathrm{H}^0(\mathcal{O}_X(D-p)) \subsetneq \mathrm{H}^0(\mathcal{O}_X(D))$, as there exists a nowhere vanishing section in one but not the other. It follows that

$$p \text{ is a basepoint } \iff h^0(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X(D-p))$$

This can be strengthened into a basepoint free criterion as follows:

Lemma 5.4.1. *$\mathcal{O}_X(D)$ is basepoint free $\iff h^0(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X(D-p)) + 1$*

Theorem 5.4.2. *Let X be a curve of genus $g \geq 1$. Then K is basepoint free, and $X \hookrightarrow \mathbb{P}^{g-1}$.*

Proof. Let K denote the canonical divisor of X. We'd like to show that $h^0(K-p) = h^0(K) - 1$ for any $p \in X$.

$$h^0(K-p) - h^0(p) = 2g - 3 - g + 1 = g - 2$$

We hope that $h^0(p) = 1$. We know that $\mathbb{C} \subset \mathrm{H}^0(\mathcal{O}_X(p))$. If you have a nonconstant meromorphic function, this defines a function $X \to \mathbb{P}^1$ with degree 1, it follows that X and $\mathbb{P}^1$ are birational, and indeed they are isomorphic. However, X is genus $g \geq 1$, and $\mathbb{P}^1$ has trivial genus, so we have a contradiction. Thus, $\mathbb{C} = \mathrm{H}^0(\mathcal{O}_X(p))$, and thus $h^0(p) = 1$. It follows that

$$h^0(K - p) = g - 1 = h^0(K) - 1$$

□

Question: What is the smallest degree non-constant meromorphic function on a Riemann Surface of genus g? We don't know the answer at this point, but we do know that there is a function of degree $\leq g + 1$. To see this, notice that if $\deg D \geq g + 1$, then

$$h^0(D) - h^0(K - D) = \deg(D) - g + 1 \geq 2$$

We've associated $\mathrm{H}^0(K)$ to the vector space of meromorphic 1-forms. Thus, $\mathrm{H}^0(K - p_1 - p_2 - \cdots - p_m)$ is the vector space of meromorphic 1-forms vanishing on all the p_i. If $p_1, \ldots, p_m$ are general points, it follows that $h^0(K - p_1 - \cdots - p_m) = g - m$ (this follows from the fact that, if p is a general point, then $h^0(D - p) = \max(0, h^0(D) - 1)$). It follows from here that there must be a meromorphic 1-form of degree $\leq g + 1$.

The next question you may want to ask is, when do we get an **embedding** from X into $\mathbb{P}^n$? we currently just have an inclusion map.

Theorem 5.4.3. *Let X be a compact Riemann Surface of genus > 0. Then the map associated $\mathcal{O}_X(D)$ (as we defined in the previous theorem) is an embedding in $\mathbb{P}^n \iff h^0(D - p - q) = h^0(D) - 2\ \forall p, q \in X$, where possibly $p = q$.*

This theorem gives rise to the following definition: A Riemann Surface X is called ***hyperelliptic*** if it admits a map $X \to \mathbb{P}^1$ of degree 2. Elliptic curves generally fit this framework, and all hyperelliptic curves are of the form $y^2 = p(x)$ for $p \in \mathbb{C}[x]$. For example, all genus 2 curves are hyperelliptic, since the map $|K_X| : X \to \mathbb{P}^1$ is a map of degree 2 as $\deg K_X = g = 2$.

Theorem 5.4.4. *Let X be a curve of genus $g \geq 3$ that is not hyperelliptic. Then $|K_X| : X \hookrightarrow \mathbb{P}^{g-1}$ is an embedding.*

Proof.

$$h^0(K - p - q) - h^0(p + q) = 2g - 4 - g + 1 = g - 3$$

We also have that $\mathrm{H}^0(\mathcal{O}_X(p + q)) \supset \mathbb{C}$. Suppose this contains a non-constant function f. This determines a basepoint free map $X \to \mathbb{P}^1$ of degree 2, as p, q lie over $\infty \in \mathbb{P}^1$, but we assumed that X is not hyperelliptic, so we have arrived at a contradiction. It follows that $h^0(p + q) = 1$ for any p, q pairs, even when $p = q$, It follows that

$$h^0(K - p - q) = g - 2 = h^0(K) - 2$$

□

The image under this embedding corresponds to the ***canonical curve*** of X in projective space.

