Geometric Invariant Theory

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Abstract

Geometric invariant theory (GIT) provides a way to construct quotients of varieties by group actions of reductive algebraic groups. We compute several new examples of linear GIT quotients in which the underlying group is non-abelian. Additionally, we prove a theorem relating the linear GIT quotient by a given non-abelian group to a linear GIT quotient by a maximal torus of that group.

Introduction

Why GIT? Geometric invariant theory (GIT) is a highly useful tool for producing quotients of varieties by group actions. In these notes, we will focus on *linear GIT*, where $X = \mathbb{A}^n_{\mathbb{C}} = \mathbb{A}^n$. We also consider only the $\mathbb{C}$-valued points of this variety, which are identified with the complex vector space $\mathbb{C}^n$.

Let G be a group that is connected, complex algebraic, and reductive - in every example we will consider, G is a subgroup of $GL_2(\mathbb{C})$. To set up a “linear GIT datum”, we fix an action of G on X that is linear, so every $g \in G$ determines a linear isomorphism on $\mathbb{A}^n$ with respect to the vector space structure. Essentially without loss of generality, we assume the kernel of this action is always finite.

Our goal is to define a quotient variety X/G of X with respect to G. Ideally, this quotient would satisfy two properties:

1. X/G should satisfy a universal property: any G-invariant morphism $\varphi : X \to Y$ should factor through a fixed ‘projection’ morphism $\pi : X \to X/G$.
2. X/G should be an orbit space: its $\mathbb{C}$-valued points should be in bijection under π^{-1} with G-orbits of $\mathbb{C}$-valued points of X.

In most cases, no X/G satisfying these properties exists. In particular, continuity of π would imply that every orbit $Gx \subset X$ corresponding to a closed point in X/G is also closed.

Example: Projective Space Let $X = \mathbb{A}^2$ and $G = GL_1(\mathbb{C}) \cong \mathbb{C}^\times$, with action $t \cdot (x_1, x_2) = (tx_1, tx_2)$. G fixes the origin, while the orbit of any nonzero (x_1, x_2) is a line punctured at the origin, which is not closed in the Zariski topology.

On the other hand, replacing $\mathbb{A}^2$ with $\mathbb{A}^2 \setminus \{0\}$ simultaneously makes every orbit closed, and the natural identification $(\mathbb{A}^2 \setminus \{0\})/\mathbb{C}^\times \cong \mathbb{P}^1$ satisfies properties 1 and 2 above.

This operation of deleting a locus of 'bad points' before taking the quotient is the key idea of GIT.

1 Definitions

In general, there are multiple choices for the locus of 'bad points', each producing potentially non-isomorphic quotients. In GIT, the choice of locus is parameterized by a character θ (an element of $\chi^*(G) := \mathrm{Hom}(G, \mathbb{C}^\times))$[1], and a full linear GIT datum consists of a triple (X, G, θ).

We will also need 1-parameter subgroups, which are elements of $\chi_*(G) = \mathrm{Hom}(\mathbb{C}^\times, G)$. There is a natural pairing of characters and 1-parameter subgroups: given a 1-parameter subgroup λ and a character θ, $\theta \circ \lambda : \mathbb{C}^\times \to \mathbb{C}^\times$ is algebraic and therefore necessarily of the form $x \to x^a$ with $a \in \mathbb{Z}$, so define $\langle \lambda, \theta \rangle = a$. We can now give the key definitions of GIT for a datum (X, G, θ); these definitions can be found in [Kin94], and are equivalent to certain statements about nonvanishing of G-invariant sections.

Definition 1. *A point $x \in X$ is semistable if for all 1-parameter subgroups λ of G such that $\lim_{t\to 0} \lambda(t) \cdot x$ exists (in the Euclidean topology on X), $\langle \lambda, \theta \rangle \geq 0$. x is stable if for all such λ, $\langle \lambda, \theta \rangle > 0$, and x is unstable if it is not semistable*[2].

Denoting the respective semistable, stable, and unstable loci as $X^{ss}(G)$, $X^s(G)$, and $X^{us}(G)$, we have $(X^{ss}(G))^C = X^{us}(G)$ and $X^s(G) \subset X^{ss}(G)$. The unstable locus plays the role of the 'bad points'.

