

Class 1: September 5th

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Problem 1. Show that each of the following is equivalent to R being Noetherian:

- Every nonempty family of ideals $\{I_\lambda\}_{\lambda \in \Lambda}$ has a maximal element.
- Every ascending chain of finitely generated ideals eventually stabilizes.
- Given any generating set S for an ideal I , the ideal I is generated by a finite subset of S .
- Every ideal of R is finitely generated.

Solution: First we show that (a) \iff (b). Given an ascending chain

$$I_1 \subset I_2 \subset \dots$$

where each $I_j \subset R$ is finitely generated. Note that $\{I_j\}_{j \in \mathbb{N}}$ is a nonempty family of ideals, so by (a) it has a maximal element. Thus, there exists some I_N such that $I_j \subset I_N$ for all j . Since by construction $I_N \subset I_{N+i}$ for any $i \in \mathbb{N}$, it follows that $I_N = I_{N+i}$, so the chain stabilizes, and (a) \implies (b). The converse immediately follows from Zorn's Lemma.

(c) \iff (d) is immediate, so it is sufficient to show that (b) \iff (d). To show the forward case, assume (b), and choose $I = (a_1, a_2, \dots)$. We wish to show that I is indeed finitely generated. Consider the chain of ideals

$$I_1 = (a_1) \subset I_2 = (a_1, a_2) \subset \dots$$

Each of these ideals is clearly finitely generated. By (b), this chain eventually stabilizes at some N . It follows then that $(a_1, \dots, a_N) = I$, so I is finitely generated. Now for the converse, suppose that every ideal is finitely generated. We now wish to show any chain of ideals eventually stabilizes. Suppose we have a chain

$$I_1 \subset I_2 \subset \dots$$

Now consider $\bigcup_{j \in \mathbb{N}} I_j$ for some indexing set \mathcal{I} . We know this is finitely generated, so we let (a_1, \dots, a_n) be its generators. We know that each a_i is contained in some I_j for some j . Choose j_i such that $a_i \in I_{j_i}$, and j_i is minimal (there exists a minimal element because \mathbb{N} is discrete, and bounded below by 1). Let $m = \max(j_1, \dots, j_n)$. I_m thus contains every a_1, \dots, a_n , so $(a_1, \dots, a_n) \subset I_m$. But these are the generators of the union the entire chain, so we can conclude that $(a_1, \dots, a_n) = I_m = \bigcup_{j \in \mathcal{I}} I_j$. It follows that the chain stabilizes at m .

Problem 2.

- a) Show that if R is Noetherian and $I \subseteq R$ is an ideal, then R/I is Noetherian.
- b) Show that if $R/(f)$ is Noetherian for all $f \in R \setminus \{0\}$, then R is Noetherian.

Solution:

- a) Choose $J \subset R/I$ an ideal. We claim that this is finitely generated. Let $\{\bar{a}_i\}_{i \in \mathcal{I}} = \{a_i + I\}_{i \in \mathcal{I}}$ be a list of generators for J , for \mathcal{I} an indexing set. Here, each a_i is contained in R . Consider the ideal $\{a_i\}_{i \in \mathcal{I}}$ in R , which must be finitely generated. Let b_1, \dots, b_n denote its generators. It follows then that $b_1 + I, \dots, b_n + I$ can generate J , so J is finitely generated.
- b) Choose an ideal $I = (a_1, \dots) \subset R$. Let $\pi : R \rightarrow R/(a_1)$ be the standard projection map. $\pi(I) = (\bar{a}_2, \dots)$. But, this is an ideal in $R/(a_1)$, implying that $\pi(I)$ is finitely generated. Let $(\bar{b}_1, \dots, \bar{b}_n)$ denote its generators. It follows then that $I = (a_1, b_1, \dots, b_n)$, so I is finitely generated.

Problem 3. Show that fields and principal ideal domains are Noetherian.

Solution: A field F has only two ideals, $0 = (0)$ and $F = (1)$. Thus any chain of ideals has length at most 2, and stabilizes. Thus, F is Noetherian.

Ideals in PIDs are generated by only 1 element, so they are finitely generated. Since all ideals are finitely generated, PIDs are Noetherian.

Problem 4. Which of the following rings are Noetherian?

- (a) \mathbb{Q}
- (b) \mathbb{Z}
- (c) $\mathbb{R}[x]$
- (d) $\mathcal{C}(\mathbb{R}, \mathbb{R})$, the ring of continuous real-valued functions in one variable.
- (e) $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$, the ring of smooth real-valued functions in one variable.
- (f) $\mathbb{C}\{z\}$, the subring of $\mathbb{C}[[z]]$ consisting of functions holomorphic in a neighborhood of $0 \in \mathbb{C}$.
- (g) The polynomial ring $\mathbb{C}[x_1, x_2, x_3, \dots]$ in countably many variables.

Solution:

- (a) \mathbb{Q} is a field, so it is Noetherian.
- (b) \mathbb{Z} is a PID, so it is Noetherian.
- (c) $\mathbb{R}[x]$ is a PID, so it is Noetherian.
- (d) $\mathcal{C}(\mathbb{R}, \mathbb{R})$ is not Noetherian. Consider the ideals of functions that vanish on $[0, 1]$. This is contained strictly in the ideal of functions that vanish on $[0, 1/2]$. In general, denote I_n the ideal of functions that vanish on $[0, 1/n]$. We have a chain

$$I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_n \subsetneq I_{n+1} \subsetneq \cdots$$

Which does not stabilize.

- (e) $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ is not Noetherian by previous logic.
- (f) $\mathbb{C}\{z\}$ is Noetherian. This is a subring of the power series ring, which we know to be Noetherian.
- (g) The polynomial ring $\mathbb{C}[x_1, x_2, x_3, \dots]$ is not Noetherian, since the ideal (x_1, x_2, \dots) is not finitely generated.

Problem 5*. Show that R is noetherian if and only if every prime ideal of R is finitely generated.

Solution:

Problem 6. If the set of prime ideals in R satisfies ACC, must R be noetherian?

Solution: If prime ideals satisfy the ascending chain condition, then every prime ideal must be finitely generated, by similar logic to the $(b) \Rightarrow (d)$ case in Problem 1. From problem 5, we know that R is Noetherian.

Problem 7. Show that R is a Noetherian ring if and only if R is a Noetherian R -module.

Solution: R -submodules of R are precisely ideals of R . Thus, an ascending chain of submodules stabilizing as submodules is equivalent to the chain stabilizing as ideals. Thus, if every chain of submodules stabilizes, then so does every ascending chain of ideals, and vice-versa.

Problem 8. Convince yourself that essentially the same argument as in (1) shows that each of the following are equivalent to M being Noetherian:

- a) Every nonempty family of submodules $\{M_\lambda\}_{\lambda \in \Lambda}$ has a maximal element.
- b) Every ascending chain of finitely generated submodules eventually stabilizes.
- c) Given any generating set S for a submodule N , the submodule N is generated by a finite subset of S .
- d) Every submodule of M (including M itself!) is finitely generated.

Solution: I am convinced. Replacing every ideal I_j with a submodule M_j in the proof of problem 1 will yield a proof of the above statement.

Problem 9. Let $N \subset M$ be R modules.

- (a) Show that if M is Noetherian, then N and M/N are both Noetherian.
- (b) Use the Lemma below to show that if N and M/N are both Noetherian, then M is Noetherian.

Lemma: Let M be a module, and let $M' \subset M''$ and N all be submodules of M . Then, $M' = M''$ if and only if $M' \cap N = M'' \cap N$ and $M'/(M' \cap N) = M''/(M'' \cap N)$.

Solution:

- (a) Any submodule of N is a submodule of M . Since any submodule of M is finitely generated, any submodule of N is also finitely generated, so N is Noetherian. M/N is Noetherian by similar logic to R/I being Noetherian in problem 2.
- (b) Consider a chain of ideals $M_1 \subset M_2 \subset \dots$ in M . Projecting down to N gives us a sequence $M_1 \cap N \subset M_2 \cap N \subset \dots$, and projecting to M/N gives us a sequence $M_1/(M_1 \cap N) \subset M_2/(M_2 \cap N) \subset \dots$. Since N and M/N are Noetherian, both of these chains stabilize. Suppose they stabilize at m_1, m_2 respectively. Let $m = \max(m_1, m_2)$. It follows that $M_m/(M_m \cap N) \subset M_{m+1}/(M_{m+1} \cap N)$ and $M_m \cap N = M_{m+1} \cap N$. By the lemma, we know that $M_m = M_{m+1}$. Thus, we know that the chain stabilizes in M , so M is Noetherian.

Problem 10. Let R be a Noetherian ring, and M an R -module. Show that M is Noetherian if and only if it is finitely generated. Conclude that in a Noetherian ring, every submodule of a finitely generated module is finitely generated.

Solution: If M is Noetherian, it is finitely generated by Problem 8, so it is sufficient to prove the converse. Let M be a finitely generated module over a Noetherian ring R , and let N be a submodule of M . Suppose N is generated by $\{a_i\}_{i \in \mathcal{I}}$. Since R is Noetherian, it is a Noetherian R -module. Furthermore, Ra_i is a Noetherian R -module for each a_i . We construct the chain

$$R \subset Ra_1 \subset Ra_1 \cup Ra_2 \subset \dots$$

Since these are all finitely generated submodules of M , this sequence stabilizes at some n . It follows that

$$N = \bigcup_{i \in \mathcal{I}} Ra_i = \bigcup_{i \leq n} Ra_i$$

So N is finitely generated, implying that M is Noetherian.

Problem 11*. Find examples of each of the following: a ring R and an R -module M such that

- (a) R and M are both Noetherian.
- (b) R is Noetherian but M is not.
- (c) R is not Noetherian and M is.
- (d) Neither R nor M are Noetherian.

Solution:

- (a) $R = M = \mathbb{Z}$
- (b) $\mathbb{R}[x_1, \dots]$ is an abelian group with respect to addition, and thus is a \mathbb{Z} -module, but is not Noetherian, though \mathbb{Z} is.
- (c) Let R be any non-Noetherian ring, and let $M = \{0\}$, or the 0 module over that ring. This is clearly a Noetherian module that is over a non-Noetherian ring.
- (d) $R = M = \mathbb{R}[x_1, \dots]$

Problem 12. This problem outlines the proof of the Hilbert Basis Theorem (R is Noetherian implies $R[x]$ is Noetherian).

- (a) For a polynomial $f(x) = r_n x^n + \dots + r_0$, with $r_i \in R$ and $r_n \neq 0$, we set $\text{LT}(f) = r_n \in R$. Show that if $I \subset R[x]$ is an ideal, then $\text{LT}(I) := \{\text{LT}(f) \mid f \in I\}$ is a (possibly improper) finitely generated ideal of R .
- (b) Pick $f_1, \dots, f_t \in I$ such that $\text{LT}(I) = (\text{LT}(f_1), \dots, \text{LT}(f_t))$, and set $N = \max_i(\deg(f_i))$. Show that every element $f \in I$ can be written as $f = \sum_i r_i f_i + g$ with $g \in I$ of degree $< N$.
- (c) Show that the set of $g \in I$ of degree $< N$ is a finitely generated R -module.

(d) Finish the proof.

Solution:

(a) Noticing that

$$\text{LT}(f) + \text{LT}(g) = \begin{cases} \text{LT}(f + gx^{\deg(f)-\deg(g)}) & \deg(f) > \deg(g) \\ \text{LT}(g + fx^{\deg(g)-\deg(f)}) & \deg(f) < \deg(g) \\ \text{LT}(f + g) & \deg(f) = \deg(g) \end{cases}$$

and $\text{LT}(f) \cdot \text{LT}(g) = \text{LT}(f \cdot g)$, we see that $\text{LT}(I)$ is clearly an abelian group. To show it is an ideal, choose $r \in R$, $\text{LT}(f) \in \text{LT}(I)$. Suppose that $f = r_n x^n + \dots + r_0$. Since $f \in I$, $rf \in I$, since $r \in R[x]$. Well, $rf = rr_n x^n + \dots + rr_0$, implying that $rr_n \in \text{LT}(I)$. well, $rr_n = r\text{LT}(f)$, so we can conclude that $\text{LT}(I)$ is an ideal. Since we are assuming that R is Noetherian, $\text{LT}(I)$ is finitely generated.

(b) Without loss of generality, let f be a monomial. If the claim holds for any monomial, it will easily hold for any sum of monomials, and any $f \in R[x]$ is a sum of monomials. Furthermore, we suppose that f is of degree $> N$, as the result immediately follows otherwise.

Now, let $I_t = \{r \in \text{LT}(I) \mid r \text{ is the leading term of a polynomial of degree } t\}$. Notice that $I_1 \subset I_2 \subset \dots$ since the leading term of a polynomial is the same as the leading term of x times the polynomial. This chain eventually halts by Noetherianity of R . Without loss of generality, it must halt before I_N . It follows that for any polynomial $f \in I$, we can subtract away linear combinations of the f_i to get a polynomial of degree $< N$, implying that $f - \sum_i r_i f_i = g$. Adding $\sum_i r_i f_i$ to both sides allows us to conclude.

(c) This is generated by a finite set of linear combinations of x^1, \dots, x^N as an R -module.

(d) We've just shown that every $f \in I \subset R[x]$ can be written as a linear combination of finitely many polynomials. All but 1 are pre-determined by I , (In particular, the f_i) and the last is also finitely generated by elements of $R[x]$, implying that I itself is finitely generated. Since every ideal of $R[x]$ is finitely generated, $R[x]$ is Noetherian.

Problem 13. Prove the corollary: If R is Noetherian, then every finitely generated R -algebra is Noetherian. In particular, finitely generated algebras over fields are Noetherian.

Solution: If A is a finitely generated R -algebra, then there exists $a_1, \dots, a_n \in R$ such that $A = R[a_1, \dots, a_n]$. By Hilbert's Basis Theorem, R being Noetherian implies that $R[a_1]$, and thus $R[a_1, a_2]$, and thus $R[a_1, \dots, a_n]$ is Noetherian. Thus, A is Noetherian.

Math 614: Commutative Algebra

IBL Lab Notebook

Class 2: September 10th

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Problem 2. The Zariski Topology.

- (a) Prove that $\mathbb{V}(S) \cap \mathbb{V}(S') = \mathbb{V}(S \cup S')$.
- (b) Prove that $\mathbb{V}(S) \cup \mathbb{V}(S') = \mathbb{V}(SS')$, where SS' denotes the subset $\{fg \mid f \in S, g \in S'\}$.
- (c) Prove that $\text{Spec}(R)$ has the structure of a topological space, whose closed sets are those of the form $\mathbb{V}(S)$ for some subset S . This is the Zariski Topology on $\text{Spec}(R)$.

Solution:

(a)

$$P \in \mathbb{V}(S) \cap \mathbb{V}(S') \iff S, S' \subset P \iff S \cup S' \subset P \iff P \in \mathbb{V}(S \cup S')$$

(b)

$$P \in \mathbb{V}(SS') \iff SS' \subset P \iff S \subset P \text{ or } S' \subset P \iff P \in \mathbb{V}(S) \cup \mathbb{V}(S')$$

The claim $SS' \subset P \iff S \subset P \text{ or } S' \subset P$ follows from P being prime.

- (c) Before we continue, we prove the following Lemma. If $I \subset J$, then $\mathbb{V}(I) \supset \mathbb{V}(J)$. Choose $P \in \mathbb{V}(J)$. Then $J \subset P$, implying that $I \subset J \subset P$, so $P \in \mathbb{V}(I)$.

From this lemma, we can see that $\mathbb{V}(R) = \emptyset, \mathbb{V}(\emptyset) = R$. Thus, these sets are closed. We have closure under arbitrary intersections:

$$\bigcap_{j \in J} \mathbb{V}(S_j) = \mathbb{V} \left(\bigcup_{j \in J} S_j \right)$$

and under finite unions:

$$\bigcup_{i=1}^n \mathbb{V}(S_i) = \mathbb{V} \left(\prod_{i=1}^n S_i \right)$$

Since the arbitrary union of ideals is still an ideal, and the finite product of ideals is still an ideal. Thus, we can conclude that this forms a topology on $\text{Spec}(R)$.

Problem 4. Recall: the **Radical** of an ideal I of R is the ideal $\sqrt{I} := \{f \in R \mid f^n \in I \text{ for some } n \in \mathbb{N}\}$.