Chapter 6

Algebraic Curves

6.1 Compact Riemann Surfaces are Algebraic Curves

Lemma 6.1.1. *Let D be a divisor such that* $\deg(D) < 0$*. Then* $h^0(D) = 0$*.*

Proof. The only $f \in \mathrm{H}^0(\mathcal{O}_X(D))$ are those such that $\mathrm{div}(f) + D \geq 0$. $\square$

Lemma 6.1.2. *If* $\deg(D) = 0$ *and* $h^0(D) > 0$*, then* $D \sim 0$*.*

Lemma 6.1.3. *Let D be a divisor of degree* $\geq 2g - 1$*. Then* $h^1(D) = 0$*.*

Lemma 6.1.4. *Let D be a divisor of degree* $d \geq 2g - 1$*. Then* $h^0(D) = \deg D - g + 1$*.*

Theorem 6.1.5. *Let X be a compact Riemann surface of genus g with divisor D. Then if* $\deg(D) \geq 2g$*, then* $\mathcal{O}_X(D)$ *is globally generated and you get a map* $f : X \to \mathbb{P}^{\deg(D)-g}$*. If* $\deg(D) \geq 2g + 1$*, then* $\mathcal{O}_X(D)$ *is very ample, and you get an embedding* $f : X \to \mathbb{P}^{\deg(D)-g}$*.*

Combined with Chow's lemma (saying that every closed subset of $\mathbb{P}^n$ is algebraic) we get that every compact Riemann Surface is an algebraic curve.

Proof. Without loss of generality, assume that D is effective. $\mathcal{O}_X(D)$ is globally generated if and only if $h^0(D) = h^0(D - p) + 1\ \forall p \in X$. $h^0(D) = \deg D - g + 1$, whereas $h^0(D - p) = \deg D - 1 + g + 1$ when D is of degree $D \geq 2g$, by the directly preceeding lemma, so we're done. For the next part, $\mathcal{O}_X(D)$ is very ample if and only if $h^0(D - p - q) = h^0(D) - 2\ \forall p, q \in X$. $h^0(D) = \deg(D) - g + 1$ and $h^0(D - p - q) = \deg(D) - 2 - g + 1$ by the same preceeding lemma when $\deg(D) \geq 2g + 1$ (as only then can we apply the lemma to $h^0(D - p - q)$). $\square$

It's worth noting that these inequalities are sharp. Let $D = K + p$, for instance. Then $\deg(D) = 2g - 1$, and is not globally generated at p. $h^0(D - p) = h^0(K) = g$ and $h^0(K + p) = 2g - 1 - g + 1 = g$. In this case, $h^0(D - p) = h^0(D)$. Conversely if D is degree $2g - 1$ and is not globally generated, then $g = h^0(D - p) = h^0(D)$ for some p. $D - p$ has degree $2g - 2$, $h^1(D - p) = 1$, so $D - p \sim K$ or $D \sim K + p$.

6.2 Weierstraß Points

Let $p \in X$, for X a compact Riemann surface of genus $g \geq 2$. We expect that

$$h^0(np) = \begin{cases} 1 & n \leq g \\ n-g & n > g \end{cases}$$

$$h^1(np) = h^0(K-p) = \begin{cases} g-n & n \leq g \\ 0 & n > g \end{cases}$$

A ***Weierstraß Point*** is a point $p \in X$ where $h^0(np)$ differs from the expectation. We can look at $n \in \mathbb{N}$ where $h^0(np) > h^0((n-1)p)$. This is a semigroup with g gaps. There is a ***weight*** of a Weierstraß point, defined as $\sum_{n\in\mathbb{N}}(h^0(np_w) - h(np_{exp}))$. In general, we know that

$$\sum_{\text{all Weierstraß points}} \sum_{n\in\mathbb{N}} (h^0(np_w) - h(np_{exp})) = (g-1)g(g+1) = g^3 - g$$

We can also think of a Weierstraß point as a ramification point of the canonical curve.

6.2.1 Weierstraß Points of Hyperelliptic Curves

Let $f : X \to \mathbb{P}^1$ be a $2:1$ canonical map. By Riemann-Hurwitz, this map has $2g+2$ ramification points. We claim that these are precisely the Weierstraß points. Here is a table of $h^0(np)$ for genus g.

n	$g=2$	$g=3$
1	1	1
2	2	2
3	2	2
4	3	3
5		3
6		4

Jumps occur at even p multiplicity.

6.2.2 Special Divisors

Let X be a compact Riemann Surface of genus g. A ***linear system*** on X is a subspace $V \subset \mathrm{H}^0(\mathcal{O}_X(D))$ for some D. We say V is g^r_d if $\dim V = r+1$ and $\deg(D) = d$. If V is basepoint free we get a map $X \to \mathbb{P}^r$. An effective divisor D is ***special*** if $h^1(D) \neq 0$. Let $|D| = \mathbb{P}(\mathrm{H}^0(\mathcal{O}_X(D)^*)$ denote the (projectivized) linear system of D we're mapping to.