Definition 2. *Given a linear GIT datum (X, G, θ), the GIT quotient $X/\!\!/_\theta G$ is the global quotient stack $[X^{ss}(G)/G]$.*

We don't need the precise definition of a quotient stack, although it satisfies properties (1) and (2) with respect to $X^{ss}(G)$, and is in fact a variety in many reasonable cases. If furthermore $X^{ss}(G) = X^s(G)$, then the $\mathbb{C}$-valued points of $[X^{ss}(G)/G]$ will all be closed, and this condition is a necessary assumption for many theorems about GIT quotients.

2 Computing Examples of Linear GIT

One problem which we considered was to produce as many examples of linear GIT as possible and study the semistable and stable loci in these examples. It is known in general that a complex reductive algebraic group G can be written as a quotient $G = (H \times D)/Z(H)$, where H is semisimple, D is diagonalizable, and $Z(H)$ is the center of H, viewed as a subgroup of the product for some choice of inclusion $Z(H) \longrightarrow H \times D$. Note that $Z(H)$ is finite by the semisimplicity of H. Then, the following proposition implies that we only need to consider the linear GIT with respect to reductive groups of form $H \times D$:

Proposition 1. *Let $q : G \longrightarrow G/H$ be a quotient map of a complex reductive group with a finite kernel. If $(G/H, X, \theta)$ is a linear GIT datum, composition of the action map $a : G/H \longrightarrow \mathrm{GL}(X)$ and q yields another linear GIT datum $(G, X, \theta' := \theta \circ q)$. Then, the loci X^{ss} (resp. X^s) with respect to these two data are the same.*

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[1] We require all characters and 1-parameter subgroups to be (algebraic) morphisms of varieties, where $\mathbb{C}^\times$ is identified with $\mathrm{Spec}(\mathbb{C}[t, t^{-1}])$.

[2] If G is not assumed to have finite kernel, a slightly different definition is needed.

Motivated by this proposition, we considered the following example of a group which takes the form $H \times D$:

Example 1. *The scaling action of* $\mathbb{G}_m$ *on* $\mathbb{A}^2$ *and the natural action of* $\mathrm{SO}(2)$ *define an action of* $\mathrm{SO}(2) \times \mathbb{G}_m$ *on* $X = (\mathbb{A}^2)^n$. *For the triple* $(\mathrm{SO}(2) \times \mathbb{G}_m, X, id_{\mathbb{G}_m})$, *we have*

$$X^s = X^{ss} = \left(\bigcup_{g \in \mathrm{SO}(2)} g \cdot (X_1 \cup X_2) \right)^c ,$$

where we define $X_1 = V(x_1 = x_2 = ... = x_n = 0)$ *and* $X_2 = V(y_1 = y_2 = ... = y_n = 0)$ *inside* $X = \mathrm{Spec}(k[x_1, y_1] \otimes k[x_2, y_2] \otimes ... \otimes k[x_n, y_n])$.

We will present two further examples:

Example 2. *Given any natural number* n, $X = M_{n \times n}$, $G = \mathrm{GL}(n)$, *and* θ *is the determinant character, where* $g \in G$ *acts on* $x \in X$ *by* $g \cdot x = gxg^{-1}$. *In this case,* $X^{ss} = X^s = \varnothing$.

Proof. In the following proof, denote $S = \mathrm{diag}(t^{-1}, 1, 1, \cdots, 1)$ We first show that $X^s \subset X^{ss} \subset$ full rank matrices by showing for any x that has lower rank, $x \notin X^{ss}$. Consider the case where the first row of x is all zero, let $\lambda(t) = S$. Then we have

$$\lambda(t) \cdot x = \lambda(t) x \lambda(t)^{-1} = x \lambda(t)^{-1}$$

Since the entries of $\lambda(t)^{-1}$ are either constant or positive power of t, we can see that after multiplying with fixed x, $\lim_{t \to 0} \lambda(t) \cdot x$ exists. But the pairing $\langle \theta, \lambda \rangle = -1 < 0$. So $x \notin X^{ss}$. When x is a general lower rank matrix, we know that there exists a $g \in \mathrm{GL}(n)$ such that gx is a matrix whose first row is all zero. Then, similarly, by taking $\lambda(t) = g^{-1} S g$ proves that $x \notin X^{ss}$.