- (a) Show that $\mathbb{V}(S) = \mathbb{V}(\langle S \rangle)$.
- (b) Prove that $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$ for any ideal $I \subset R$.

Solution:

- (a) Since $S \subset \langle S \rangle$, by the lemma we have that $\mathbb{V}(\langle S \rangle) \subset \mathbb{V}(S)$. Now, we choose $P \in \mathbb{V}(S)$. Thus, $S \subset P$. Since P is an ideal, it follows that if any elements are in P , then their products and sums are in P . Thus, $\langle S \rangle \subset P$, so $P \in \mathbb{V}(\langle S \rangle)$.
- (b) $I \subset \sqrt{I}$, so $\mathbb{V}(\sqrt{I}) \subset \mathbb{V}(I)$. For the converse, choose $P \in \mathbb{V}(I)$. Thus, $I \subset P$. We wish to show that $\sqrt{I} \subset P$. Choose $f \in \sqrt{I}$. $f^n \in I \subset P$ for some n . Well, since P is prime, and because $f^n = f^{n-1}f$, either $f \in P$ or $f^{n-1} \in P$. If $f^{n-1} \in P$, repeat the process with f, f^{n-2} . It follows from this logic that $f \in P$, so $\sqrt{I} \subset P$, so $P \in \mathbb{V}(\sqrt{I})$, so $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$.

Problem 6. Let $\varphi : R \rightarrow S$ be a ring homomorphism.

- (a) Prove that if $P \in \text{Spec}(S)$, then $\varphi^{-1}(P) \in \text{Spec}(R)$.
- (b) Prove that the map

$$\text{Spec}(S) \xrightarrow{\varphi^*} \text{Spec}(R) \quad P \mapsto \varphi^{-1}(P)$$

is continuous in the Zariski Topology.

Solution:

- (a) First, we verify that it is an abelian group. $0 \in \varphi^{-1}(P)$, and if $a, b \in \varphi^{-1}(P)$, then $\varphi(a), \varphi(b) \in P$, so $\varphi(a) + \varphi(b) = \varphi(a + b) \in P$, so $a + b \in \varphi^{-1}(P)$. Similar logic can be used to show that $\varphi^{-1}(P)$ is in fact an ideal.

We now show that this is in fact prime. Choose $ab \in \varphi^{-1}(P)$. This means that $\varphi(ab) = \varphi(a)\varphi(b) \in P$. Since P is prime, either $\varphi(a) \in P$ or $\varphi(b) \in P$, implying that a or b is contained in $\varphi^{-1}(P)$. This implies that $\varphi^{-1}(P)$ is prime.

- (b) We need to verify that the inverse image of a closed set is closed. Choose $\mathbb{V}(X) \subset \text{Spec}(R)$. We claim that $(\varphi^*)^{-1}(\mathbb{V}(X)) = \mathbb{V}(\varphi(X))$. Choose $P \in \mathbb{V}(S)$. This means that $X \subset P$, so $\varphi(X) \subset \varphi(P)$. Thus, $\varphi(P) \in \mathbb{V}(\varphi(X))$, and $(\varphi^*)^{-1}(\mathbb{V}(X)) \subset \mathbb{V}(\varphi(X))$. Now choose $Q \in \mathbb{V}(\varphi(X))$. This implies that $\varphi(X) \subset Q$, so $X \subset \varphi^{-1}(Q)$. Thus, $\varphi^{-1}(Q) \in \mathbb{V}(X)$, implying that Q is in the preimage of the induced map from Spec . In other words, $Q \in (\varphi^*)^{-1}(\mathbb{V}(X))$. Thus, $(\varphi^*)^{-1}(\mathbb{V}(X)) = \mathbb{V}(\varphi(X))$, and the inverse image of a closed set is closed, implying that φ^* is continuous.

Problem 7. The Functor Spec. Consider the mapping

$$\text{CommRing} \rightarrow \text{Top}$$

where $R \mapsto \text{Spec}(R)$.

- Explain using (6) how this mapping defines a contravariant functor from $\text{CommRing} \rightarrow \text{Top}$. This is the $\text{Spec}(-)$ functor.
- Do isomorphic rings have Spectra that are homeomorphic topological spaces? Explain.
- If two rings have homeomorphic spectra, are they necessarily isomorphic rings? Explain.

Solution:

- Recall that morphisms in CommRing are ring homomorphisms, and morphisms in Top are continuous maps. It follows that $\text{Spec}(-)$ takes objects to objects (commutative rings to topological spaces) and morphisms to morphisms. As for contravariance, observe that for $\varphi : R \rightarrow S$, $\text{Spec}(\varphi) : \text{Spec}(S) \rightarrow \text{Spec}(R)$. Implying that if $\varphi : R \rightarrow S$, $\psi : S \rightarrow T$, then $\text{Spec}(\psi \circ \varphi) = \text{Spec}(\varphi) \circ \text{Spec}(\psi)$.
- Yes. If $R \cong S$, there exist maps $\varphi : R \rightarrow S$, $\varphi^{-1} : S \rightarrow R$. The image of both these maps under $\text{Spec}(-)$ are continuous, and they remain inverses. This establishes two mutually inverse maps between $\text{Spec}(R)$ and $\text{Spec}(S)$, so $\text{Spec}(R)$ is homeomorphic to $\text{Spec}(S)$.
- No. Both \mathbb{Q} and \mathbb{R} have the same Spectra (single points) but are clearly not isomorphic as rings.

Problem 11. For $f \in R$, define $D(f) := \{P \in \text{Spec}(R) \mid f \notin P\}$.

- Prove that $D(f)$ is an open set in $\text{Spec}(S)$.
- Prove that open sets of the form $D(f)$ form a basis for the Zariski topology on $\text{Spec}(R)$.
- For any ideal $I \subset R$, prove that $\mathbb{V}(I) = \emptyset$ if and only if $1 \in I$.

Solution:

- $D(f) = \text{Spec}(R) \setminus \mathbb{V}(f)$.
- For $P \in \text{Spec}(R)$, let $P = (f_1, f_2, \dots)$.

$$\mathbb{V}(P) = \mathbb{V}(f_1, f_2, \dots) = \mathbb{V}(f_1) \cap \mathbb{V}(f_2) \cap \dots$$

Implying that for a open set of the form $\text{Spec}(R) \setminus \mathbb{V}(P)$,

$$\begin{aligned}\text{Spec}(R) \setminus \mathbb{V}(P) &= \text{Spec}(R) \setminus (\mathbb{V}(f_1) \cap \mathbb{V}(f_2) \cap \dots) \\ &= \text{Spec}(R) \setminus \mathbb{V}(f_1) \cup \text{Spec}(R) \setminus \mathbb{V}(f_2) \cup \dots \\ &= D(f_1) \cup D(f_2) \cup \dots\end{aligned}$$

(c) It follows from the lemma that $V(I) = \emptyset$ if and only if $I = R$.

Math 614: Commutative Algebra

IBL Lab Notebook

Class 3: September 12th

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Problem 1. For a point $p = (a_1, \dots, a_n) \in K^n$, consider the "evaluation at p " map $e_p : K[x_1, \dots, x_n] \rightarrow K$ sending $f \mapsto f(p)$.

- Show that e_p is a ring homomorphism and find generators for its kernel, m_p .
- Show that evaluation gives a bijection between K^n and the subset of $\text{Spec}(K[x_1, \dots, x_n])$ consisting of maximal ideals whose residue field is isomorphic to K .
- Find a maximal ideal of $\mathbb{R}[x, y]$ not of the form m_p for any $p \in \mathbb{R}^2$.
- Use Hilbert's Nullstellensatz (below) to describe the points of $\text{MaxSpec}(\mathbb{C}[x_1, \dots, x_n]/I)$.

Hilbert's Nullstellensatz: If K is algebraically closed, then every maximal ideal of $K[x_1, \dots, x_n]$ is of the form m_p for some $p \in K^n$.

Solution:

(a)

$$\begin{aligned} e_p(f + g) &= (f + g)(p) = f(p) + g(p) = e_p(f) + e_p(g) \\ e_p(fg) &= (fg)(p) = f(p)g(p) = e_p(f)e_p(g) \end{aligned}$$

$\ker(e_p)$ consists of all $f \in K[x_1, \dots, x_n]$ such that $f(p) = 0$. This is generated by the polynomials $(x_1 - a_1, x_2 - a_2, \dots, x_n - a_n)$. Thus,

$$\ker(e_p) = m_p = (x_1 - a_1, x_2 - a_2, \dots, x_n - a_n)$$

(b) First, observe that e_p is surjective, since for any $k \in K$, $e_p(k) = k$. Thus, by the first isomorphism theorem:

$$K[x_1, \dots, x_n]/(x_1 - a_1, \dots, x_n - a_n) \cong K$$

Implying that $(x_1 - a_1, \dots, x_n - a_n)$ is maximal. It follows that $e_{(-)}$ maps a point to such an ideal $(x_1 - a_1, \dots, x_n - a_n)$, and such an ideal is uniquely defined by a point $p = (a_1, \dots, a_n)$.

- $I = (x^2 + 1, y^2 + 1)$. Both of these are irreducible in $\mathbb{R}[x, y]$, and thus generate a prime ideal, and is clearly maximal. However, it is not of the form m_p .
- $\text{MaxSpec}(\mathbb{C}[x_1, \dots, x_n])$ consists entirely of ideals of the form m_p by the Nullstellensatz. Since $\text{Spec}(R/I)$ is in direct bijection with the subset of $\text{Spec}(R)$ containing the prime ideals which contain I , it follows that $\text{MaxSpec}(\mathbb{C}[x_1, \dots, x_n]/I)$ contains all ideals of the form m_p such that $I \subset m_p$.

Problem 2. Algebraic Sets. Let S be any collection of polynomials in $K[x_1, \dots, x_n]$. Define the corresponding **algebraic set** in K^n as

$$\mathbb{V}(S) := \{p \in K^n \mid f(p) = 0 \forall f \in S\}$$

- (a) Prove that \mathbb{V} is order reversing: if $S \subset S'$, then $\mathbb{V}(S') \subset \mathbb{V}(S)$.
- (b) Show that $\mathbb{V}(S) = \mathbb{V}(\langle S \rangle) = \mathbb{V}(\sqrt{\langle S \rangle})$.
- (c) Prove that every algebraic set can be defined by finitely many polynomials.
- (d) Describe each algebraic set in the cases below:
- (1) $S = \{x^2 - y\} \subset \mathbb{R}[x, y]$.
 - (2) $S = \{n(x^2 + y^2 - z^2) \mid n \in \mathbb{N}\} \subset \mathbb{R}[x, y, z]$.
 - (3) For K arbitrary, S any collection of linear polynomials in n variables.
 - (4) $S = \{x^2 + 1\} \subset \mathbb{R}[x]$.
 - (5) $S = \{x^2 + 1\} \subset \mathbb{C}[x]$.
 - (6) $S = \langle f \rangle$, where f is an arbitrary non-zero, non-constant polynomial in $\mathbb{C}[x]$.
 - (7) K algebraically closed, $S \subset K[x]$ arbitrary (three cases).
- (e) Explain why $\text{SL}_n(K)$ has the structure of an algebraic set.
- (f) Explain why the set of $m \times n$ matrices of rank $< t$ is an algebraic set (contained in $K^{m \times n}$).

Solution:

- (a) Choose $p \in \mathbb{V}(S')$. This means that $f(p) = 0 \forall f \in S'$. Since $S \subset S'$, p vanishes on any $f \in S \subset S'$, so $p \in \mathbb{V}(S)$.
- (b) Since $S \subset \langle S \rangle \subset \sqrt{\langle S \rangle}$, $\mathbb{V}(\sqrt{\langle S \rangle}) \subset \mathbb{V}(\langle S \rangle) \subset \mathbb{V}(S)$. Thus, it is sufficient to verify inclusions in the opposite direction. Suppose $p \in \mathbb{V}(S)$. Then $f(p) = 0 \forall f \in S$. This implies that for all $f, g \in S$, $fg(p) = f(p)g(p) = 0$. It follows that $p \in \mathbb{V}(\langle S \rangle)$. Furthermore, if $f^n(p) = 0$, then $f(p) = 0$. It follows that $p \in \mathbb{V}(\sqrt{\langle S \rangle})$. Therefore, $\mathbb{V}(S) = \mathbb{V}(\langle S \rangle) = \mathbb{V}(\sqrt{\langle S \rangle})$.
- (c) By Hilbert's Basis Theorem, $K[x_1, \dots, x_n]$ is Noetherian. Thus, for any ideal $I \subset K[x_1, \dots, x_n]$, $I = (f_1, \dots, f_n)$. Since any algebraic set is of the form $\mathbb{V}(I)$, it follows that
- $$\mathbb{V}(I) = \mathbb{V}((f_1, \dots, f_n)) = \mathbb{V}(f_1, \dots, f_n)$$
- Thus, $\mathbb{V}(I)$ is defined by finitely many elements, namely f_1, \dots, f_n .
- (d) (1) Parabola for $y = x^2$ in \mathbb{R}^2 .

- (2) Cone around z -axis, with point at $(0, 0, 0)$ in \mathbb{R}^3 . To see why, when z is fixed, the zero set corresponds to a circle of radius z . When z varies, the radius of the circle varies, yielding a cone shape.
- (3) A single point.
- (4) \emptyset .
- (5) $x = i, -i$.
- (6) If f is irreducible (of the form $x - a$) then $\mathbb{V}(S)$ is precisely a point. If f is not irreducible, since \mathbb{C} is algebraically closed, Factor f into $(x - a_1) \dots (x - a_n)$. Then $\mathbb{V}(S)$ contains precisely the points a_1, \dots, a_n .
- (7) Since $K[x]$ is a PID, we can reduce to the above case if f is nonzero and nonconstant. If f is zero, then $\mathbb{V}(S) = K[x]$. If f is constant (and nonzero), then $\mathbb{V}(S) = \emptyset$.
- (e) $\text{SL}_n(K)$ contains all matrices A where $\text{Det}(A) = 1$. $\text{Det}(A)$ is a polynomial of A , so it follows that $\text{SL}_n(K) = \mathbb{V}(\text{Det}(A) - 1)$ for $A \in K^{n \times n}$.
- (f) A matrix is of rank $< t$ if the determinant of every $t \times t$ minor is 0. For a given matrix A , the determinant of a fixed $t \times t$ minor is a polynomial of A . Thus, this set is the vanishing set of all such polynomials, varying over each $t \times t$ minor.

Problem 3. Let $V \subset K^n$ be an algebraic set. Define an ideal in $K[x_1, \dots, x_n]$ by

$$\mathbb{I}(V) := \{f \in K[x_1, \dots, x_n] \mid f(p) = 0 \forall p \in V\}$$

- (a) Show that $\mathbb{I}(V)$ is a radical ideal in $K[x_1, \dots, x_n]$.
- (b) For $V = \mathbb{V}(S)$, show that $\langle S \rangle \subset \sqrt{\langle S \rangle} \subset \mathbb{I}(V)$.
- (c) For $V = \mathbb{V}(S)$, show that $\mathbb{V}(S) = \mathbb{V}(\mathbb{I}(V))$.
- (d) Give an example (when $K = \mathbb{R}$, say) to show that $\sqrt{\langle S \rangle} \subset \mathbb{I}(V)$ can be proper.
- (e) When K is algebraically closed, use the Nullstellensatz (below) to find explicit bijections between algebraic sets in K^n and radical ideals in $K[x_1, \dots, x_n]$.

Hilbert's Nullstellensatz: If K is algebraically closed, then for all algebraic sets $V \subset K^n$, $\mathbb{I}(V) = \sqrt{\langle S \rangle}$ as ideal in $K[x_1, \dots, x_n]$.