Theorem 6.2.1. ***(Clifford's Theorem)*** *Let D be a special effective divisor. Then,*

$$\dim|D| \leq \frac{1}{2}\deg(D)$$

Equality holds if and only if $D = 0, K$ or X is hyperelliptic and $D \sim mg^1_2$.

As a fun application of this theorem, let $X \subset \mathbb{P}^n$ be a non-degenerate curve (i.e. it doesn't lie in a hyperplane). Further suppose that X is a smooth curve of degree n. Then, X is the rational normal curve of degree n, and is isomorphic to $\mathbb{P}^1$.

To see this, take a hyperplane H. Then $X \cap H = D$ a divisor (by non-degeneracy) and $\dim(D) = n$. By Clifford's Theorem, D is not special. $h^0(D) \geq n+1$, but $h^0(D) = \deg D - g + 1 = n + 1 - g$. It follows that, as $g \geq 0$, $g = 0$ for both of these to hol. We already know that a genus 0 smooth non-degenerate curve of degree n is the rational normal curve.

6.3 Canonical Maps

If X is not hyperelliptic, then $|K| : X \to \mathbb{P}^{g-1}$ is an embedding, with image being the canonical curve. We have yet to discuss what happens when X *is* hyperelliptic. We still get a map $|K| : X \to \mathbb{P}^{g-1}$. but it factors via a $2:1$ map to $\mathbb{P}^1$ as follows:

$$\begin{tikzcd} X \arrow[r, "|K|"] \arrow[dr, "2:1"'] & \mathbb{P}^{g-1} \\ & \mathbb{P}^1 \arrow[u, hook] \end{tikzcd}$$

Yielding a $2:1$ map onto the rational normal curve.

Lemma 6.3.1. *$K \sim (g-1)g^1_2$, where g^1_2 (the $2:1$ map above) is unique.*

Proof. Let f denote the canonical map $|K|$ for simpler notation. Since X is hyperelliptic, $\exists g^1_2$. For $p + q \in g^1_2$, then by Riemann Roch, p (or equivalently, q) is a basepoint of $|K - p|$. Then $f(p) = f(q)$.

Let $\mathrm{im}(f) = X'$. Then $f : X \to X' \subset \mathbb{P}^{g-1}$ is not necessarily birational, but is a $\mu : 1$ mapping onto X'. $\mu(\deg(X)) = 2g - 2$, and $\mu \geq 2$, so $\deg(X') \leq g - 1$. It follows that X' is precisely the rational normal curve. To see this, take the normalization $\widetilde{X'}$ and let $g(\widetilde{X'}) = h$. Then take a linear system of degree d on $\widetilde{X'}$ and dimension $\geq g$ such that $\widetilde{X'} \cong \mathbb{P}^1$. Then

$$g \geq h^0(D) - h^1(D) = d - h + 11 \leq g - 1 - h + 1 = g - h$$

As $h^1(D) = 0$ by Clifford's Theorem, it follows that $h = 0, h^0(D) = g, \deg(D) = g - 1$, and $\mu = 2$. Consequently, g^1_2 is unique.

6.4 Equations of Curves

Suppose $g = 3$ and X is nonhyperelliptic. $|K| : X \to \mathbb{P}^2$ is a plane quartic curve. Conversely, any smooth plane quartic curve is a genus 3 canonical curve.

To prove this (and other facts about canonical curves) one needs adjunction.

6.4.1 Adjunction

Suppose $D \subset Y$ is a smooth divisor of a smooth projective variety. Then $K_D \sim (K_Y + D)|_D$. More generally, if $E \to Y$ is a vector bundle of rank r over Y and Z is a smooth zero section $V(s)$ of $s \in \mathrm{H}^0(E)$, where Z is of expected dimension. Then, $N_{Z/Y} \simeq E|_Z$. Taking the normal sequence and plugging this in gives us short exact sequence

$$T|Z \to T_Y|_Z \to E|_Z \to 0$$

Implying that $c_1(T_z) = c_1(T_Y)|_Z - c_1(E|_Z)$, so $K_Z = -c_1(T_Z)$. Back to our example above, viewing $i : X \hookrightarrow \mathbb{P}^2$ where $i^*\mathcal{O}_{\mathbb{P}^2}(1) \cong K_X$, we see that $K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$. $\square$

6.5 Classifying Canonical Curves

6.5.1 Genus $g = 3$

For $g = 3$, the canonical image of a non-hyperelliptic curve is a smooth plane quartic, and conversely, every smooth plane quartic is the canonical image of a genus 3 curve.

Lemma 6.5.1. 2 *canonical curves* C_1, C_2 *are isomorphic* $\iff \exists M \in \mathrm{PGL}_g$ *such that* $MC_1 = C_2$.