After showing X^{ss} is included in the full rank matrices, we can also show that for all $g \in \mathrm{GL}(n)$ whose determinant is positive, if $x \in X^{ss}$, then $gxg^{-1} \in X^{ss}$. As a $1-$ parameter subgroup, λ's image is in the form of $\lambda(t) = h \, \mathrm{diag}(t^{a_1}, \cdots, t^{a_n}) h^{-1}$ for some $a_i \in \mathbb{Z}$ and $h \in \mathrm{GL}(n)$. Now, as a known fact, we know that if $\lim_{t \to 0} \lambda(t) \cdot (gxg^{-1})$ exists, then

$$\lim_{t \to 0} (g^{-1} \lambda(t)) \cdot gxg^{-1} = \lim_{t \to 0} (g^{-1} \lambda(t) g) \cdot x$$

exists.

Since $x \in X^{ss}$, we know that $\langle \theta, g^{-1} \lambda g \rangle > 0$. But as θ is a group homomorphism, $\langle \theta, g^{-1} \lambda g \rangle = \langle \theta, \lambda \rangle > 0$. Therefore $gxg^{-1} \in X^{ss}$.

So now, since we know we could write all full rank diagonalizable matrices A as $A = PBP^{-1}$ for some diagonal matrix B and $\det(P) > 0$, to see diagonalizable matrices are not in X^{ss}, it suffices to see whether the diagonal matrices are in X^{ss}.

Given any y full rank and diagonal, take $\lambda(t) = S$. We then have $\lambda(t) \cdot y = \lambda(t) y \lambda(t)^{-1} = y$, which shows that $\lim_{t \to 0} \lambda(t) \cdot y$ exists, but $\langle \theta, \lambda \rangle = -1 < 0$. So diagonal matrices are not in X^{ss}.

So, diagonalizable matrices are not in X^{ss} as desired. Now, since the unstable locus is closed and diagonalizable matrices are a dense subset of X, we know that X must be contained in the unstable locus, which proves that $X^s \subset X^{ss} = \varnothing$ as desired.

□

Example 3. *Since GIT gives us a set of rules to remove a closed subset so that the quotient will become better behaved, we are interested in computing the semistable and stable loci, as well as the stabilizer, in the following example. Let* $X = \mathrm{Sym}^n(\mathbb{C}^k), G = GL(k)$*. Let* θ *be the* n^{th} *power of the inverse of the determinant character.*

We will discuss the problem for $k = 2$ *and* n *arbitrary i.e.,* X *is the vector space of homogeneous polynomials of degree* n *in the two variables* x *and* y *with an ordered basis given by* $(x^n, x^{n-1}y, x^{n-2}y^2, \ldots, xy^{n-1}, y^n)$*. Under the above identifications, we write an element* $p \in X$ *as a polynomial* $p((x, y))$*. Now, using the fact that* $\mathbb{C}$ *is algebraically closed, we can say that any* $p \in \mathrm{Sym}^n(\mathbb{C}^2)$ *can be written as a product of the following linear factors*

$$(a_1x + b_1y)(a_2x + b_2y)\ldots(a_{n-1}x + b_{n-1}y)(a_nx + b_ny)$$

and these factors are unique up to scaling and reordering. Let $G' = GL(2)$ *act on* X *by*

$$g \cdot p = p(g^{-1}(x, y))$$

So, we concluded that, if p *has linear factors as above, then the linear factors of* $g \cdot p$ *are given by*

$$(a_i, b_i)g^{-1}(x, y)^T$$

We also found that

$$\mu_n = \mathrm{span}\{\mathrm{diag}(\xi_n, \xi_n)\}$$

is the kernel of the G'*-action, where* ξ_n *is a primitive nth root of unity.*

Now, define $G := G'/\mu_n$*. Then* θ *descends to a character of* G *because* θ *is the identity function on* μ_n*. We were able to prove the following lemma:*

Lemma 1. *If* n *is odd, then* $X^{ss}_\theta(G') = X^s_\theta(G') = \{p(x, y)|$ *the linear factors of* p *are distinct* $\}$.