Solution:

- (a) It is sufficient to verify that if $f^n \in \mathbb{I}(V)$, then $f \in \mathbb{I}(V)$.

$$f^n \in \mathbb{I}(V) \Rightarrow f^n(p) = 0 \forall p \in V \Rightarrow f(p) = 0 \forall p \in V \Rightarrow f \in \mathbb{I}(V)$$

- (b) We know that $\langle S \rangle \subset \sqrt{\langle S \rangle}$, so it is sufficient to show that $\sqrt{\langle S \rangle} \subset \mathbb{I}(V)$. Choose $f^n \in \sqrt{\langle S \rangle}$, where $f \in \langle S \rangle$. $f^n(p) = 0 \forall p \in \mathbb{V}(\langle S \rangle) = \mathbb{V}(S)$, since $f(p) = 0 \forall p \in \mathbb{V}(\langle S \rangle) = \mathbb{V}(S)$, since $f \in \langle S \rangle$. So, $f^n \in \mathbb{I}(V)$.
- (c) Since $S \subset \langle S \rangle \subset \mathbb{I}(V)$, $\mathbb{V}(\mathbb{I}(V)) \subset \mathbb{V}(S)$. To conclude, we need to verify that $\mathbb{V}(S) \subset \mathbb{V}(\mathbb{I}(V))$. Choose $p \in \mathbb{V}(S)$. This implies that $f(p) = 0 \forall f \in \langle S \rangle$. We want to show that $f(p) = 0 \forall f \in \mathbb{I}(V)$. This is equivalent to showing that $f(p) = 0 \forall f$ such that $f(p') = 0$ for $p' \in \mathbb{V}(S)$. Well, $p \in \mathbb{V}(S)$, so $f(p) = 0$, so $f \in \mathbb{V}(\mathbb{I}(V))$ implying that $\mathbb{V}(S) = \mathbb{V}(\mathbb{I}(V))$.
- (d) Let $S = (x^2+1)$. $\mathbb{V}(S) = \emptyset$, implying that $\mathbb{I}(\mathbb{V}(S)) = \mathbb{R}[x]$. Since $\sqrt{(x^2+1)} \subsetneq \mathbb{R}[x]$, the result follows.
- (e) If an ideal is radical, $S = \sqrt{S}$. It then follows that the maps $\mathbb{V} : K[x_1, \dots, x_n] \rightarrow K^n$ where $\mathbb{V} : S \mapsto \mathbb{V}(S)$ and $\mathbb{I} : K^n \rightarrow K[x_1, \dots, x_n]$ where $\mathbb{I} : X \mapsto \mathbb{I}(X)$ are mutually inverse, as $\mathbb{I}(\mathbb{V}(S)) = S$. Similarly, for any algebraic set $X \subset K^n$, X is of the form $\mathbb{V}(S)$. We know that $\mathbb{V}(\mathbb{I}(\mathbb{V}(S))) = \mathbb{V}(S)$, implying that $\mathbb{V}(\mathbb{I}(X)) = X$. Thus, $\mathbb{V} \circ \mathbb{I} = \text{Id}$ and $\mathbb{I} \circ \mathbb{V} = \text{Id}$. Therefore, we have a bijection from K^n to $K[x_1, \dots, x_n]$.

Problem 4. Let $V \subset K^n$ be an algebraic set. Let \mathcal{R} be the ring of all K -valued functions on the set V . Consider the restriction map $\rho : K[x_1, \dots, x_n] \rightarrow \mathcal{R}$ sending $f \mapsto f|_V$.

- (a) Verify that ρ is a ring homomorphism and compute its kernel. The image is called the **coordinate ring** of V , and is denoted $K[V]$.
- (b) Prove that \mathcal{R} , and thus $K[V]$, is reduced.
- (c) Give a K -algebra presentation for $K[V]$. Explain why $K[V]$ is called the coordinate ring. [Note: the i th coordinate function on K^n is the function x_i which takes a point $p = (a_1, \dots, a_n) \in V$ to its i th coordinate a_i .]
- (d) Describe the coordinate ring for the following algebraic sets:
- (1) $\mathbb{V}(y - x^2) \subset K^2$
 - (2) $\mathbb{V}(y^2, xy, y^{17}) \subset K^2$
 - (3) (*) The set of 3×3 matrices over \mathbb{F}_5 of determinant 1 and trace 0.

Solution:

- (a) ρ is a ring homomorphism, since $f|_V + g|_V = (f + g)|_V$, and $f|_V g|_V = (fg)|_V$, and because $1|_V$ is still the multiplicative identity of \mathcal{R} . The kernel consists of polynomials f such that $f|_V = 0$. In other words, $f(p) = 0 \forall p \in V$. Thus, $\ker(\rho) = \mathbb{I}(V)$. This implies (via the first isomorphism theorem) that

$$K[V] \cong K[x_1, \dots, x_n]/\mathbb{I}(V)$$

- (b) Choose $f \in \mathcal{R}$ such that $f^n = 0$. We wish to show that $f = 0$. Choose $p \in V$. $f^n(p) = 0 \in K$. This implies that $f(p) = 0$, since K is a field with no nilpotents. Well, p was chosen arbitrarily, so $f = 0$. Thus, \mathcal{R} , and therefore $K[V]$, is reduced.
- (c) The presentation of $K[V]$ is $\mathbb{I}(V)$. $K[V]$ is defined to be the K -algebra (ring) generated by all of the "coordinate functions" restricted to V , and is thus called the coordinate ring of V .
- (d) (1) $K[x, y]/(y - x^2)$
 (2) $K[x, y]/(x^2, xy, y^{17}) \cong K[x]/(x^2) \oplus K[y]/(y^{17})$

Problem 6. Fix an algebraic set V in K^n . Define the Zariski topology on V by letting closed sets be sub-algebraic sets.

- (a) Make sure you can prove this really defines a topology.
- (b) Show that $D(f) = \{p \in V \mid f(p) \neq 0\} \subset V$ over each $f \in K[V]$ form a basis.
- (c) Show that the map $V \rightarrow \text{MaxSpec}(K[V])$ sending a point $p \mapsto m_p$ is a homeomorphism.
- (d) The Zariski Topology on V can be identified with the subspace of closed points in $\text{Spec}(K[V])$. Why?
- (e) Is the Zariski Topology Hausdorff?

Solution:

- (a) This is a topology, as it contains the whole space and the empty set ($\mathbb{V}(k[x_1, \dots, x_n]) = \emptyset, \mathbb{V}(0) = K^n$), and is closed under finite union and arbitrary intersection.
- (b) This is similar to the previous problem set. Choose an open set $K^n \setminus \mathbb{V}(I) = K^n \setminus \mathbb{V}(f_1, \dots, f_n)$.
- $$\begin{aligned} K^n \setminus \mathbb{V}(I) &= K^n \setminus (\mathbb{V}(f_1) \cap \mathbb{V}(f_2) \cap \dots \cap \mathbb{V}(f_n)) \\ &= K^n \setminus \mathbb{V}(f_1) \cup K^n \setminus \mathbb{V}(f_2) \cup \dots \cup K^n \setminus \mathbb{V}(f_n) \\ &= D(f_1) \cup D(f_2) \cup \dots \cup D(f_n) \end{aligned}$$
- (c) We just need to verify that the map and its inverse are well defined and continuous. It is sufficient to verify this on basis elements. Choose $D(f) \subset V$. This consists of all the points which are nonzero on f . This would map to the set of ideals in $\text{MaxSpec}(K[V])$ which do not contain f , which is an open set, and the inverse behaves similarly. To see why, notice the image of of the form
- $$\{(x_1 - a_1, \dots, x_n - a_n) \in \text{MaxSpec}(K[V]) \mid f(a_1, \dots, a_n) \neq 0\}$$
- $f(a_1, \dots, a_n) = 0$ if and only if $f \in (x_1 - a_1, \dots, x_n - a_n)$ so the claim above follows.
- (d) From part (c), we know $\text{MaxSpec}(K[V]) \cong V$, where $\text{MaxSpec}(K[V])$ is precisely the set of closed points in $\text{Spec}(K[V])$.
- (e) No.

Math 614: Commutative Algebra**IBL Lab Notebook**

Class 4: September 17th

Name: *Vignesh Jagathese*Collaborators: *Noah Luntzlara and Will Garland***Problem 1. RECALL FROM LAST TIME:**

- How is each of these maps is defined?
- Points are special cases of algebraic sets V in K^n . Describe $\mathbb{I}(V)$ in this case, both intrinsically (without giving generators) and explicitly (by describing generators).
- $\langle 0 \rangle$ and $\langle 1 \rangle$ are special cases of ideals I in $K[x_1, \dots, x_n]$. Describe $\mathbb{V}(I) \subset K^n$ in these cases.
- The ideals $\langle x, y \rangle$ and $\langle x^2 + y^2 \rangle$ are special cases of ideals I in $\mathbb{R}[x, y]$. Describe $\mathbb{V}(I) \subset \mathbb{R}^2$ in these cases.
- The ideals $\langle 0 \rangle$ and $\langle x^p - x, xy^p - x^p y \rangle$ are special cases of radical ideals I in $\mathbb{F}_p[x, y]$. Describe $\mathbb{V}(I) \subset \mathbb{F}_p^2$ in these cases.

Solution:

- $I \mapsto \mathbb{V}(I)$, where $\mathbb{V}(I) = \{p \in K^n \mid f(p) = 0 \forall f \in I\}$. $V \mapsto \mathbb{I}(V)$, where $\mathbb{I}(V) = \{f \in K[x_1, \dots, x_n] \mid f(p) = 0 \forall p \in V\}$.
- $\mathbb{I}(V)$ where V is a single point is a maximal ideal of $K[x_1, \dots, x_n]$. For a given point (a_1, \dots, a_n) , $\mathbb{I}(V) = (x_1 - a_1, \dots, x_n - a_n)$.
- $\mathbb{V}(0) = K^n$, $\mathbb{V}(1) = \emptyset$.
- (x, y) is a maximal ideal in $\mathbb{R}[x, y]$, and corresponds to the point $(0, 0)$. $\mathbb{V}(x^2 + y^2) = (0, 0)$ as well. Since \mathbb{R} is not algebraically closed, these radical ideals correspond to the same algebraic set.
- $x^p = x$ for any $x \in \mathbb{F}_p$. It follows that $\mathbb{V}(x^p - x, xy^p - x^p y) = K^n$.

Problem 2. Consider the map $\overline{\mathbb{F}}_{17}^{-2} \rightarrow \overline{\mathbb{F}}_{17}^{-3}$ sending $(u, v) \mapsto (u^2, uv, v^2)$. Prove that the image is an algebraic set in $\overline{\mathbb{F}}_{17}^{-3}$. Find the corresponding radical ideal in $\overline{\mathbb{F}}_{17}[x, y, z]$.

Solution: This corresponds to $\mathbb{V}(y^2 - 2x)$.

Problem 3. REDUCTION TO THE CORE DIFFICULTY.

- (a) Last time you showed that the composition of \mathbb{I} and \mathbb{V} in some order is the identity, whether or not $K = \overline{K}$. Which order?
- (b) Also recall your proof that, whether or not $K = \overline{K}$, for any ideal $I \subset K[x_1, \dots, x_n]$, $I \subset \sqrt{I} \subset \mathbb{I}(\mathbb{V}(I))$.
- (c) When $K = \overline{K}$, conclude that to prove the Nullstellensatz, it suffices to show that for all ideals I ,

$$\mathbb{I}(\mathbb{V}(I)) \subset \sqrt{I}.$$

Solution:

- (a) We showed that, for any algebraic set X , $\mathbb{V}(\mathbb{I}(X)) = X$.
- (b) It has been recalled.
- (c) If the above holds, then we would know that $\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}$. Replacing I with $\langle S \rangle$, and $\mathbb{V}(S)$ with V yields the statement of the Nullstellensatz in the previous worksheet.

Problem 4. THE CASE WHERE I IS MAXIMAL. We need to show that $\mathbb{I}(\mathbb{V}(m)) \subset m$.

- (a) Let $m \in K[x_1, \dots, x_n]$ be a maximal ideal. Show that $K[x_1, \dots, x_n]/m$ has countable dimension over K .
- (b) Let K be an arbitrary field, and let t be a *transcendental* element of some extension L . Prove that the elements $\{\frac{1}{t-\lambda}\}_{\lambda \in K} \subset L$ are linearly independent over K .
- (c) Assume K is uncountable. With notation as in (b), show that $[L : K]$ is uncountable.
- (d) Let K be an uncountable field. Let $m \in K[x_1, \dots, x_n]$ be a maximal ideal. Show that $K \hookrightarrow K[x_1, \dots, x_n]/m$ is an algebraic extension.
- (e) Prove that if K is an uncountable algebraically closed field, then every maximal ideal has the form $\langle x_1 - a_1, \dots, x_n - a_n \rangle$, where $a_i \in K$.
- (f) Deduce that when $K = \mathbb{C}$, the maps \mathbb{V} and \mathbb{I} define inverse bijections between points \mathbb{C}^n and maximal ideals of $\mathbb{C}[x_1, \dots, x_n]$. (Call this the **Nullstellensatz for Points**.)

Solution:

- (a) $K[x_1, \dots, x_n]$ has countable dimension as a vector space over K , as does $m \subset K[x_1, \dots, x_n]$. This implies that the quotient $K[x_1, \dots, x_n]/m$ has countable dimension. Since this is a field, we can conclude that it has countable dimension as a field extension of K .

- (b) Suppose that $\{\frac{1}{t-\lambda}\}_{\lambda \in K}$ is linearly dependent. Then, there would exist distinct $\lambda_1, \dots, \lambda_n \in K$ and $k_1, \dots, k_n \in K$ (where not all $k_i = 0$) such that

$$\sum_{i=1}^n k_i \frac{1}{t - \lambda_i} = 0$$

Clearing denominators by multiplying by $\prod_{i=1}^n (t - \lambda_i)$, we get

$$0 = \left(\sum_{i=1}^n k_i \frac{1}{t - \lambda_i} \right) \prod_{i=1}^n (t - \lambda_i) = \sum_{i=1}^n k_i \prod_{i \neq j} (t - \lambda_j)$$

If we verify that if there exists $i \leq n$ such that $f_i(t) := k_i \prod_{i \neq j} (t - \lambda_j)$ is nonzero, as then it follows that t satisfies a polynomial with coefficients in K , violating the transcendence of t , yielding a contradiction. Thus, it is sufficient to verify that $\exists i \leq n$ such that $f_i(x) \neq 0$. $\forall x \in K$.

Notice that $f_i(\lambda_k) = k_i \prod_{i \neq j} (\lambda_j - \lambda_k)$. If $k \neq i$, $f_i(\lambda_k) = 0$. If $i = k$, then $\prod_{i \neq j} (\lambda_j - \lambda_k) \neq 0$. Thus, if $f_i = 0$, then $k_i = 0$. Thus, if all $f_i = 0$, then all $k_i = 0$, violating our linear dependence condition above. Thus, there must be a nonzero f_i term, so we can conclude.

- (c) Since K is uncountable, $\{\frac{1}{t-\lambda}\}_{\lambda \in K}$ is an uncountable set. By (b) this set is linearly independent, and we know this is contained in L . Since L contains an uncountable list of linearly independent elements over K , it must be the case that $[L : K]$ is uncountable.
- (d) If $K[x_1, \dots, x_n]/m$ was not an algebraic extension, then not every element in $K[x_1, \dots, x_n]/m$ will be algebraic. Then we could choose $t \in K[x_1, \dots, x_n]/m$ such that t is transcendental, as in (b). Then by (c), $K[x_1, \dots, x_n]/m$ would have uncountable dimension over K , a contradiction to (a). Thus, $K[x_1, \dots, x_n]/m$ is an algebraic extension of K .
- (e) If K is algebraically closed, then all algebraic extensions are trivial. In particular, $K[x_1, \dots, x_n]/m \cong K$. Thus, m must necessarily be of the above form.
- (f) We've already proven that \mathbb{V}, \mathbb{I} bijects from points in \mathbb{C}^n to ideals in $\mathbb{C}[x_1, \dots, x_n]$ of the form $(x_1 - a_1, \dots, x_n - a_n)$. Since we've just shown all maximal ideals are of this form (and naturally, all ideals of this form are maximal), then \mathbb{V}, \mathbb{I} define inverse bijections between points in \mathbb{C}^n and maximal ideals of $\mathbb{C}[x_1, \dots, x_n]$.

Problem 5. THE CASE OF THE UNIT IDEAL. Let K be algebraically closed, and let $I \subset K[x_1, \dots, x_n]$ be an ideal. Assuming the Nullstellensatz for points, show that $\mathbb{V}(I)$ is empty if and only if I is the unit ideal.