Proof. Suppose we have a morphism of curves $f : C_1 \to C_2$. Canonical curves have a canonical cotangent bundle, so $f^*\Omega_2 \cong \Omega_1$. In particular $C_1 \cong C_2 \iff$ we have an isomorphism $\mathrm{H}^0(K_{c_1}) \cong \mathrm{H}^0(K_{c_2})$ induced by f. These are finite dimensional vector spaces, so they are isomorphic $\iff f$ is a linear map defined by an invertible matrix, i.e. f is of the form $M \in \mathrm{PGL}_g$. $\square$

The projectivized vector space of quadrics over $\mathbb{P}^2$ is 15 dimensional, so $\mathbb{P}^{14}/\mathrm{Disc} = \mathcal{U}^{14}$, and $\mathcal{U}^{14}$ modulo this PGL_g action yields a 6-dimensional variety of smooth quadrics of genus 3.

6.5.2 Genus $g = 4$

Moving on to $g = 4$, we get the canonical map $\varphi : C \to \mathbb{P}^3$ yielding a curve of degree 6 in $\mathbb{P}^3$. Now observe that $\deg \mathcal{O}_C(d) = 6d$. If $d \geq 2$, $6d \geq \deg(K) = 2g - 2$ It follows that in this case, $h^1(\mathcal{O}_C(d)) = 0$, so by Riemann Roch $h^0(\mathcal{O}_C(d)) = 6d - 5 + 1 = 6d - 3$

d	$h^0(\mathcal{O}_{\mathbb{P}^3}(d))$	$h^0(\mathcal{O}_C(d))$
2	10	9
3	20	15
$\vdots$	$\vdots$	$\vdots$

For a quadric Q, we have a natural map $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to \mathrm{H}^0(\mathcal{O}_C(2))$ defined by $Q \mapsto Q|_C$. $h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = 10$ but $h^0(\mathcal{O}_C(2)) = 9$, so this morphism has a kernel. Thus, there is a quadric that vanishes on C, denoted Q. For instance, if L is a linear form, then LQ vanishes on C. This quadric unique up to scaling, and we have a 4 dimensional space of forms LQ for fixed Q and L a linear form. For cubics, we thus get a map

$$\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to \mathrm{H}^0(\mathcal{O}_C(3))$$

which is a map from a 20 dimensional vector space down to a 15 dimensional one. Thus we have a kernel of dimension at least 5, with 4 dimensions of cubics of the form LQ. Thus, there is a cubic X not divisible by Q vanishing on C. Thus, $C \subset X \cap Q$. $X \cap Q$ is of degree 6. By the purity theorem, the complete intersection $X \cap Q$ is Cohen Macaulay, implying that $C = X \cap Q$. It follows that C is necessarily the intersection of a quadric and a cubic. Conversely, this implies (via adjunction) that every smooth $(2,3)$-complete intersection is a canonical genus 4 curve (it is sufficient to check that $K_C \sim \mathcal{O}_C(1)$).

Lemma 6.5.2. *There are* $2g-1$ *dimensions of hyperelliptic curves of genus* g.

Proof. Take a 2;1 map $\mathbb{C} \to \mathbb{P}^1$. By Riemann-Hurwitz, $g-1 = 2(-1) + \frac{b}{2}$, so $b = 2g+2$. Thus every hyperelliptic curve is cut out by equations $y^2 = \prod_{i=1}^{2g+2}(x-\alpha_i)$. These α_i are chosen up to some PGL_2 action, so the space of hyperelliptic curves is precisely $(\mathbb{P}^1)^{2g+2}/\,\mathrm{PGL}_2$, which is $2g-1$ dimensional.

□

In general we have dimensions

g	$\dim \mathcal{H}_g$	$\dim \mathcal{M}_g$
2	3	3
3	5	6
4	7	9
$\vdots$	$\vdots$	$\vdots$

So for large g there are far more nonhyperelliptic curves than hyperelliptic. Now suppose we have a quadratic curve $Q \subset \mathbb{P}^3$ of genus 4. It follows that only $x^2+y^2+z^2+w^2$ or $x^2+y^2+z^2$ can cut this out, since $x^2+y^2 = (x+iy)(x-iy)$, so in this case the curve would lie in the intersection of two hyperplanes; by non-degeneracy it must be contained in one of them, leading to a contradiction.