We are currently working on finding other descriptions of the locus $X^{ss}_\theta(G')$*, including descriptions of its nontrivial stabilizers.*

3 An Abelianization Theorem

Lemma 2.

$$\chi_*(G) = \bigcup_{g\in G} \chi_*(gTg^{-1})$$

Proof. It's clear that $\chi_*(T) \subset \chi_*(G)$ for any maximal torus T, as $T \subset G$. The conjugate of a maximal torus is still a maximal torus, so $\bigcup_{g\in G} \chi_*(gTg^{-1}) \subset \chi_*(G)$. Now, suppose that $\lambda \in \chi_*(G)$. By existence of maximal tori for reductive groups, $\mathrm{im}(\lambda)$ (which is abelian, as the domain of λ is abelian) is contained in some maximal torus, say T_λ, so we can identify λ with a character in $\chi_*(T_\lambda)$. As all maximal tori are conjugate, $\exists g \in G$ such that $gTg^{-1} = T_\lambda$, so $\lambda \in \chi_*(T_\lambda) = \chi_*(gTg^{-1})$, so $\lambda \in \bigcup_{g\in G} \chi_*(gTg^{-1})$. Equality follows from here. □

A maximal torus T acts on $\mathbb{A}^n$ with a natural linear action, suggesting that it has a finite dimensional $\mathbb{C}$-representation. In particular, there are finitely many characters $\zeta_1, \ldots, \zeta_n$ of T with indexing subspaces $X_{\zeta_i} \subset X$ such that $X = \bigoplus X_{\zeta_i}$. This is known as a weight space decomposition, where each $X_{\zeta_i} := \{x \in X \mid g \cdot x = \zeta_i(g)x \ \forall g \in G\}$ is a weight space of X with respect to this representation. In particular, this weight space decomposition yields a nice construction of the unstable locus of T, which can be extended to a construction for the general G-action.

Lemma 3. *Let G be a complex reductive group and $T \subset G$ be some fixed maximal torus, with $\mu \in \chi_*(T)$, and $X^{\mu \geq 0} := \bigoplus_{\langle \mu, \zeta_i \rangle \geq 0} X_{\zeta_i}$ with ζ_i as above. Then,*

$$X_\theta^{us}(T) = \bigcup_{\langle \mu, \theta \rangle < 0} X^{\mu \geq 0} \qquad \textit{and} \qquad X_\theta^{us}(G) = \bigcup_{\langle \mu, \theta \rangle < 0} G \cdot X^{\mu \geq 0}$$

Proof. Omitted for brevity, though the result follows once one checks that $X^{\mu \geq 0}$ is precisely the set of all $x \in X$ such that $\lim_{t \to 0} \mu(t) \cdot x$ exists. The first lemma is than used to generalize this argument to computing $X_\theta^{us}(G)$. □

Lemma 4. *Let $\lambda \in \chi_*(G)$ and $\lambda = g\mu(t)g^{-1}$ where $\mu \in \chi_*(T)$ and $g \in G$. Then:*

(a) $\langle \theta, \lambda \rangle = \langle \theta, \mu \rangle$

(b) $g^{-1} \cdot X^{\lambda \geq 0} = X^{\mu \geq 0} \subset G \cdot X^{\lambda \geq 0}$

Proof. (a) This part follows from the fact that $\mathbb{G}_m$ is abelian:

$$\theta \circ \lambda(t) = \theta(g\mu(t)g^{-1}) = \theta(g)\theta(\mu(t))\theta(g^{-1}) = \theta(g)\theta(g^{-1})\theta(\mu(t)) = \theta \circ \mu(t)$$

implying $\langle \theta, \lambda \rangle = \langle \theta, \mu \rangle$.