Solution: If I is the unit ideal, we already know that $\mathbb{V}(I) = \emptyset$. Now let $\mathbb{V}(I) = \emptyset$ for some ideal I . We want to show that $I = (1)$. Well, suppose that $I \subsetneq (1)$. Then,

there exists a maximal ideal m such that $I \subset m \subsetneq (1)$. Let $p \in \mathbb{V}(m)$ denote the point corresponding to $\mathbb{V}(m)$ under the Nullstellensatz for points. It follows that $\{p\} = \mathbb{V}(m) \subset \mathbb{V}(I) = \emptyset$, which is a contradiction. Thus, $I = (1)$.

Problem 6. THE GENERAL CASE. Let K be algebraically closed. We now deduce the full Nullstellensatz from the special cases already proven in (3) - (5).

Let $I = \langle f_1, \dots, f_t \rangle$ be an arbitrary ideal. We want to show $\mathbb{I}(\mathbb{V}(I)) \subset \sqrt{I}$.

- (a) Choose $g \in \mathbb{I}(\mathbb{V}(I))$. Prove that $\langle f_1, \dots, f_t, gz - 1 \rangle = \langle 1 \rangle$ in the larger polynomial ring $K[x_1, \dots, x_n, z]$.
- (b) Prove that we can write $1 = \frac{g_1 f_1 + \dots + g_t f_t}{g^T}$ (in $K(x_1, \dots, x_n)$) for some $g_1, \dots, g_t \in K[x_1, \dots, x_n]$ and some $T \in \mathbb{N}$. [Hint: Write $1 = h_1 f_1 + \dots + h_t f_t + h_{t+1}(1 - gz)$ where $h_1, \dots, h_t, h_{t+1} \in K[x_1, \dots, x_n, z]$. Apply an appropriate map $K[x_1, \dots, x_n, z] \rightarrow K[x_1, \dots, x_n][\frac{1}{g}]$ sending z to $\frac{1}{g}$.]
- (c) Conclude that $g \in \sqrt{I}$. The Nullstellensatz is proved, at least over \mathbb{C} !

Solution:

- (a) From the previous question, it is sufficient to verify that $\mathbb{V}(\langle f_1, \dots, f_t, gz - 1 \rangle) = \emptyset$. First, note that $\mathbb{V}(\langle f_1, \dots, f_t, gz - 1 \rangle) \subset \mathbb{V}(I)$. If $f_i(p) = 0 \forall i \leq t$, it follows that $g(p) = 0$, as $g \in \mathbb{I}(\mathbb{V}(I))$. However, this would imply that $(gz - 1)(p) = -1 \neq 0$. Thus, any point where $f_i(p) = 0$ for any $i \leq t$ will not vanish under $gz - 1$, implying that $\mathbb{V}(\langle f_1, \dots, f_t, gz - 1 \rangle) = \emptyset$ as desired.
- (b) Following the hint, we write $1 = h_1 f_1 + \dots + h_t f_t + h_{t+1}(1 - gz)$ where $h_1, \dots, h_t, h_{t+1} \in K[x_1, \dots, x_n, z]$, and pass this through the aforementioned map (hereafter denoted as φ). Since $(gz - 1) \mapsto (g(1/g) - 1) = 0$, it follows that the image of the expression under φ is of the form $1 = \varphi(h_1 f_1) + \dots + \varphi(h_t f_t)$. Since $f_1, \dots, f_t \in K[x_1, \dots, x_n]$, they are fixed under φ . Thus, the image is of the form $1 = \varphi(h_1) f_1 + \dots + \varphi(h_t) f_t$. Note that each $\varphi(h_i)$ for $i \leq t$ can be written as g_i/g^{n_i} for some $g_i \in K[x_1, \dots, x_n]$. It follows that

$$1 = \varphi(h_1) f_1 + \dots + \varphi(h_t) f_t = \frac{g_1}{g^{n_1}} f_1 + \dots + \frac{g_t}{g^{n_t}} f_t$$

Multiplying both sides by $\frac{g^{n_1 + \dots + n_t}}{g^{n_1 + \dots + n_t}}$ clears denominators, and yields an expression of the desired form.

$$1 = \left(\frac{g_1}{g^{n_1}} f_1 + \dots + \frac{g_t}{g^{n_t}} f_t \right) \frac{g^{n_1 + \dots + n_t}}{g^{n_1 + \dots + n_t}} = \frac{\sum_{i=1}^t g^{\sum_{j \neq i} n_j} g_i f_i}{g^{n_1 + \dots + n_t}}$$

- (c) It follows from part (b) that we can write $g^T = g_1 f_1 + \dots + g_t f_t$, implying that $g^T \in I$. Thus, $g \in \sqrt{I}$.

Math 614: Commutative Algebra**IBL Lab Notebook**

Class 5: September 19th

Name: Vignesh Jagathese

Collaborators: Don't Remember

Problem 1. Review localization and its universal property:

- (a) What is the ring structure on $U^{-1}R$ and why is it well defined? Describe the maps in the factorization explicitly and understand why they are uniquely determined.
- (b) Discuss good ways to understand $U^{-1}R$ explicitly in the following cases:
- (1) $R = \mathbb{Z}$, $U = \mathbb{Z} \setminus \{0\}$.
 - (2) $R = \mathbb{Z}$, $U = \{2^n \mid n \in \mathbb{Z}_{\geq 0}\}$.
 - (3) $R = \mathbb{Z}$, $U = \mathbb{Z} \setminus 2\mathbb{Z}$.
 - (4) $R = \mathbb{Z}/12\mathbb{Z}$, $U = \{\bar{1}, \bar{2}, \bar{4}, \bar{8}\}$.
 - (5) For a fixed non-zero integer n , $R = \mathbb{Z}[x]/\langle nx - 1 \rangle$, $U = \{n^k \mid k \in \mathbb{Z}_{\geq 0}\}$.
- (c) Explain how to think of $U^{-1}R$ as a quotient of the polynomial ring $R[\{X_s\}_{s \in U}]$.

Solution:

- (a) The operations work as follows:

$$r/d + s/e = (re + sd)/de, \quad r/d \cdot s/e = rs/de$$

If $r/d = r'/d'$, $s/e = s'/e'$, then there are $f_1, f_2 \in U$ such that $(rd' - r'd)f_1 = 0$ and $(se' - s'e)f_2 = 0$. Manipulating these two equations, we get $((re + sd)d'e' - (r'e' + s'd')de)f_1f_2 = 0$. So the addition is well-defined. Similarly, $(rsd'e' - r's'de)(f_1 + f_2) = 0$. Thus, multiplication is also well defined.

- (b) (1) \mathbb{Q} . The "denominators" are nonzero integers.
- (2) Fractions of the form $p/q \in \mathbb{Q}$ such that p is odd, and q is of the form 2^n for some n . p is even if and only if $q = 1$. Let this localization be denoted T for future use.
- (3) Since $0 \in \mathbb{Z}/2\mathbb{Z}$, localizing by it yields the zero ring.
- (4) As we will soon see, localizing and quotienting are compatible actions. (4) is simply the case of (2), but in $\mathbb{Z}/12\mathbb{Z}$. Thus, the localization is of the form $T/\overline{12\mathbb{Z}}$, where $\overline{12\mathbb{Z}}$ denoted the image of $12\mathbb{Z}$ in the localization.
- (5) In R , $nx = 1$. Since n is inverted, we know that $x = 1/n$. Thus, the localization of the form $\mathbb{Z}[1/n]$.
- (c) Localization is essentially choosing a set of elements and inverting them. If we are inverting just one element f , then $U^{-1}R$ looks like $R[1/f]$. In this sense, inverting elements is akin to adjoining elements that are canonically their inverse. Thus, $U^{-1}R$ can be thought of as $R[\{1/f\}_{f \in U}]$.

Problem 2. THE KERNEL OF LOCALIZATION

- (a) Prove that localization $R \rightarrow U^{-1}R$ is injective if and only if U contains no zero-divisors.
- (b) For a multiplicative set U , define $I := \{r \in R \mid ru = 0 \text{ for some } u \in U\}$. Prove that the localization map $R \rightarrow U^{-1}R$ factors through the quotient map $R \rightarrow R/I$.
- (c) Examine what this says about the example in (1) (b) (iv).

Solution:

- (a) We prove an equivalent statement, derived from taking the contrapositive both ways. We show that $R \rightarrow U^{-1}R$ has nontrivial kernel if and only if U does contain zero divisors.

$$\begin{aligned}
 &U \text{ contains zero divisors} \\
 &\iff \exists u \in U \text{ nonzero, } r \in R \text{ such that } ru = 0 \\
 &\iff (r \cdot u) \cdot (1/u) = 0, 1/u \neq 0 \text{ in } U^{-1}R \\
 &\iff r = 0 \text{ in } U^{-1}R \\
 &\iff 0 \neq r \in \ker(R \rightarrow U^{-1}R)
 \end{aligned}$$

- (b) Let $\varphi : R \rightarrow U^{-1}R$ denote the localization map. By construction, $\varphi(I) = 0$. From the universal property of quotients, it follows that there exists a unique $\bar{\varphi}$ that makes the following diagram commute:

$$\begin{array}{ccc}
 R & \xrightarrow{\pi} & R/I \\
 & \searrow \varphi & \downarrow \bar{\varphi} \\
 & & U^{-1}R
 \end{array}$$

Thus, the localization map factors through the quotient map.

- (c) I guess I kind of cheated and assumed this earlier....Oh well.

Problem 3. CONTRACTION AND EXPANSION. Let $R \xrightarrow{\phi} S$ be a ring homomorphism. The **contraction** of an ideal J in S , denoted $J \cap R$, is the ideal $\phi^{-1}(J) \subset R$. The **expansion** of an ideal I to S , denoted IS , is the ideal of S generated by $\phi(I)$. [These are denoted J^c and I^e , respectively, in Atiyah-MacDonald.]

- (a) Show that $(J \cap R)S \subset J$ for all ideals $J \subset S$;
- (b) Show that $I \subset (IS) \cap R$ for all ideals $I \subset R$.
- (c) Given an example of a ring map $R \rightarrow S$ and a prime ideal $P \subset R$ such that PS is not prime.

- (d) Suppose $P \in \text{Spec } R$ is disjoint from a multiplicative set $U \subset R$. Prove that $PU^{-1}R$ is prime. What happens if $P \cap U \neq \emptyset$?

Solution:

- (a) $J \cap R \subset J$, implying that $(J \cap R)S \subset JS = J$.
- (b) $I \subset IS$ since $1 \in S$, and $I \subset R$, so $I \subset IS \cap R$.
- (c) Consider $\varphi : \mathbb{Z} \hookrightarrow \mathbb{Q}$. $(p\mathbb{Z})\mathbb{Q} = \mathbb{Q}$, which is not prime.
- (d) Choose $ab/u \in PU^{-1}R$. Since $ab \in P$, $a \in P$ or $b \in P$. Without loss of generality, then let $a \in P$. Then $ab/u = a(b/u)$, where $a \in P$, and thus $a \in PU^{-1}R$, implying that $PU^{-1}R$ is prime (Notation is being abused quite a bit here, when necessary P is $P \subset R$ and when necessary it denotes the projection of P into the localization). If $P \cap U \neq \emptyset$, then $PU^{-1}R$ is not prime. The above case is an example of this, where $R = \mathbb{Z}, U = \mathbb{Z} \setminus \{0\}$.

Problem 4. Let $U \subset R$ be a multiplicative set.

- (a) Show that $\text{Spec } U^{-1}R$ is homeomorphic to $Y = \{P \in \text{Spec } R \mid P \cap U = \emptyset\}$ with the subspace topology. [Hint: the maps are given by contraction and expansion.]
- (b) For $f \in R$, show that $\text{Spec } R[\frac{1}{f}]$ is homeomorphic to $D(f) \subset \text{Spec } R$. [By $R[\frac{1}{f}]$, we mean the localization at the multiplicative set $\{f^n \mid n \in \mathbb{N}\}$.]
- (c) For $P \in \text{Spec } R$, prove that $U = R \setminus P$ is a multiplicative set and that $U^{-1}R$ is a **local ring**, meaning that it has a unique maximal ideal. We denote this ring R_P and call it the **localization of R at P** .

Solution:

- (a) We've just shown that $PU^{-1}R$ is prime if P is prime and $P \cap U = \emptyset$. Thus, the maps $PU^{-1}R \mapsto PU^{-1}R \cap R \supset P$ and $P \mapsto PU^{-1}R$ are well defined, and by the previous part, are also mutually inverse. It is sufficient to verify that contraction and expansion are continuous. Choose $\mathbb{V}(I) \subset \text{Spec}(U^{-1}R)$. Notice that $\mathbb{V}(I) \mapsto \mathbb{V}(IU^{-1}R)$ under the expansion map, and $\mathbb{V}(J) \subset Y$ maps to $\mathbb{V}(J \cap U^{-1}R)$.
- (b) Notice that $R[1/f] = U^{-1}R$ for $U = \{f\}$. let ρ define the localization map. Thus, $\text{Spec}(R[1/f])$ bijective corresponds to the set

$$\{P \in \text{Spec}(R) \mid P \cap \{f\} = \emptyset\} = \{P \in \text{Spec}(R) \mid f \notin P\} = D(f)$$

Thus, $\rho^* : \text{Spec}(R[1/f]) \rightarrow \text{Spec}(R)$ corresponds to the inclusion map $D(f) \rightarrow \text{Spec}(R)$. In particular, $D(f)$ is homeomorphic to $\text{Spec}(R[1/f])$.

- (c) Choose $u, v \in R \setminus P$. Since P is prime, if $uv \in P$ then one of u, v must be in P . Since neither are, we can assume that $uv \notin P$, implying that $uv \in R \setminus P$. Thus, $R \setminus P$ is multiplicative.

We now verify that R_P is local. Notice that

$$\text{Spec}(R_P) = \{Q \in \text{Spec}(R) \mid Q \cap (R \setminus P) = \emptyset\} = \{Q \in \text{Spec}(R) \mid Q \subset P\}$$

It is clear from this construction that all prime ideals of R_P are contained in the projection of P into R_P , or PR_P . Since any ideal is contained in a prime ideal, it follows that PR_P is the unique maximal ideal of R_P .

Problem 5.

- (a) Show that f is nilpotent if and only if f is contained in every prime ideal of R . [Hint: Characterize nilpotency in terms of $R[\frac{1}{f}]$ and use the fact that every non-zero ring has a maximal ideal].
- (b) Conclude that the nilradical of R is

$$\text{Nil}(R) = \bigcap_{P \in \text{Spec } R} P = \bigcap_{P \in \text{minSpec } R} P$$

- (c) Conclude that R is reduced if and only if $\bigcap_{P \in \text{minSpec } R} P = 0$.
- (d) Conclude that for any ideal I in R , the radical $\sqrt{I} = \bigcap_{P \in \mathbb{V}(I)} P$.

Solution:

- (a) We consider the forward case first. If $f^n = 0$, then since $0 \in P$, and since prime ideals are radical, $f \in P$, for any P prime.

We now prove the reverse case. If f were not nilpotent, then we can localize by f , getting the ring $R[1/f]$. Prime ideals in $R[1/f]$ directly correspond to prime ideals $P \subset R$ where P does not contain any powers of f , including f itself. Well, choose m a maximal ideal in $R[1/f]$. This ideal is prime, so we can look at the ideal m' it corresponds to in R . This ideal is prime, but does not contain f . Thus, if f is not nilpotent, there exists a prime ideal which does not contain it, so we can conclude the reverse case via contrapositive.

- (b) The forward case implies that if $f \in \text{Nil}(R)$, then $f \in \bigcap_{P \in \text{Spec}(R)} P$, and the reverse case implies that if $f \in \bigcap_{P \in \text{Spec}(R)} P$, then $f \in \text{Nil}(R)$. Therefore, we can conclude that $\text{Nil}(R) = \bigcap_{P \in \text{Spec}(R)} P$. The intersection over all primes certainly is contained in the intersection of the minimal primes, since they are themselves primes. Furthermore, by minimality, each minimal prime lies in a given $P \in \text{Spec}(R)$, implying that the intersection over minSpec is contained in the intersection of Spec . Thus,
- $$\bigcap_{P \in \text{Spec}(R)} P = \bigcap_{P \in \text{minSpec}(R)} P$$

- (c) If R is reduced, it has no nilpotents, so $\text{Nil}(R) = 0$. From the previous part, the result follows.
- (d) \sqrt{I} corresponds to the preimage of $\text{Nil}(R/I)$ under the quotient map R/I . Since $\text{Nil}(R/I) = \bigcap_{P \in \text{Spec}(R/I)} P$, and since $\text{Spec}(R/I)$ can be identified with the prime ideals which contain I , by identifying both $\text{Nil}(R/I)$ and $\bigcap_{P \in \text{Spec}(R/I)} P$ with their counterparts in R , we find that $\sqrt{I} = \bigcap_{P \in \mathbb{V}(I)} P$.