6.5.3 Genus $g = 6$

We'd like to understand genus 6 canonical curves that are not hyperelliptic. It follows that $\deg K = 10$, so $X \hookrightarrow \mathbb{P}^5$ is a degree 10 embedding.

d	$h^0(\mathcal{O}_{\mathbb{P}^5}(d))$	$h^0(\mathcal{O}_X(d))$
2	21	15

It follows that X is contained in a ≥ 6 dimensional family of quadrics. For general $\mathbb{P}^n$, $|\mathcal{O}_{\mathbb{P}^n}(2)| = \mathbb{P}^{\binom{n+2}{2}-1}$. There are finitely many $\mathrm{PGL}(n+1)$ orbits, determined by the rank of the quadric. Let $X_{n+1-s} \subset \mathbb{P}^{\binom{n+2}{2}-1}$ be the space of quadrics of rank $\leq n+1-s$. This has codimension $\frac{s(s+1)}{2}$, which can be computed, so X is contained in a quadric Q of rank ≤ 4. The space of such quadrics is a codimension 3 space, so there is a dimension 3 family of rank 4 quadrics containing X.

Consequently, there is a quadric of rank 4 whose vertex meets with X, and if through the point there is a 1 parameter family of quadrics of rank ≤ 4 containing X, then we can project X through this point p and get a curve contained in a 1 parameter family of quadrics. Thus, $X \subset Q_1 \cap Q_2$, two quadrics in the 1-parameter family, so X is contained in a Del Pezzo Surface.

6.6 Geometric Riemann Roch

Theorem 6.6.1. *Let C be a trigonal curve (i.e. C admits a map $f : C \to \mathbb{P}^1$ that is $3:1$). Then the fibers of f are collinear in the canonical curve. The canonical curve lies in the rational normal surface scroll.*

Suppose that C has a g^1_4. **Geometric Riemann Roch** states that

$$\underbrace{h^0(g^1_3)}_{2} - h^0(K - g^1_3) = \deg(g^1_3) - g + 13 - g + 1 = 4 - g$$

It follows that $h^0(K - g^1_3) = g - 2$, so setting $g^1_3 = p + q + r$ for points p, q, r, it follows that

$$h^0(K - p - q - r) = g - 2$$

6.7 Projective Embeddings

Theorem 6.7.1. *If X is a smooth projective variety of dimension d, then X embeds into $\mathbb{P}^{2d+1}$. In particular, a Riemann Surface always embeds into $\mathbb{P}^3$.*

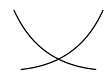
Proof. We know that $X \hookrightarrow \mathbb{P}^n$ for some n, as it is a projective variety. If $n \leq 2d+1$, then we're done. If not, take the general projection from a point p, inducing a map $\pi_p : X \hookrightarrow \mathbb{P}^{n-1}$. This is an embedding $\iff$ $\pi_p(x) \neq \pi_p(y)$ if $x \neq y$, and $\pi_p(T_x)$ is an isomorphism onto its image for all $x \in X$. This holds if p not contained in either the Secant Variety or the Tangent Variety of x. $\dim(\mathrm{Sec}(X)) \leq 2\dim X + 1$ (derived via an incidence correspondence), and similarly one can derive that $\dim \tan(X) \leq 2\dim X$. Thus, if $n > 2d+1$, then projection from a general point (i.e. one that is not contained in the secant or tangent variety) $\pi_p : X \hookrightarrow \mathbb{P}^{n-1}$ is an embedding.

It follows that we can keep projecting from points until we hit $n = 2d+1$, so we can conclude. □

While this seems to be a nice bound, we can do better.

Theorem 6.7.2. *Any Riemann Surface maps into $\mathbb{P}^2$, in such a way that the image has only nodes as singularities.*

Recall a node is a singularity that locally analytically looks like



i.e. it is a plane singularity of multiplicity 2, where the tangent cone consists of 2 distinct lines. We now prove the above theorem.

Proof. We already have an embedding of a Riemann Surface X into $\mathbb{P}^3$, we'd just like to project down to $\mathbb{P}^2$ in a sufficiently nice way. We need to check that this projection only has finitely many singularities. We construct this via projecting down from a point p. We'd like to pick p such that:

- No tangent line to X passes through p
- There are finitely many secant lines passing through p.

Thus, each of these lines needs to just meet the curve at exactly 2 points. In addition, if a secant line S is secant to X at x and y, then the tangent lines T_x and T_y should not intersect, lest they be coplanar and project to the same line (this would project to a cuspidal singularity, not a nodal singularity). Who says this point exists?

Theorem 6.7.3. ***(General Position Theorem)*** *Suppose $C \subset \mathbb{P}^r$ is an irreducible, non-degenerate curve of degree d, and $r \geq 2$. A general hyperplane meets C in d points, any r of which are linearly independent.*

This suggests that for $C \subset \mathbb{P}^r$ and $r \geq 3$, a general hyperplane meets C in d points, where no 3 of which are collinear (this is also known as the ***tri-secant lemma***). This implies that one can choose a secant line which is not tri-secant. It follows that such a point p (in the above proof) exists. □

It's fairly clear that the General Position Theorem implies the tri-secant lemma, but in fact these are equivalent with a bit of work. In fact, both statements are equivalent to saying that all secant lines are tri-secant.