(b) Recall that $X^{\lambda \geq 0} = \{x \in X \mid \lim_{t \to 0} \lambda(t)x \text{ exists}\}$ (and similarly for μ).

$$\begin{aligned} x \in X^{\mu \geq 0} &\iff \lim_{t \to 0} \mu(t) \cdot x \text{ exists} \iff \lim_{t \to 0} \mu(t)g^{-1}(g \cdot x) \text{ exists} \\ &\overset{(*)}{\iff} \lim_{t \to 0} g \cdot \mu(t)g^{-1}(g \cdot x) \text{ exists} \iff \lim_{t \to 0} \lambda(t)(g \cdot x) \text{ exists} \\ &\iff g \cdot x \in X^{\lambda \geq 0} \iff x \in g^{-1} \cdot X^{\lambda \geq 0} \end{aligned}$$

The only nontrivial step here is (*), but this follows from the fact that g acts on X continuously, so it necessarily preserves limits. It follows that $g^{-1} \cdot X^{\lambda \geq 0} = X^{\mu \geq 0}$, with the final $\subset$ condition immediately following from this equality.

□

Theorem 1. *Let G be a complex reductive group acting on $X = \mathbb{A}^n$, with $T \subset G$ a maximal torus. Let θ be a character of G, which also defines a character on T via restriction. Suppose that $X_\theta^{ss}(G)$ and $X_\theta^{ss}(T)$ are nonempty. Then*

$$X_\theta^{ss}(G) \neq X \iff X_\theta^{ss}(T) \neq X$$

Proof. First, suppose that $X_\theta^{ss}(G) \neq X$, i.e. X has nontrivial unstable locus. We know that

$$X_\theta^{us}(G) = \bigcup_{\langle \lambda, \theta \rangle < 0} G \cdot X^{\lambda \geq 0}$$

suggesting that we can choose $\lambda \in \chi_*(G)$ such that $\langle \lambda, \theta \rangle < 0$ and $G \cdot X^{\lambda \geq 0}$ is nonempty. $\chi_*(G) = \bigcup_{g \in G} \chi_*(gTg^{-1})$, so we can assume that λ is a 1-parameter subgroup over some maximal torus gTg^{-1}. Thus, we can write $\lambda(t) = g\mu(t)g^{-1}$ for $\mu \in \chi_*(T)$. $X^{\lambda \geq 0}$ is precisely the set of $x \in X$ where $\lim_{t \to 0} \lambda(t)x$ exists, and $G \cdot X^{\lambda \geq 0}$ is the set of gx, with x as above, for any $g \in G$. It follows that $X^{\lambda \geq 0}$ is nonempty (as its orbit is nonempty), further implying that $g^{-1} \cdot X^{\lambda \geq 0} \neq \varnothing$, which is just $X^{\mu \geq 0}$ by a previous lemma. As $0 > \langle \theta, \lambda \rangle = \langle \theta, \mu \rangle$, it follows that $\bigcup_{\langle \mu, \theta \rangle < 0} X^{\mu \geq 0} = X_\theta^{us}(T)$ is nonempty. Therefore, T has nontrivial unstable locus, so $X_\theta^{ss}(T) \neq X$.

Now onto the converse, which follows very similarly. Suppose that $X_\theta^{ss}(T) \neq X$. Then $\bigcup_{\langle \mu, \theta \rangle < 0} X^{\mu \geq 0} = X_\theta^{us}(T)$ is nonempty, so we can choose $x \in X^{\mu \geq 0}$ for a $\mu \in \chi_*(T)$ where $\langle \theta, \mu \rangle < 0$. Defining $\lambda(t) = g\mu(t)g^{-1}$, we see that $x \in G \cdot X^{\lambda \geq 0}$ from a previous lemma. as $\langle \theta, \lambda \rangle = \langle \theta, \mu \rangle < 0$, it follows that $x \in \bigcup_{\langle \lambda, \theta \rangle < 0} G \cdot X^{\lambda \geq 0} = X_\theta^{us}(G)$, so $X_\theta^{us}(G)$ is nonempty. Therefore, X has nontrivial G-unstable locus, so $X_\theta^{ss}(G) \neq X$. The result follows. ☐

References

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