Problem 6.

- (a) Show that $\text{Spec}(R \times S)$ is homeomorphic to the disjoint union $\text{Spec } R \cup \text{Spec } S$.
- (b) Let R be the product ring $\mathbb{Z} \times \mathbb{Z}/\langle 24 \rangle \times \mathbb{Q}[x]/\langle x^2 - 1 \rangle$. Describe $\text{Spec } R$.
- (c) For R as in (b), let $P \in \text{Spec } R$ be the prime ideal $\mathbb{Z} \times \mathbb{Z}/\langle 24 \rangle \times \overline{\langle (x-1) \rangle}$. Describe the set $\text{Spec } R_P$.

Solution:

- (a) Recall that prime ideals in $R \times S$ are of the form $P \times S$, $R \times Q$ for $P \in \text{Spec}(R)$, $Q \in \text{Spec}(S)$. We call this "form R " and "form S " respectively. It follows that for any closed set $\mathbb{V}(I) \subset \text{Spec}(R \times S)$ can be partitioned into $X_R \cup X_S$, where X_R consists of the prime ideals of form R and X_S consists of the prime ideals of form S . Since X_R contains ideals of the form $P \times S$, we can identify this to a closed set in $\text{Spec}(R)$ via the map $P \times S \mapsto P$, and similarly for S . Concretely, we have the following identifications:

$$\begin{aligned} \mathbb{V}(I \times J) &= \{X \in \text{Spec}(R \times S) \mid I \times J \subset X\} \\ &= \{P \times S \subset \text{Spec}(R \times S) \mid I \times J \subset P \times S\} \sqcup \{R \times Q \in \text{Spec}(R \times S) \mid I \times J \subset R \times Q\} \\ &\longleftrightarrow \{P \subset \text{Spec}(R) \mid I \times J \subset P \times S\} \sqcup \{Q \in \text{Spec}(S) \mid I \times J \subset R \times Q\} \\ &= \mathbb{V}(I) \sqcup \mathbb{V}(J) \end{aligned}$$

$\{P \subset \text{Spec}(R) \mid I \times J \subset P \times S\} = \mathbb{V}(I)$ because $I \subset P \iff I \times J \subset P \times S$, and similarly with $\{Q \in \text{Spec}(S) \mid I \times J \subset R \times Q\}$ and $\mathbb{V}(J)$. It is easy to see that the homeomorphism follows.

- (b) $\text{Spec}(\mathbb{Z} \times \mathbb{Z}/\langle 24 \rangle \times \mathbb{Q}[x]/\langle x^2 - 1 \rangle) = \text{Spec}(\mathbb{Z}) \sqcup \text{Spec}(\mathbb{Z}/24\mathbb{Z}) \sqcup \text{Spec}(\mathbb{Q}[x]/\langle x^2 - 1 \rangle)$.
 $\text{Spec}(\mathbb{Z}) = \{p\mathbb{Z}\}$, and $\text{Spec}(\mathbb{Z}/24\mathbb{Z}) = \{p\mathbb{Z} \mid 24\mathbb{Z} \subset p\mathbb{Z}\} = \{2\mathbb{Z}/24\mathbb{Z}, 3\mathbb{Z}/24\mathbb{Z}\}$.

Recall that $\mathbb{Q}[x]$ is a PID. $\text{Spec}(\mathbb{Q}[x]/\langle x^2 - 1 \rangle) = \{(f) \in \text{Spec}(\mathbb{Q}[x]) \mid f \mid (x^2 - 1)\} = \{x+1, x-1\}$. Thus,

$$\text{Spec}(\mathbb{Z} \times \mathbb{Z}/\langle 24 \rangle \times \mathbb{Q}[x]/\langle x^2 - 1 \rangle) = \{p\mathbb{Z} \mid p \text{ prime}\} \sqcup \{2\mathbb{Z}/24\mathbb{Z}, 3\mathbb{Z}/24\mathbb{Z}\} \sqcup \{x+1, x-1\}$$

- (c) This corresponds to the prime ideals of R that are disjoint from P^c , implying that $\text{Spec}(R_P)$ corresponds to the the prime ideals of R contained in P . It follows that

$$\text{Spec}(R_P) = \{p\mathbb{Z} \mid p \text{ prime}\} \sqcup \{2\mathbb{Z}/24\mathbb{Z}, 3\mathbb{Z}/24\mathbb{Z}\} \sqcup \{x-1\}$$

Problem 7. (a) Discuss the following two diagrams in the category **CommRing**. Some arrows are natural quotient or localization maps; describe these with elements. Other arrows use the universal property of localizations or quotients. Explain. [Here, \bar{U} denotes $\pi(U)$ where $\pi : R \rightarrow R/I$ is the natural quotient map.] To what extent are the diagrams different?

$$\begin{array}{ccc} R & \longrightarrow & U^{-1}R \\ \downarrow & & \downarrow \\ R/I & \longrightarrow & \bar{U}^{-1}R/I \end{array} \qquad \begin{array}{ccc} R & \longrightarrow & U^{-1}R \\ \downarrow & & \downarrow \\ R/I & \longrightarrow & U^{-1}R/IU^{-1}R \end{array}$$

- (b) Prove that there is a unique isomorphism identifying residue class field of the local ring R_P at its unique maximal ideal with the fraction field of R/P . This field is called the residue field at $P \in \text{Spec } R$ and denoted $k(P)$.
- (c) Explain why there is a bijection between $\text{Spec}(U^{-1}R/IU^{-1}R)$ and the primes in R containing I and disjoint from U . What does this say about $\text{Spec } k(P)$?

Solution:

- (a) Maps $R \rightarrow R/I$ send $r \mapsto \bar{r}$, denoting the image in the quotient. $R \rightarrow U^{-1}R$ maps $r \mapsto r/1$, denoting the equivalence class of r in $U^{-1}R$.

These diagrams are constructed by applying the universal property of localization and the universal property of quotients. To see how, Consider the natural projection map $\pi_1 : R \rightarrow R/I \rightarrow \bar{U}^{-1}R/I$, which sends $r \mapsto \bar{r} \mapsto \bar{r}/\bar{1}$. It is easy to see that $\pi_1(U) \subset (\bar{U}^{-1}R/I)^\times$, implying this map factors through the localization:

$$\begin{array}{ccc} R & \longrightarrow & U^{-1}R \\ & \searrow \pi_1 & \downarrow \text{---} \\ & & \bar{U}^{-1}R/I \end{array}$$

Furthermore, $\pi_1(I) = 0$, implying this also factors through the quotient:

$$\begin{array}{ccc} R & \longrightarrow & U^{-1}R \\ \downarrow & \searrow \pi_1 & \downarrow \text{---} \\ R/I & \dashrightarrow & \bar{U}^{-1}R/I \end{array}$$

Furthermore, we can take the projection map $\pi_2 : R \rightarrow U^{-1}R \rightarrow R/IU^{-1}R$, sending $r \mapsto r/1 \mapsto \bar{r}/\bar{1}$. It is easy to see that $\pi_2(U) \subset (R/IU^{-1}R)^\times$, and that $\pi_2(I) = 0$, implying that the diagrams commute.

$$\begin{array}{ccc}
R & \xrightarrow{\quad} & U^{-1}R \\
\downarrow & \searrow \pi_2 & \downarrow \\
R/I & \dashrightarrow & R/IU^{-1}R
\end{array}$$

The difference in these diagrams comes from the difference between π_1, π_2 . π_1 projects to the quotient then localizes, then π_2 projects to the localization and then quotients.

- (b) The unique maximal ideal of R_P is PR_P . Thus, the residue field of this localization is of the form R_P/PR_P . This is the ring formed from localizing by P^C then quotienting by PR_P . This is equivalent to quotienting by P , then localizing by $\overline{P^C}$, which is the image of the complement of P in the quotient R/P . Note that $\overline{P^C}$ is precisely all the elements not sent to zero by the quotient map R/P . It follows then that $\overline{P^C} = (R/P) \setminus \{0\}$, implying that

$$R_P/PR_P = \overline{P^C}^{-1}(R/P) = ((R/P) \setminus \{0\})^{-1}R/P = \text{Frac}(R/P)$$

Thus, $\kappa(P)$ can be viewed as R_P/PR_P and as $\text{Frac}(R/P)$.

- (c) $\text{Spec}(R/I)$ is in bijection with the prime ideals of R that contain I . $\text{Spec}(U^{-1}R)$ is in bijection with the prime ideals of R that are disjoint from U . Thus, $\text{Spec}(U^{-1}R/IU^{-1}R)$ is in bijection with the prime ideals of $U^{-1}R$ which contain I , which are in bijection of the ideals of R which are disjoint from U , but also contain I . It follows then that

$$\text{Spec}(\kappa(P)) = \text{Spec}(R_P/PR_P) = \{Q \in \text{Spec}(R) \mid P \subset Q, Q \cap (P^C) = \emptyset\} = \{P\}$$

This is because $P \subset Q$, but Q is disjoint from the complement of P , so $Q \subset P$. Thus, $Q = P$. This makes sense, since $\kappa(P)$ is a field, so its spectrum should just be a single point.

Math 614: Commutative Algebra

IBL Lab Notebook

Class 6: September 24th

Name: Vignesh Jagathese

Collaborators: Don't Remember

Problem 1. Fix a ring R , multiplicative set $U \subset R$, and ideal $I \subset R$. Let \bar{U} denote the image of U under the natural quotient map $R \rightarrow R/I$.

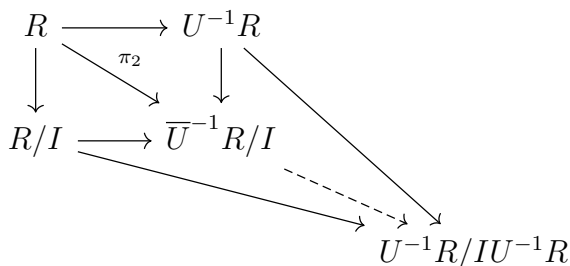
- Describe the canonical isomorphism $U^{-1}R/IU^{-1}R \cong \bar{U}^{-1}R/I$ both explicitly (saying what elements correspond) and using universal properties.
- For the natural quotient map $R \rightarrow R/I$, review the induced map $\text{Spec}(R/I) \rightarrow \text{Spec}(R)$. Why is it injective and what is the image?
- For the natural localization map $R \rightarrow U^{-1}R$, review the induced map $\text{Spec}(U^{-1}R) \rightarrow \text{Spec}(R)$. Why is it injective and what is the image?
- For the natural localization map $R \rightarrow R_P$, review the induced map $\text{Spec}(R_P) \rightarrow \text{Spec}(R)$. Why is it injective and what is its image?
- For the natural localization map $R \rightarrow R[1/f]$, explain why the induced map of spectra can be thought of as the inclusion of a basic open set $D(f) \subset \text{Spec}(R)$.
- Consider the natural map $R \rightarrow U^{-1}R/IU^{-1}R$, which can be viewed either as a localization followed by a quotient map, or vice versa. Discuss the induced map of spectra $\text{Spec}(U^{-1}R/IU^{-1}R) \rightarrow \text{Spec}(R)$. Is it injective? What is the image?
- For $P \in \text{Spec}(R)$, define the residue field of P to be $\kappa(P) = R_P/PR_P$. There is a canonical map $R \rightarrow \kappa(P)$ (why?). Describe the induced map of Spectra.

Solution:

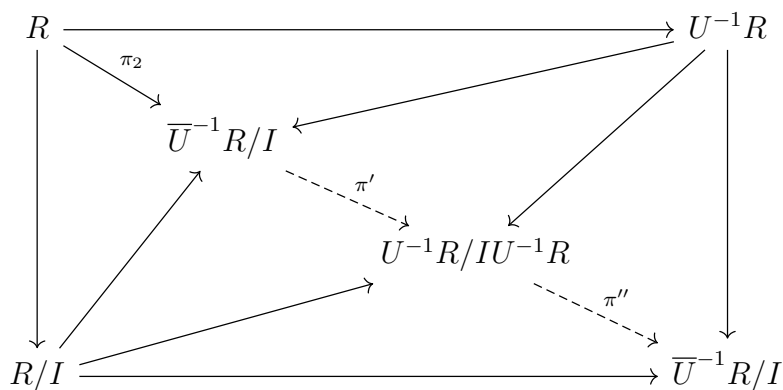
- (a) The canonical isomorphism sends an element of the form $\overline{\left(\frac{r}{u}\right)} \mapsto \frac{\bar{r}}{\bar{u}}$ Where \bar{r} denotes the image of r in the quotient R/I , and similarly with $u \in U, r/u \in U^{-1}R$. In terms of universal properties, the isomorphism follows from the following diagrams:

$$\begin{array}{ccc}
 R & \longrightarrow & U^{-1}R \\
 \downarrow & & \downarrow \\
 R/I & \longrightarrow & \bar{U}^{-1}R/I
 \end{array}
 \qquad
 \begin{array}{ccc}
 R & \longrightarrow & U^{-1}R \\
 \downarrow & & \downarrow \\
 R/I & \longrightarrow & U^{-1}R/IU^{-1}R
 \end{array}$$

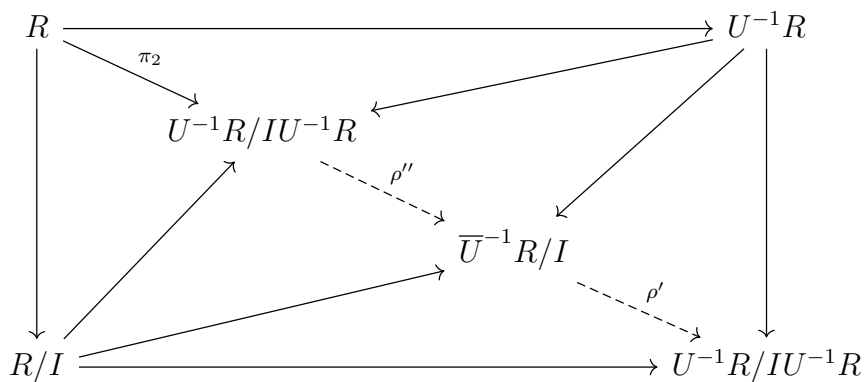
An explanation for why these diagrams commute can be found in problem 7(a) in the previous worksheet. Composing the diagrams yield



Observe that under the map $\pi : R/I \rightarrow U^{-1}R/IU^{-1}R$, $\pi(\bar{U}) \subset (U^{-1}R/IU^{-1}R)^\times$, implying that the map factors through the localization. Thus, there exists a unique map $\pi' : \bar{U}^{-1}R/I \rightarrow U^{-1}R/IU^{-1}R$. Composing further,



A similar argument to above yields a second map $\pi'' : U^{-1}R/IU^{-1}R \rightarrow \bar{U}^{-1}R/I$ that (uniquely) makes the diagram commute, and they must compose to the unique map $\bar{U}^{-1}R/I \rightarrow \bar{U}^{-1}R/I$ which makes the diagram commute. Well, Id satisfies this criterion, so by uniqueness it follows that $\pi'' \circ \pi' = \text{Id}$. We can also construct the following diagram similarly:



By uniqueness, it follows that $\rho' = \pi'$, $\rho'' = \pi''$, and $\pi' \circ \pi'' = \text{Id}$. It follows that $U^{-1}R/IU^{-1}R \cong \bar{U}^{-1}R/I$.