Chapter 7

Further Topics

7.1 Monodromy

Theorem 7.1.1. ***(Uniform Position Theorem)*** *Suppose that $C \subset \mathbb{P}^r$ is a degree d irreducible non-degenerate curve, and $C \cap H$ is a general hyperplane section. Then d points on $C \cap H$ have a subset of size r which are linearly independent.*

Suppose that $\pi : \mathcal{X} \to B$ is a flat family of varieties parameterised by a base B, which is smooth and irreducible. Let D be the discriminant locus of B, and let $\mathcal{U}$ be all points in $\mathcal{X}$ NOT lying over D (so $\mathcal{U} = X \backslash \pi^{-1}(D)$). This gives us a covering $\mathcal{U} \to B \backslash D$, and $\pi_1(B \backslash D)$ acts on covers.

When $\mathcal{X}$ is a finite cover (say of degree d), then there is a natural mapping $\pi_1(B \backslash D) \to \mathfrak{S}_d$, which is the ***Monodromy Representation*** of the family. The question now is, what is the image in $\mathfrak{S}_d$?

Lemma 7.1.2. *The monodromy group of this family is the full symmetric group.*

Such a lemma actually implies the Uniform Position Theorem, as this is in a sense saying that you can move around any points in the fiber of π as we please.

Proof. Let

$$I(m) = \{(p_1, \ldots, p_m, H) \mid H \in \mathcal{U}, p_1, \ldots, p_m \in H \cap C \text{ distinct}\}$$

If $I(m)$ is connected, then there is a path connecting $(p_1, \ldots, p_m, H) \to (p_{i_1}, \ldots, p_{i_m}, H)$. Thus there is an element of $\pi_1(\mathbb{P}^{r*} \backslash D)$ lifting to this path, so the monodromy action acts m-transitively on the fibers, which would allow us to conclude. For $\pi_1(B \backslash D)$ to equal S_d, this action need only be 2-transitive. Thus, to prove the lemma it is sufficient to check that $I(2)$ is connected and contains a simple transposition.

Take a hyperplane tangent to the curve such that it intersects the curve at $d - 2$ points. Moving it in a little bit, the tangent point moves down to 2 separate intersections. Moving along the curve will just swap these points, so it follows that $I(2)$ has a simple transposition. To check connectedness, take a projection $I(2) \to (C \times C)/\Delta$. the fibers are open sets in $\mathbb{P}^{r-2}$, so this projection is continuous. $C \times C \backslash \Delta$ is connected, so it follows that $I(2)$ must also be connected. $\square$

Theorem 7.1.3. ***(Generalized Uniform Position Theorem)*** *Suppose that $C \subset \mathbb{P}^r$ for $r \geq 2$ is a non-degenerate irreducible curve of degree d. If $\mathcal{D}$ is any linear system on C and $\Gamma = H \cap C$ is a general hyperplane section, then all subsets of points in Γ of a given size m impose the same number of conditions on $\mathcal{D}$.*

7.2 Castelnuovo's Bound

Let C be a smooth curve that admits a birational map onto a non-degenerate curve of degree d in $\mathbb{P}^r$. Then

$$g(C) \leq \pi(d,r) := \frac{m(m-1)}{2}(r-1) + m \cdot \varepsilon$$

For $m = \lfloor \frac{d-1}{r-1} \rfloor$ and ε is the remainder when you divide $d-1$ by m (i.e. $d-1 = m(r-1)+\varepsilon$).

7.2.1 Examples

- When $r = 2$, then $m = d-1$ and $\varepsilon = 0$. It follows that $g(C) \leq \binom{d-1}{2} = \frac{(d-1)(d-2)}{2}$. This is already known (this is an equality for smooth curves, and singular curves have strictly smaller geometric genus), but we can arrive at this via Castelnuovo's bound.

- When $r = 3$, we have two choices. If d is odd, then $d = 2m+1$ and $\varepsilon = 0$. Then

$$g(C) = \frac{(d-1)(d-2)}{2} \cdot 2$$

  When d is even, $\varepsilon = 1$ and $m = \frac{d-2}{2}$. In this case,

$$g(C) = 2\left(\frac{d-2}{2}\right)\left(\frac{d-4}{2}\right) + \frac{d-1}{2}$$

We've checked that this bound is sharp, though we have yet to prove such a bound exists. Using the notation above, we see that X has Hilbert polynomial $P_X(m) = dm + (1-p_a)$. We'd like to bound the Hilbert polynomial from below in same way utilizing the Castelnuovo bound. Choose $E_\ell \subset |\ell D|$ be the linear subspace cut out by hypersurfaces of degree ℓ. Let $\beta_\ell = \dim E_\ell$, and let Γ be a hyperplane section of X. Then, $\beta_\ell - \beta_{\ell-1} \geq \dim E_\ell - \dim E_\ell(-\Gamma)$.

Lemma 7.2.1. *Let $p_1, \ldots, p_d$ be a set of points in $\mathbb{P}^{r-1}$ such that any r among them are linearly independent. Then, they impose at least* $\min(d, k(r-1)+1)$ *conditions on polynomials of degree k.*

This lemma bounds $E_\ell - \dim E_\ell(-\Gamma)$, implying that $\beta_\ell - \beta_{\ell-1} \geq \min(d, \ell(r-1)+1)$ for any ℓ. A telescoping sum argument on each of the terms above yields the result.

Theorem 7.2.2. *Suppose that C is non-hyperelliptic. Then we have a surjcetive mapping*

$$\mathrm{Sym}^\ell(\mathrm{H}^0(C.K)) \twoheadrightarrow \mathrm{H}^0(C, \ell K)$$

Consider the $\ell = 2$ case. As $\dim \mathrm{Sym}^2(\mathrm{H}^0(C.K)) = \binom{g+1}{2}$ and $\dim \mathrm{H}^0(C, 2K) = 3g - 3$, then

$$\begin{aligned}\binom{g+1}{2} - 3g + 3 &= \frac{(g+1)g - 6(g-1)}{2} \\ &= \frac{g^2 - 5g + 6}{2} \\ &= \frac{(g-2)(g-3)}{2}\end{aligned}$$

This corresponds to the dimension of the kernel.

Theorem 7.2.3. ***(Castelnuovo)*** *Suppose that $r \geq 3$ and $d \geq 2r + 1$. Let $m = \lfloor \frac{d-1}{r-1} \rfloor$ and $d - 1 = m(r-1) + \varepsilon$ like before. Then, extremal curves exist and are precisely either of the following forms:*

(a) The image of a smooth plane curve $C \subset \mathbb{P}^2$ of degree k under the $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ Veronese embedding.

(b) A nonsingular member of $|mH + L|$, on a rational normal surface scroll (here $\varepsilon = 0$)

(c) A nonsingular member of $|(m+1)H - (r - \varepsilon - 2)L|$ on a rational normal surface scroll.

7.3 Hirzebruch Surfaces

Let $\mathbb{F}_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$ define the ***Hirzebruch Surface*** of order e. These correspond to rational normal scrolls in projective space.

Explicitly, let's choose a rational normal curve of degree a in $\mathbb{P}^a$, and a rational normal curve of degree b in $\mathbb{P}^b$, and suppose both $\mathbb{P}^a, \mathbb{P}^b$ exist inside a larger $\mathbb{P}^n = \mathbb{P}^{a+b+1}$. Fix an isomorphism between both of these rational normal curves (such a map exists because, abstractly, both are isomorphic to $\mathbb{P}^1$). This defines a rational normal surface scroll on $S_{a,b} \subset \mathbb{P}^n$.

Theorem 7.3.1. *Suppose that $E + eF$ is basepoint free and maps $\mathbb{F}_e \to \mathbb{P}^{e+1}$ as a cone over a rational normal curve of degree e. Then $E + mF$ is very ample for $m > e$, and the mapping $\mathbb{F}_e \to \mathbb{P}^{2m-e+1}$ has image that is precisely the scroll $S_{m-e,m}$.*

Recall that if $X \subset \mathbb{P}^r$ is a non-degenerate surface, then $\deg(X) \geq r + 1$. Equality holds when X is a Veronese surface in $\mathbb{P}^5$ or $S_{a,b} \subset \mathbb{P}^{a+b+1}$.

7.4 Extremal Curves

To prove the Castelnuovo bound, one needs to look at the points along a general hyperplane section of a curve, and count how many conditions they impose on various varieties. Using the Castelnuovo bound, we want to maximize genus by minimizing the Hilbert Polynomial, minimizing the conditions imposed by these points at each step. Thus, if $g = \frac{m(m-1)(r-1)}{2} + m\varepsilon$ then the points on the hyperplane section must impose the minimum possible conditions on hypersurfaces.