- (b) For $\pi : R \rightarrow R/I$, and $P \in \text{Spec}(R/I)$, this map is defined as $\pi^* : P \mapsto \pi^{-1}(P)$, and is injective because if $P \in \ker(\pi^*)$, then $\pi^{-1}(P) = 0 \Rightarrow P = \pi(0) \Rightarrow P = 0$. Thus, π^* has trivial kernel. The image is of the form

$$\text{im}(\pi^*) = \mathbb{V}(I) = \{P \in \text{Spec}(R) \mid I \subset P\}$$

- (c) For $\rho : R \rightarrow U^{-1}R$, and $P \in \text{Spec}(U^{-1}R)$, $\rho^* : P \mapsto \rho^{-1}(P)$. Choosing $P \in \ker(\rho^*)$, we have that $\rho^{-1}(P) = 0$, implying that $P = \rho(0) = 0$. Thus, ρ^* has trivial kernel, and is injective. The image is of the form

$$\text{im}(\rho^*) = \{P \in \text{Spec}(R) \mid P \cap U = \emptyset\}$$

Notice that because injective maps biject onto images, this allows us to characterize $\text{Spec}(R/I)$ and $\text{Spec}(U^{-1}R)$ in terms of prime ideals of R .

- (d) Notice that $R_P = (P^C)^{-1}R$, making this analogous to the previous case. let ρ define the localization map again. ρ^* still sends $Q \mapsto \rho^{-1}(Q)$, and is clearly still injective.

$$\text{im}(\rho^*) = \{Q \in \text{Spec}(R) \mid Q \cap P^C = \emptyset\} = \{Q \in \text{Spec}(R) \mid Q \subset P\}$$

- (e) Notice that $R[1/f] = U^{-1}R$ for $U = \{f\}$. let ρ define the localization map again. Thus, $\text{Spec}(R[1/f])$ bijective corresponds to the set

$$\{P \in \text{Spec}(R) \mid P \cap \{f\} = \emptyset\} = \{P \in \text{Spec}(R) \mid f \notin P\} = D(f)$$

Thus, $\rho^* : \text{Spec}(R[1/f]) \rightarrow \text{Spec}(R)$ corresponds to the inclusion map $D(f) \rightarrow \text{Spec}(R)$.

- (f) The induced map of Spectra is indeed injective. To see why, consider the sequence of maps

$$R \longrightarrow U^{-1}R \longrightarrow U^{-1}R/IU^{-1}R$$

Recall that $\text{Spec}(-)$ is a contravariant functor. Thus this induces a sequence of maps

$$\text{Spec}(R) \longleftarrow \text{Spec}(U^{-1}R) \longleftarrow \text{Spec}(U^{-1}R/IU^{-1}R)$$

Since each of these maps is injective, the composition is injective. Recall that $\text{Spec}(R/I)$ is in bijection with the prime ideals of R that contain I . $\text{Spec}(U^{-1}R)$ is in bijection with the prime ideals of R that are disjoint from U . Thus, $\text{Spec}(U^{-1}R/IU^{-1}R)$ is in bijection with the prime ideals of $U^{-1}R$ which contain I , which are in bijection of the ideals of R which are disjoint from U , but also contain I . Thus, the image is of the form

$$\{P \in \text{Spec}(R) \mid I \subset P, P \cap U = \emptyset\}$$

- (g) There is a canonical map $R \rightarrow R_P \rightarrow R_P/PR_P$. We've proven that localizing and quotienting is equivalent to quotienting then localizing, so it follows that $R \rightarrow R_P \rightarrow R_P/PR_P$ is the same map as $R \rightarrow (R/P) \rightarrow \text{Frac}(R/P)$.

$$\text{Spec}(\kappa(P)) = \text{Spec}(R_P/PR_P) = \{Q \in \text{Spec}(R) \mid P \subset Q, Q \cap (P^C) = \emptyset\} = \{P\}$$

Thus the induced map on spectra maps the single point to P .

Problem 2. Fibers. Let $f : Y \rightarrow X$ be the map of spectra induced by each of the ring homomorphisms $\varphi : R \rightarrow S$ below. Use the Theorem to (or otherwise) describe the fibers over each point. Note that there are different kinds of points to consider.

- (a) $\mathbb{Z} \rightarrow \mathbb{Q}$.
- (b) $\mathbb{C}[x] \rightarrow \mathbb{C}[x, y]$.
- (c) $\mathbb{Z} \rightarrow \mathbb{Z}[x]$.
- (d) $\mathbb{R}[x] \rightarrow \mathbb{C}[x]$.

Solution:

- (a) Since \mathbb{Q} is a field, the map of spectra maps the only point $0 \in \text{Spec}(\mathbb{Q})$ to the prime ideal $0 \in \text{Spec}(\mathbb{Z})$. Thus, the fiber over 0 is 0, and any fibers over nonzero primes in \mathbb{Z} are the empty set.
- (b) The only prime ideals in $\mathbb{C}[x]$ are of the form (0) and $(x - a)$ for some $a \in \mathbb{C}$. The 0 ideal (via the theorem) have fibers that are isomorphic to

$$\text{Spec}(\mathbb{C}(x) \otimes_{\mathbb{C}[x]} \mathbb{C}[x, y]) \cong \text{Spec}(\mathbb{C}(x)[y])$$

the ideals in the second case have fibers that are isomorphic to

$$\text{Spec}((\mathbb{C}[x]_{(x-a)}/(x-a)) \otimes_{\mathbb{C}[x]} \mathbb{C}[x, y]) \cong \text{Spec}(\mathbb{C} \otimes_{\mathbb{C}[x]} \mathbb{C}[x, y]) \cong \text{Spec}(\mathbb{C}[y])$$

the first isomorphism follows from the fact that residues of prime ideals over algebraic extensions of \mathbb{C} , and since \mathbb{C} is algebraically closed, we know that all residues are just \mathbb{C} , as algebraic extensions of algebraically closed fields are trivial.

- (c) The 0 fiber is of the form $\text{Spec}(\mathbb{Q}[x])$, and the fibers over p are of the form $\text{Spec}(\mathbb{F}_p[x])$.
- (d) Similar to the \mathbb{C} case, prime ideals of $\mathbb{R}[x]$ are of the form 0 or (f) for f irreducible. It follows that the fibers over (f) are precisely the ideals generated by the roots that f now has in \mathbb{C} , and fibers over ideals of the form $(x - a)$ for $a \in \mathbb{R}$ are just $(x - a)$ itself, but $a \in \mathbb{R} \subset \mathbb{C}$. Over 0, the fiber is just 0, as expected.

Problem 3. Let $f : Y \rightarrow X$ be the map of spectra induced by the ring homomorphism $\varphi : R \rightarrow S$.

- (a) Show that for any $P \in X$, the closure of P in $\text{Spec}(R)$ is $\mathbb{V}(P)$.
- (b) Show that for any $P \in X$, $\text{Spec}(S/PS)$ can be identified with $\{Q \in \text{Spec}(S) \mid f(Q) \supset P\}$.
- (c) For any $P \in X$, let U be the image of $R \setminus P$ under the map $\varphi : R \rightarrow S$. Prove that U is a multiplicative set in S . Now show there is a natural bijection between $\text{Spec}(U^{-1}S)$ and the set $\{Q \in \text{Spec}(S) \mid f(Q) \subset P\}$.

- (d) Show that for any $P \in X$, the fiber over P is homeomorphic to $\text{Spec}(U^{-1}S/PU^{-1}S)$ (with U as in (c)). That is, the fiber over P is $\text{Spec}(S \otimes_R \kappa(P))$ where $\kappa(P)$ is the residue field of P .

Solution:

- (a) It is sufficient to verify that $\mathbb{V}(P)$ is the minimal closed set that contains P . Suppose that $P \in \mathbb{V}(I)$ for some I . Then, $I \subset P$, implying that $\mathbb{V}(P) \subset \mathbb{V}(I)$. Thus $\mathbb{V}(P)$ is contained in any closed set that contains P , meaning it is the smallest such set. Thus, the closure of P is $\mathbb{V}(P)$.

- (b)

$$\begin{aligned} \text{Spec}(S/PS) &= \{Q \in \text{Spec}(S) \mid PS \subset Q\} \\ &= \{Q \in \text{Spec}(S) \mid f^{-1}(P) \subset Q\} \\ &= \{Q \in \text{Spec}(S) \mid P \subset f(Q)\} \end{aligned}$$

- (c) Since $R \setminus P$ is a multiplicative set, it follows that its image under a ring homomorphism is multiplicative. To see why, suppose that $\varphi(u), \varphi(v) \in U$. Then $\varphi(u)\varphi(v) = \varphi(uv) \in U$, since $uv \in R \setminus P$.

$$\begin{aligned} \text{Spec}(U^{-1}S) &= \{Q \in \text{Spec}(S) \mid Q \cap U = \emptyset\} \\ &= \{Q \in \text{Spec}(S) \mid Q \cap f^{-1}(R \setminus P) = \emptyset\} \\ &= \{Q \in \text{Spec}(S) \mid f(Q) \cap (R \setminus P) = \emptyset\} \\ &= \{Q \in \text{Spec}(S) \mid f(Q) \subset P\} \end{aligned}$$

- (d) Consider the maps $S \rightarrow U^{-1}S \rightarrow U^{-1}S/PU^{-1}S$. Since Spec is a contravariant functor, this gives us the sequence of maps

$$\text{Spec}(U^{-1}S/PU^{-1}S) \rightarrow \text{Spec}(U^{-1}S) \rightarrow \text{Spec}(S)$$

With images of the form

$$\begin{aligned} \{P \in \text{Spec}(S)\} &\rightarrow \{P \in \text{Spec}(S) \mid f(Q) \subset P\} \rightarrow \{P \in \text{Spec}(S) \mid f(Q) \subset P, P \subset f(Q)\} \\ &= \{P \in \text{Spec}(S) \mid f(Q) = P\} \end{aligned}$$

Thus, $\text{Spec}(U^{-1}S/PU^{-1}S)$ precisely corresponds to the fiber over P .

Math 614: Commutative Algebra**IBL Lab Notebook**

Class 7: September 26th

Name: *Vignesh Jagathese*Collaborators: *Don't Remember*

Problem 1. For each algebra below, discuss whether or not the algebra has the following properties: finitely generated, module finite, and/or integral.

- (a) The \mathbb{Z} -algebra of Gaussian integers $\mathbb{Z}[i]$.
- (b) The \mathbb{Q} -algebra \mathbb{C} .
- (c) The \mathbb{R} -algebra \mathbb{C} .
- (d) The \mathbb{Z} -algebra $\mathbb{Z}[\{\sqrt{n} \mid n \in \mathbb{N}\}]$.
- (e) The quotient $\mathbb{C}[x, y] \rightarrow \mathbb{C}[x, y]/\langle x^2 - y^{17} \rangle$
- (f) The inclusion $\mathbb{C}[x] \subset \mathbb{C}[x, y]$
- (g) The localization $\mathbb{Q}[x] \rightarrow \mathbb{Q}[x, \frac{1}{x}]$
- (h) The polynomial ring $\overline{\mathbb{F}_2}[x_1, \dots, x_n, \dots]$ in infinity many variables over the algebraic closure of \mathbb{F}_2 .
- (i) The power series ring $\mathbb{C}(t)[[x]]$ over the field $\mathbb{C}(t)$.

Solution:

Ring	Finitely Generated?	Module Finite?	Integral?
$\mathbb{Z}[i]$	Y	Y	Y
\mathbb{C} over \mathbb{Q}	N	N	N
\mathbb{C} over \mathbb{R}	Y	Y	Y
$\mathbb{Z}[\{\sqrt{n} \mid n \in \mathbb{N}\}]$	N	N	Y
$\mathbb{C}[x, y]/\langle x^2 - y^{17} \rangle$	Y	Y	Y
$\mathbb{C}[x, y]$ over $\mathbb{C}[x]$	Y	N	N
$\mathbb{Q}[x, 1/x]$ over $\mathbb{Q}[x]$	Y	N	N
$\overline{\mathbb{F}_2}[x_1, \dots]$ over $\overline{\mathbb{F}_2}$	N	N	N
$\mathbb{C}(t)[[x]]$ over $\mathbb{C}(t)$	N	N	N

Problem 2. Prove that every module-finite R -algebra A is finitely generated as an R -algebra. That is, show that “module-finite” implies “algebra-finite”. Is the converse true?

Solution: Suppose that A is module finite. Then $\forall x \in A \exists a_1, \dots, a_n$ such that $x = \sum_{i=1}^n r_i a_i$. Notice that $\sum_{i=1}^n r_i a_i$ is a polynomial in a_1, \dots, a_n . Thus, x can be written as a polynomial in finitely many elements over A . Thus, A is algebra finite.

The converse is NOT true, see problem 1 for some counterexamples. In particular, one can construct a finitely generated algebra which is not integral.

Problem 3. Let R be Noetherian. Prove that every finitely generated R -algebra is a Noetherian ring.

Solution: If A is a finitely generated R -algebra, there exists a surjection

$$\pi : R[x_1, \dots, x_n] \rightarrow A$$

It follows from the first isomorphism theorem that

$$R[x_1, \dots, x_n]/\ker(\pi) \cong A$$

Since R is Noetherian, $R[x_1, \dots, x_n]$ is Noetherian, so $R[x_1, \dots, x_n]/\ker(\pi)$ is Noetherian, so A is Noetherian.

Problem 4. Suppose that $A \rightarrow B \rightarrow C$ are ring homomorphisms.

- Prove that if B is algebra finite over A and C is algebra finite over B , then C is a finitely generated A -algebra.
- Prove that if C is a finitely generated A -algebra, then also C is a finitely generated B -algebra.
- Give an example to show that, even when the maps $A \rightarrow B \rightarrow C$ are injective, finite generation of C over A does not imply that finite generation of B over A . [Hint: Consider the K -sub-algebra $B = K[x, xy, xy^2, xy^3, \dots]$ of $K[x, y]$.]

Solution:

- B being finitely generated over A implies that there exists a surjection $A[x_1, \dots, x_n] \rightarrow B$. Similarly, if C is finitely generated over B , there exists a surjection $B[y_1, \dots, y_m] \rightarrow C$. Thus, there exists a surjection

$$A[x_1, \dots, x_n][y_1, \dots, y_m] \rightarrow C$$

Thus, C is algebra finite over A .

- (b) Choose $\varphi : A \rightarrow B, \psi : B \rightarrow C$. Choose $c \in C$. $\exists f \in A[x_1, \dots, x_m]$ such that for some $c_1, \dots, c_m \in C$, $f(c_1, \dots, c_m) = c$. Well, notice that A acts on C by the composition $a \cdot c = (\psi \circ \varphi)(a)c$. Thus,

$$c = f(c_1, \dots, c_m) = \sum a_{ij} \cdot \prod_{i=1}^m c_i^{n_i} = \sum (\psi \circ \varphi)(a_{ij}) \prod_{i=1}^m c_i^{n_i} = \sum \varphi(a_{ij}) \cdot \prod_{i=1}^m c_i^{n_i}$$

Where the second \cdot action corresponds to the action of B on C , and $\varphi(a_{ij}) \in B$. Thus, since this decomposition holds for any $c \in C$, it follows that C is finitely generated as a B algebra.

- (c) Consider the case where $A = k[x], B = k[x, xy, xy^2, \dots], C = k[x, y]$. We have ring homomorphisms $A \rightarrow B \rightarrow C$, where C is clearly finitely generated over A . However, B is infinitely generated over A .

Problem 5. INTEGRAL EXTENSIONS

- (a) Show that a composition of module finite ring maps is module finite.
 (b) Let $R \rightarrow S$ be a ring map. Prove that if $s \in S$ is integral over R , then $R[s]$ is module finite over R .
 (c) Prove that if an R -algebra A can be generated by finitely many integral elements $\{a_1, \dots, a_n\}$, then A is module finite over R .

Solution:

- (a) Suppose $A \rightarrow B \rightarrow C$. If B is module finite over A , there exists b_1, \dots, b_n such that $\forall b \in B, \sum_{i=1}^n a_i b_i = b$. Similarly, C being module finite over B means that $\exists c_1, \dots, c_m$ such that $\forall c \in C, \sum_{j=1}^m b_j c_j = c$. Thus,

$$c = \sum_{j=1}^m b_j c_j = \sum_{i,j=1}^{n,m} a_{ij} c_j$$

Because there exists a decomposition over A for each b_j . Thus, C is module-finite over A .

- (b) Choose $\sum_{i=1}^n r_i s^i \in R[s]$. Since s is integral over R , there exists a monic polynomial $f \in R[x]$ such that $f(s) = 0$. Suppose that $\deg(f) = m$. Then, s^i for $i \geq m$ can be written as a polynomial of s over $R[x]$ of degree $m - 1$. Thus, any $f \in R[s]$ can be written in the form

$$\sum_{i=1}^{m-1} r_i s^i$$

Thus, $R[s]$ is generated by $1, s, s^2, \dots, s^{m-1}$ as an R -module, implying that $R[s]$ is module finite.

- (c) We induct on the number of generators. When $n = 1$, we have the case proved in (b). We now proceed with the inductive step, for $n = k$.

By the inductive hypothesis, $R[a_1, \dots, a_{k-1}]$ is module finite over R . By part (b), $R[a_1, \dots, a_{k-1}][a_k] = R[a_1, \dots, a_k]$ is module finite over $R[a_1, \dots, a_{k-1}]$, which is module finite over R . Then, by part (a), $R[a_1, \dots, a_k]$ must be module finite over R , completing the proof.

Problem 6. THE DETERMINANT TRICK. Let B be an $n \times n$ matrix with entries in R . We will now prove the following useful matrix equation (which is well-known when R is a field):

$$B \times \text{adj}(B) = \det B \times I_{n \times n}$$

where $\text{adj}(B)$ denotes the adjugate matrix of B —that is, the transpose of the $n \times n$ matrix whose ij -th entry is $(-1)^{i+j}$ times the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from B .

- Observe that an arbitrary ring has a canonical \mathbb{Z} -algebra structure. Use this to show that R is isomorphic to a quotient of a polynomial ring $\mathbb{Z}[\mathbf{X}]$ over \mathbb{Z} (possibly in infinitely many variables).
- Show that to prove the matrix equation for R , it suffices to prove the formula for $\mathbb{Z}[\mathbf{X}]$. [Hint: use the surjection $\mathbb{Z}[\mathbf{X}] \rightarrow R$ to lift each entry b_{ij} of B to a_{ij} in $\mathbb{Z}[\mathbf{X}]$.]
- Show that to prove the matrix equation over $\mathbb{Z}[\mathbf{X}]$, it suffices to prove it over its fraction field. [Hint: Don't forget that two matrices are equal means that their corresponding entries are equal.]
- Viola! Conclude the matrix equation above from the well-known corresponding linear algebra fact (over fields). Be sure to check the field case too, if it's not familiar to you. It's easy, if you know about computing determinants by "expanding along a row or column".

Solution:

- Since any abelian group is a \mathbb{Z} -module, it follows that any ring is an (associative) \mathbb{Z} -algebra. Let $\{a_i\}_{i \in I}$ denote a set of generators for R over \mathbb{Z} . It follows that there exist a surjection $\mathbb{Z}[x_i]_{i \in I} \rightarrow R$, implying that R is isomorphic to a quotient of $\mathbb{Z}[x_i]_{i \in I}$. We denote $\mathbb{Z}[X] := \mathbb{Z}[x_i]_{i \in I}$.
- Suppose we wish to show a matrix equation holds with coordinate in R , denoted $E(\{b_{ij}\})$, $b_{ij} \in R$. Following the hint, we lift each b_{ij} to a coordinate $a_{ij} \in \mathbb{Z}[X]$, since there exists a surjection $\varphi : \mathbb{Z}[X] \rightarrow R$. We know the a_{ij} satisfy the equation by hypothesis, implying that the equation evaluates to zero. Thus, we say that $E(\{a_{ij}\}) = 0$. Then, passing the entire equation through the ring homomorphism, we get $\varphi(E(\{a_{ij}\})) = \varphi(0) = 0$. However, because φ is a ring homomorphism, and because matrix equations are linear equations, $\varphi(E(\{a_{ij}\})) = E(\{\varphi(a_{ij})\}) = E(\{b_{ij}\})$. In particular, $E(\{b_{ij}\}) = 0$, and the matrix equation is in fact satisfied over R . Thus, if the equation is satisfied over $\mathbb{Z}[X]$, it is satisfied over R .

- (c) Let $E/1$ denote the term in $\text{Frac}(\mathbb{Z}[X])$ corresponding to the matrix equation. Suppose that this equals zero (i.e. the equation is satisfied over the fraction field). Recall that in any ring R , $0 = r/1 \in U^{-1}R$ if and only if $0 = r \in R$. Since taking the fraction field is simply a localization, it follows that if $E/1 = 0$, then $E = 0$ over $\mathbb{Z}[X]$. In particular, if the equation holds over the fraction field of $\mathbb{Z}[X]$, then it holds over $\mathbb{Z}[X]$.
- (d) Since we know the result to hold over fields, it (in particular) holds over the fraction field of $\mathbb{Z}[X]$. We've proven that this implies that the equation above holds over $\mathbb{Z}[X]$, and thus over R . Thus the matrix equation is proven!

Problem 7. MODULE FINITE IMPLIES INTEGRAL. Let $R \rightarrow A$ be a module finite ring homomorphism. Our goal is to prove that it is integral. [You have considered the converse in Problem 5.]

Our strategy: Pick any $a \in A$. The idea is a Cayley-Hamilton type result: the R -linear map $A \rightarrow A$ given by multiplication by a satisfies its own characteristic polynomial over R .

- (a) Take any $a \in A$. Prove that the map $A \rightarrow A$ sending $b \mapsto ab$ is R -linear.
- (b) Fix R -module generators a_1, a_2, \dots, a_n for A . **Without loss of generality, assume $a_1 = 1$.** Explain how to think of any element of A as a column vector with entries in R and any R -linear map $A \rightarrow A$ as given by (left) multiplication by some $n \times n$ matrix with entries in R .
- (c) For $a \in A$, let A be the matrix of the map “multiplication by a ”. Prove the matrix equation in A
- $$a I_{n \times n} v = A v,$$
- where v is the column vector $[1 \ a_2 \ \dots \ a_n]^{tr}$ made up of our R -module generators of A .
- (d) Let $B = (A - a I_{n \times n})$. Show that $\det B = 0$. [Hint: Multiply both sides of the matrix equation $Bv = 0$ by $\text{adj}(B)$. **If you are thinking A is a diagonal matrix you are making a big mistake!**]
- (e) Prove that $a \in A$ satisfies a monic polynomial with coefficients in R .
- (f) Conclude that the Theorem in page 1 is proved!

Solution:

- (a) Define this map φ_a . Let $\psi : R \rightarrow A$ be the above module finite ring homomorphism.

$$\begin{aligned} \varphi_a(b_1 + b_2) &= a(b_1 + b_2) = ab_1 + ab_2 = \varphi_a(b_1) + \varphi_a(b_2) \\ \varphi_a(r \cdot b) &= a(r \cdot b) = a\psi(r)b = \psi(r)ab = r \cdot ab = r \cdot \varphi_a(b) \end{aligned}$$

- (b) Given a_1, \dots, a_n are R -module generators of A , $\forall a \in A \exists r_1, \dots, r_n$ such that $b = \sum_{i=1}^n r_i a_i$. Given that these a_1, \dots, a_n are known, we can formulate a vector representation of A , given that we've fixed R -module generators, and denote A as follows:

$$\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$$

Since φ_a is R -linear, it is completely determined by where it sends generators a_1, \dots, a_n . Well, the image of a_i is itself an element of A , and has its own generator expansion; i.e. $\varphi_a(a_i) = \sum_{j=1}^n b_{ji} a_j$. Since $\{b_{ji}\}_{j \leq n}$ Completely determines where a_i is sent by φ_a , it follows that φ_a is completely determined by $A = \{b_{ij}\}_{1 \leq i, j \leq n}$. In this way, we can write coefficients in an $n \times n$ matrix. It is easy to see that, from how we've chosen our coefficients, matrix multiplication is in the linear algebra case works just as intended here. In other words, $A \cdot \vec{b} = \varphi_a(b)$.

- (c) In this notation, what we defined above as $[1, 1, \dots, 1]^T$ is of the form $[a_1, a_2, \dots, a_n]^T$. Thus, $a I_{n \times n} v = [aa_1, aa_2, \dots, aa_n]^T$, whereas $Av = \varphi_a(v) = av = \sum_{i=1}^n aa_i$, which has the vector representation equivalent to the above expression.
- (d) Following the hint, $(B \times \text{adj}(B))(v) = (\det(B) \times I_{n \times n})v$. Since $Bv = 0$, the left hand side of the equation is zero, implying that $(\det(B) \times I_{n \times n})v = 0$. In particular, since $v \neq 0$ and $I_{n \times n} v = v$, it follows that $\det(B) = 0$.
- (e) $\det(B) \times I_{n \times n}$ is a polynomial in its coefficients, which live in R . Since any element in $a \in A$ has a vector representation v , and $(\det(B) \times I_{n \times n})(v) = 0$, it is clear that a satisfies a polynomial in R .

Math 614: Commutative Algebra

IBL Lab Notebook

Class 8: October 1st

Name: Vignesh Jagathese

Collaborators: Don't Remember

Problem 1. For each of the rings below, find the dimension using the definition/theorem:

- (a) The field $\mathbb{C}(t)$ of meromorphic functions on the Riemann sphere.
- (b) \mathbb{Z}
- (c) The algebraic integers \mathcal{A} , i.e., the ring of all elements in \mathbb{C} which satisfy an equation of integral dependence over \mathbb{Z} .
- (d) $\mathbb{Q}[x]$
- (e) $\mathbb{Q}[x, y]/\langle y^{17} - x^2y^3 + 5x^5 \rangle$
- (f) The polynomial ring $K[x_1, \dots, x_n]$. For this one, just guess, and prove your guess is a lower bound. To show it's an upper bound is quite hard; we'll do it soon enough.
- (g) The polynomial ring $K[x_1, \dots, x_n, \dots]$
- (h) $\mathbb{Q}[x, y, z, w]/\langle w^3 - w^2x - yz, z^4 - xy \rangle$ (use (f)).

Solution:

- (a) 0, it is a field.
- (b) 1, it is a PID.
- (c) 1, since this is definitionally an integral extension of \mathbb{Z} , which has dimension 1.
- (d) 1, it is a PID.
- (e) 1, it is an integral extension of $\mathbb{Q}[x]$, since y satisfies the monic polynomial $y^{17} - x^2y^3 + 5x^5$ over $\mathbb{Q}[x]$.
- (f) $\geq n$.
- (g) Infinite. Consider the chain $(x_1) \subsetneq (x_1, x_2) \subsetneq \dots$
- (h) w and z satisfy a monic polynomial over $\mathbb{Q}[x, y]$ (in particular, $w^3 - w^2x - yz$ and $z^4 - xy$), so this is an integral extension of $\mathbb{Q}[x, y]$, and has dimension 2.

Problem 2. Explain why $\dim W^{-1}R \leq \dim R$ and $\dim R/I \leq \dim R$.

Solution: $\text{Spec}(W^{-1}R)$ and $\text{Spec}(R/I)$ bijectively correspond to subsets of $\text{Spec}(R)$. Thus, any saturated chain in $\text{Spec}(W^{-1}R)$ and $\text{Spec}(R/I)$ lift to a chain in $\text{Spec}(R)$, implying that for any chain of height t in $\text{Spec}(W^{-1}R)$ or $\text{Spec}(R/I)$, there must be a corresponding chain of height $\geq t$ in $\text{Spec}(R)$, yielding a dimension bound.

Problem 3. Explain why every principle ideal domain has dimension one.

Solution: Let R be a PID. If $(r) \subsetneq (s)$, then $r|s$, implying that (s) was not generated by an irreducible element, and is thus not prime. Therefore, every prime must be of height at most 1, so R has krull dimension at most 1. Assuming that this PID is not a field, then it has krull dimension exactly 1.

Problem 4. Prove that the ring $R \times S$ has dimension equal to the larger of $\dim R$ and $\dim S$.

Solution: $\text{Spec}(R \times S) = \text{Spec}(R) \sqcup \text{Spec}(S)$. Without loss of generality, let R have higher dimension than S . It follows then that any chain in $\text{Spec}(R \times S)$ must correspond to either a chain in $\text{Spec}(R)$ or a chain in $\text{Spec}(S)$. Thus, the saturated chain in $\text{Spec}(R \times S)$ must correspond to a chain in $\text{Spec}(R)$ (Since $\dim(R) \geq \dim(S)$), implying that $\dim(R \times S) \leq \dim(R)$. Similarly, a chain in $P_1, \dots, P_t \in \text{Spec}(R)$ corresponds to a chain in $(P_1, S), \dots, (P_t, S) \in \text{Spec}(R \times S)$, so it follows that $\dim(R) \leq \dim(R \times S)$. Thus, $\dim(R) = \dim(R \times S)$.

Problem 5. Prove that the ring R and its reduced ring R_{red} have the same dimension.

Solution: Recall that $R_{red} = R/\text{Nil}(R)$. Since $\text{Nil}(R)$ is precisely the intersection of all prime ideals, $P \in \text{Spec}(R) \Rightarrow \text{Nil}(R) \subset P$. It follows then that $P \in \text{Spec}(R) \iff P \in \text{Spec}(R_{red})$, since we know that prime ideals of $R/\text{Nil}(R)$ are precisely the prime ideals of R which contain $\text{Nil}(R)$. Thus $\text{Spec}(R) = \text{Spec}(R_{red})$. Thus, they clearly must have the same dimension.

Problem 6. PROOFS OF THE COROLLARIES. Let $R \hookrightarrow S$ be an integral extension of rings. Assume the Lying Over Theorem.

- (a) Suppose that $Q_0 \in \text{Spec } S$ contracts to $P_0 \in \text{Spec } R$. Prove that the map $R/P_0 \hookrightarrow S/Q_0$ is also an integral extension.
- (b) Given a chain of primes $P_0 \subset P_1$ in R , prove that there exists a chain of primes $Q_0 \subset Q_1$ in S such that $P_i = Q_i \cap R$ for $i = 0, 1$. [Hint: Use (a)]

- (c) Show that the Going Up Corollary follows from the Lying Over Theorem.
 (d) Show also that the Dimension Corollary follows.

Solution:

- (a) We want to verify that $R/(Q_0 \cap R) \rightarrow S/Q_0$ is integral. Choose $\bar{s} \in S/Q_0$ and lift it to $s \in S$. $\exists f \in R[x]$ monic such that $f(s) = 0$, since $R \rightarrow S$ is integral. Now, consider the projection $\bar{f} \in (R/(Q_0 \cap R))[x]$. It is clear that $f(s) = 0 \Rightarrow \bar{f}(\bar{s}) = 0$. Thus, \bar{s} is integral, and thus so is all of S/Q_0 , over $R/(Q_0 \cap R)$.
- (b) Since $R/P_0 \rightarrow S/Q_0$ is an integral extension, by the lying over theorem, $\text{Spec}(S/Q_0) \rightarrow \text{Spec}(R/P_0)$ is surjective. Thus, for $P_1 \in \text{Spec}(R/P_0)$, there exists some $Q_1 \in \text{Spec}(S/Q_0)$ such that $P_1 = Q_1 \cap R/P_0$. Notice that $P_1 \in \text{Spec}(R/P_0)$ implies that $P_0 \subset P_1$, and similarly $Q_0 \subset Q_1$.
- (c) The process used above to construct Q_1 from P_1 given facts about P_0, Q_0 can be repeated to construct Q_n from P_n . From part (b) and part (a), we know that $R/P_1 \rightarrow S/Q_1$ is an integral extension. mimicing the proof of (b), we find that there exists a $Q_2 \in \text{Spec}(S/Q_1)$, and thus $\in \text{Spec}(S)$, such that $Q_2 = P_2 \cap R$. This process can be continued through P_t , proving existence of $Q_t \in \text{Spec}(S)$ such that $P_t = Q_t \cap R$. This yields the going up corollary.
- (d) The going up corollary gives us that any chain in R can be lifted to a chain in S , and the lying over theorem asserts that if the chain in R consists of incomparable primes, then the chain in S must also consist of incomparable primes. Thus, for any chain of length t in R , there exists a chain of length t in S , implying that $\dim(R) \leq \dim(S)$. $\dim(S) \leq \dim(R)$ is clear, so it follows that $\dim(R) = \dim(S)$.

Problem 7. INCOMPARABLE FIBERS. Let $R \hookrightarrow S$ be an integral extension.