For instance, on quadrics the minimum number of conditions imposed is $\min(d, 2(r-1)+1)$ and the codimension of quadrics vanishing on such points is $2r-1$.So the hyperplane sections lie on a rational normal curve. It follows that all the quadrics that vanish on a Riemann Surface X must determine a surface. As $\binom{d}{2}$ quadrics vanish on the natural normal curve, that implies that they impose the following number of conditions:

$$\binom{d+2}{2} - \binom{d}{2} = \frac{d^2 - 3d + 2 - d^2 + d}{2} = 2d + 1$$

Thus the aforementioned surface is the Veronese Surface contained in $\mathbb{P}^5$ or a rational normal scroll.

Now recall the Max Noether theorem, saying that for canonical curves (i.e. where $g \geq 3$), $\operatorname{Sym}^m \mathrm{H}^0(C, K) \twoheadrightarrow \mathrm{H}^0(C, 2K)$. In the case where $m = 2$, then $\dim \mathrm{H}^0(C, K) = g$, so $\dim \operatorname{Sym}^2 \mathrm{H}^0(C, K) = \binom{g+1}{2}$ and $\dim \mathrm{H}^0(C, 2K) = 3g - 3$. It follows that every canonical curve is contained in $\binom{g+1}{2} - (3g-3) = \binom{g-2}{2}$ quadrics. Thus,

g	# Quadrics
3	0
4	Unique Quadrics
5	3 dimensions of Quadrics
6	6 dimensions of Quadrics
7	10 dimensions of Quadrics
$\vdots$	$\vdots$

This induces the following question: Do the quadrics generate the ideal? If so, can you write them down?

Theorem 7.4.1. ***(Petri's Theorem)*** *Let $C \subset \mathbb{P}^{g-1}$ be a canonical curve of $g \geq 5$. Then $I(C)$ is generated by quadrics unless C is trigonal or C is isomorphic to a plane quintic curve. In these cases, the ideal is generated by quadrics and/or cubics.*

Proof. Suppose that $p_1, \ldots, p_g \in C$ are general. Notice that $h^0\left(K - \sum_{j \neq i} p_j\right) = 1$, so it is spanned by some nonzero element w_i, where $\{w_i, \ldots, w_g\}$ is a basis of $\mathrm{H}^0(K)$. Now consider the map $\psi_n : \mathrm{H}^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(n)) \twoheadrightarrow \mathrm{H}^0(C, nK)$, which exists by the Max Noether Theorem. Take a hypersurface of dimension n. This maps to $X|_C$. Let x_i be a basis of $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^{g-1}}(1))$ such that

$$\psi_n \left(\prod_{i=1}^{g} x_i^{a_i} \right) = \prod_{i=1}^{g} w_i^{a_i}$$

Writing $I(C) = \bigoplus I_k$ via its grading and $S = \mathbb{C}[x_1, \ldots, x_g]$, it's clear that $R(C) = S/I$, where $R(C)$ is the coordinate ring of C. Letting $D = p_3 + \cdots + p_g$, it follows that $h^0(K - D) = 2$, which is spanned by w_1, w_2. Thus,

$$\mathrm{H}^0(C, mK) \subset \mathrm{H}^0(C, nK - D) \subset \mathrm{H}^0(C, nK - 2) \subset \cdots \subset \mathrm{H}^0(C, nK + (-n+1)D)$$

Where $h^0(C, nK - sD) = n(2h-2) - s(g-2) - g + 1$ for $0 \leq s \leq n-1$, and $\deg(nK - sD) = n(2g-2) - s(g-2) = (2n-s)g - 2n + 2s$. This implies that the sequence of subsets above is in fact a vector space filtration of $\mathrm{H}^0(nK)$, where each piece has codimension $g-2$ in the previous piece. Now consider the map

$$\mathrm{H}^0((n-1)K + (-s+1)D) \otimes \mathrm{H}^0(K - D) \to \mathrm{H}^0(nK - D)$$

Defined via multiplying vectors, where $h^0(K - D) = 2$ and

$$h^0((n-1)K + (-s+1)D) \otimes \mathrm{H}^0(K - D) = (n-1)(2g-2) + (-s+1)(g-2) - g + 1$$

To conclude the theorem, we need to use the *basepoint free pencil trick*. Let C be a smooth curve, $\mathcal{L}$ a line bundle, $\mathscr{F}$ a torsion free $\mathcal{O}_C$-module. Let s_1, s_2 be linearly independent sections of $\mathcal{L}$, and $V \subset \mathrm{H}^0(C, \mathcal{L})$ the subspace they generate. We will see that

$$\ker\left(V \otimes \mathrm{H}^0(C, \mathscr{F}) \to \mathrm{H}^0(C, \mathscr{F} \otimes \mathcal{L})\right) = \mathrm{H}^0(C, \mathscr{F} \otimes \mathcal{L}^{-1}(B))$$

Where B is the base locus of V. This completely characterizes the map above and allows us to conclude the result. $\square$