- (a) Prove that if R and S are domains, then every $u \in S$ has a non-zero multiple in R .
 (b) Show that if $Q_0 \subset Q_1$ in $\text{Spec } S$ both contract to P in $\text{Spec } R$, then $Q_0 = Q_1$. [Hint: Use 5a.]
 (c) Conclude that the fibers of the map $\text{Spec } S \rightarrow \text{Spec } R$ consist of *incomparable* primes, and that the ring $S \otimes_R R_P/P R_P$ has Krull dimension zero.

Solution:

- (a) By integrality, $\exists f = \sum r_i x^i \in R[x]$ such that $r_n u^n + r_{n-1} u^{n-1} + \dots + r_0 = 0$. This implies that $u(r_n u^{n-1} + \dots + r_1) = -r_0$. Since R and S are domains it follows that $-r_0 \neq 0$. Thus, u has a non-zero multiple in R .

- (b) Suppose that $Q_0 \subset Q_1$, and $P = Q_0 \cap R = Q_1 \cap R$. We wish to show that $Q_0 = Q_1$. Suppose that $Q_0 \subsetneq Q_1$. By problem 6(a), we know that $R/P \rightarrow S/Q_0$ is an integral extension. By the previous part, for any $u \in S/Q_0$, u has a multiple in R . In particular, $(u) \cap R \neq 0$. Choose $u \in \overline{Q_1}$, where $\overline{Q_1}$ is the projection of Q_1 into S/Q_0 , which is nonzero since $Q_0 \subsetneq Q_1$. Thus, it follows that $0 \neq (u) \cap R \subset \overline{Q_1} \cap R = 0$, since $Q_1 \cap R = P$, a contradiction. Thus, $Q_0 = Q_1$.
- (c) We have shown above that if $Q_0, Q_1 \in f^{-1}(P)$ (meaning that $Q_0 \cap R = Q_1 \cap R = P$), and Q_0 and Q_1 are comparable (meaning either $Q_0 \subset Q_1$ or $Q_1 \subset Q_0$), then $Q_0 = Q_1$. Thus, a fiber cannot contain comparable primes, so the fibers of $\text{Spec}(S) \rightarrow \text{Spec}(R)$ consist of incomparable primes. Since $f^{-1}(P)$ can be associated with $\text{Spec}(S \otimes_R \kappa(P)) = \text{Spec}(S \otimes_R R_P/PR_P)$, it follows that chains in $S \otimes_R R_P/PR_P$ must be of maximal height 0 (else they would violate incomparability), implying that it has Krull dimension 0.

Problem 8. Consider the extension $\mathbb{Z} \hookrightarrow \mathbb{Z}[\sqrt{2}]$ and the corresponding contraction map $\text{Spec } \mathbb{Z}[\sqrt{2}] \rightarrow \text{Spec } \mathbb{Z}$.

- (a) Compute the fiber over the points $\langle 0 \rangle, \langle 2 \rangle, \langle 3 \rangle, \langle 7 \rangle$ of $\text{Spec } \mathbb{Z}$.
- (b) For $p > 2$ a prime integer, prove that there are at most two points over $\langle p \rangle$, with exactly one if and only if 2 is not a square mod p .

Solution:

- (a) We compute $f^{-1}(0)$ explicitly first.

$$f^{-1}(0) = \text{Spec}(\mathbb{Z}[\sqrt{2}] \otimes_{\mathbb{Z}} \mathbb{Q}) = \text{Spec}(\mathbb{Q}[\sqrt{2}]) = \text{Spec}(\mathbb{Q}(\sqrt{2})) = \{0\}$$

Now, notice that $\kappa(p\mathbb{Z})$ is simply $\mathbb{Z}/p\mathbb{Z}$. This is because $\mathbb{Z}/p\mathbb{Z}$ is already a field, so taking the fraction field is a trivial action. Next, we compute $f^{-1}(p)$ for any prime p .

$$f^{-1}(p) = \text{Spec}(\mathbb{Z}[\sqrt{2}] \otimes_{\mathbb{Z}} \mathbb{Z}/(p)) = \text{Spec}((\mathbb{Z}/p)[a]/(a^2 - 2))$$

When $p = 2$, This yields $\text{Spec}((\mathbb{Z}/2)[a]/(a^2)) = \text{Spec}(\mathbb{Z}/2 \oplus \mathbb{Z}/2) = \{(\mathbb{Z}/2, 0), (0, \mathbb{Z}/2)\}$.

When $p = 3$, $a^2 - 2$ is irreducible over $\mathbb{Z}/p[a]$, implying that the $\mathbb{Z}/p[a]/(a^2 - 2)$ is just \mathbb{F}_9 , a field, and $\text{spec } \{(0)\}$.

When $p = 7$, $a^2 - 2$ is not irreducible, and is of the form $(a - 3)(a + 3)$. In this case, we get a case similar to the $p = 2$, case, and get 2 points, $(\mathbb{Z}/7, 0), (0, \mathbb{Z}/7)$.

- (b) This process above can be generalized. If 2 is not a square mod p , then $a^2 - 2$ is irreducible over $\mathbb{Z}/p[a]$, and $\mathbb{Z}/p[a]/(a^2 - 2) \cong \mathbb{F}_{p^2}$, implying the spec has 1 point. This means that $f^{-1}(p)$ has exactly 1 point.

In the other case, if 2 is a square modulo p , we get a factorization, and $\mathbb{Z}/p[a]/(a^2 - 2) \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$, implying its spectra has two points, so $f^{-1}(p)$ has exactly 2 points.

Problem 10. EASY LEMMA. Let S be an arbitrary ring, and let U be a multiplicative set and J an ideal of S . Consider the two subsets, $\mathbb{V}(J)$ and $\{P \mid P \cap U = \emptyset\}$, of $\text{Spec} S$. Prove that their intersection is non-empty if and only if $J \cap U = \emptyset$. [Hint: Consider the ring SU^{-1}/JSU^{-1} .]

Solution: Notice that

$$\begin{aligned} \text{Spec}(U^{-1}S/JU^{-1}S) &= \{P \in \text{Spec}(S) \mid P \cap U = \emptyset, J \subset P\} \\ &= \mathbb{V}(J) \cap \{P \in \text{Spec}(S) \mid P \cap U = \emptyset\} \end{aligned}$$

If $J \cap U \neq \emptyset$, then for any P such that $J \subset P$, $P \cap U \neq \emptyset$. Thus, $\mathbb{V}(J) \cap \{P \in \text{Spec}(S) \mid P \cap U = \emptyset\}$ must be trivial, since $P \in \mathbb{V}(J) \Rightarrow P \notin \{P \in \text{Spec}(S) \mid P \cap U = \emptyset\}$.

If $J \cap U = \emptyset$, then the quotient $U^{-1}S/JU^{-1}S$ is nonzero, implying it has nonempty spectra. Thus, $\mathbb{V}(J) \cap \{P \in \text{Spec}(S) \mid P \cap U = \emptyset\}$ is nonempty.

Problem 11. THE PROOF OF LYING OVER. Let $R \hookrightarrow S$ be an integral extension.

- Let $I \subset R$ be any ideal. Prove that $IS \cap R \subset \sqrt{I}$. In particular, if I is radical, $IS \cap R = I$. [Hint: Use 8.]
- Take $P \in \text{Spec} R$. Show that, as subsets of S , $(R \setminus P) \cap PS = \emptyset$. [Hint: Use 9.]
- Use (b) to show that for all $P \in \text{Spec} R$, there exists $Q \in \text{Spec} S$ such that $Q \cap R = P$.
- Conclude that the Lying Over Theorem holds.

Solution:

- Choose $u \in IS \cap R$. Since $x \in IS$, it satisfies a polynomial with coefficients in \sqrt{I} . It follows from 9 that $\exists r_1, \dots, r_n \in \sqrt{I}$ such that

$$u^n + r_1 u^{n-1} + \dots + r_n = 0$$

In particular,

$$u^n = -r_1 u^{n-1} - \dots - r_n$$

Since $u \in R$, $-r_1 u^{n-1} - \dots - r_n \in \sqrt{I}$. Thus, $u^n \in \sqrt{I}$, so $u \in \sqrt{I}$. Thus, $IS \cap R \subset \sqrt{I}$.

We already know that $I \subset IS \cap R$. Thus, if I is radical, then $I \subset IS \cap R \subset \sqrt{I} = I$, implying that $I = IS \cap R$.

- Notice that $R \setminus P$ is a multiplicative set and PS is an ideal. Thus, it is sufficient to verify that $\mathbb{V}(PS)$ and $\{Q \mid Q \cap (R \setminus P)\}$ have nontrivial intersection by problem 10. Notice that $\mathbb{V}(PS) = \{Q \in \text{Spec}(S) \mid PS \subset Q\}$ and $\{Q \in \text{Spec}(S) \mid Q \cap (R \setminus P)\} = \{Q \in \text{Spec}(S) \mid Q \cap R \subset P\}$. Since P is prime, it is radical, and $PS \cap R = P$. Thus, $Q = PS$ is contained in both these sets, so they have nontrivial intersection. It follows that $(R \setminus P) \cap PS = \emptyset$.

(c) As in the previous proof, take $Q = PS$.

(d) This shows the first part of the lying over theorem. Since we've verified incompatibility in a previous part, we can conclude that the whole theorem holds.

Problem 12. Let U be a multiplicative set of a ring R that contains some non-unit. Prove that $R \rightarrow U^{-1}R$ is not an integral **extension**. [Hint: Consider the map of Spec.]

Solution: Suppose that $R \rightarrow U^{-1}R$ was integral. By Lying over, $\text{Spec}(U^{-1}R) \rightarrow \text{Spec}(R)$ would be surjective. We already know that this is injective, so this would imply that $\text{Spec}(U^{-1}R) \cong \text{Spec}(R)$, which is certainly false, since U contained some non-unit.

Problem 13. Let \mathbb{F}_2^X be the ring of \mathbb{F}_2 -valued functions on an infinite set X . Prove that \mathbb{F}_2^X is integral over \mathbb{F}_2 , and compute its dimension. [Hint: Consider the polynomial $x^2 - x = 0$.]

Solution: Choose $f \in \mathbb{F}_2^X$. Notice that $f^2 = f$, $(f(p))^2 \in \mathbb{F}_2$, and $x^2 \equiv x$ modulo 2. Thus, f satisfies $x^2 - x \in \mathbb{F}_2[x]$, and \mathbb{F}_2^X is an integral extension of \mathbb{F}_2 . \mathbb{F}_2 is a field, and thus has dimension zero. Since \mathbb{F}_2^X is an integral extension of \mathbb{F}_2 , it too has dimension 0.

Math 614: Commutative Algebra

IBL Lab Notebook

Class 9: October 3rd

Name: Vignesh Jagathese

Collaborators: Don't Remember

Problem 1. Prove that for any $P \in \text{Spec } R$, the height of P equals the dimension of R_P .

Solution: Recall that

$$\text{Spec}(R_P) = \{Q \in \text{Spec}(R) \mid Q \subset P\}$$

Thus chains terminating in P are in direct bijection with chains in $\text{Spec}(R_P)$. The result follows.

Problem 2. Prove the Proposition. [Hint: Write an integral element of the fraction field in lowest terms.]

Solution: Choose $u/v \in \text{Frac}(R)$ in lowest terms (which we can do, as R is UFD). Suppose it satisfied some integral dependence over R .

$$\begin{aligned} (u/v)^n + a_{n-1}(u/v)^{n-1} \cdots + a_0 &= 0 \\ \Rightarrow u^n + a_{n-1}u^{n-1}v + \cdots + a_0v^n &= 0 \\ \Rightarrow u^n &= -v(a_{n-1}u^{n-1} + \cdots + a_0v^{n-1}) \end{aligned}$$

Now, notice that in a UFD, every nonzero non-unit has a unique factorization into primes. I claim that v is unit. If not, there exists a prime p that divides v . By above, it would also divide u^n , and thus u . This is a contradiction, as we took u/v to be in its reduced form.

Problem 3. Which of the following are normal?

- (a) \mathbb{Z}
- (b) $\mathbb{Z}[x, y, z]$
- (c) The algebraic integers \mathcal{A} , i.e., the ring of all elements in \mathbb{C} which satisfy an equation of integral dependence over \mathbb{Z} . [Hint: remember that a composition of integral extensions is integral.]
- (d) $\mathbb{C}[[x, y, z]]$.
- (e) $\mathbb{Q}[x^2, x^3]$

Solution:

- (a) Yes, it is a UFD.
- (b) Yes, it is a UFD.
- (c) Yes, any element in $\text{Frac}(\mathcal{A})$ that satisfies an equation of integral dependence over \mathcal{A} also satisfies it over \mathbb{Z} , implying that it is contained in \mathcal{A} .
- (d) Yes, it is a UFD.
- (e) Suppose \exists a polynomial f such that $f(1/x^2) = 0$.

$$(1/x^2)^n + \cdots + a_0 = 0$$

Clearing denominators:

$$1 + \cdots + a_0 x^{2n} = 0$$

Which is a contradiction, as $1, x, x^2, \dots, x^{2n}$ are linearly independent. A similar argument can be made for $1/x^3$. This implies that $\mathbb{Q}[x^2, x^3]$ is normal, since anything that satisfies a polynomial in the fraction field actually lies in $\mathbb{Q}[x^2, x^3]$.

Problem 4. Let $R \hookrightarrow S$ be an integral extension. Suppose that $Q \in \text{Spec } S$ lies over $P \in \text{Spec } R$.

- (a) Prove that $\text{height } P \geq \text{height } Q$. [Hint: Use Lying Over.]
- (b) Assuming the **Going Down** Theorem, prove its COROLLARY.

Solution:

- (a) We've shown that any chain $0 = Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_t = Q$ contracts to a chain of incompatibles $0 = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_t = P$. Thus, any chain of length t to Q induces a chain of length t to P . This implies that the height of P is at least the height of Q , as we can take a saturated chain in Q and lift it to P .
- (b) Going Down Theorem tells us that we can lift a chain of P to a chain of Q such that $Q_i \cap R = P_i$ for each P_i, Q_i in the chain. This implies (by similar logic to above) that the height of Q is at least the height of P , since a saturated chain of P can be lifted to a chain of Q . It follows from this and the previous part that the height of P and Q are equal.

Problem 5. Let $R = K[x]$ and $S = K[x, y]/\langle y^2 - y, xy \rangle$. Let $Q \subset S$ be the ideal $\langle y - 1 \rangle$.

- (a) Show that $R \subset S$ is an integral extension.
- (b) Show that $x \in Q$.

- (c) Prove Q is prime in S and that S_Q is a field.
- (d) Show that $Q \cap R = \langle x \rangle$.
- (e) Compare the height of Q and the height of its contraction to R .
- (f) Why is this not contradicting Going Down?

Solution:

- (a) y satisfies a polynomial in x , in particular, $y^2 - y = 0$.
- (b) $x(y - 1) = \overline{xy - x} \in Q$. Since we are quotienting by xy , $\overline{xy - x} = \overline{-x} \in Q$. Well, this implies that $\overline{x} \in Q$, and we can conclude.
- (c) $y - 1$ is irreducible in S , so $Q = (y - 1)$ is prime. Notice that

$$S_Q = (K[x, y]/(y^2 - y, xy))_{(y-1)} = K[x, y]_{(y-1)}/\overline{(y^2 - y, xy)} = K[x, y]_Q/QK[x, y]_Q$$

Since $QK[x, y]_Q$ is the unique maximal ideal in $K[x, y]_Q$, it follows the quotient, and thus S_Q , is a field.

- (d) We've shown that $x \in Q$, implying that $(x) \subset Q$, so $(x) \subset Q \cap R$. $k \notin (y - 1) \forall k \in K$, implying that $(x) = Q \cap R$.
- (e) Q has height 0, since S_Q has dimension 0. The height of the contraction to R is 1.
- (f) S is not torsion free, so going down does not apply.

Math 614: Commutative Algebra**IBL Lab Notebook****Class 10: October 8th***Name: Vignesh Jagathese**Collaborators: Will Garland, Michael Mueller*

Problem 1. EQUIVALENT FORMULATIONS OF ALGEBRAIC INDEPENDENCE. Let $\mathcal{S} = \{y_1, \dots, y_t, \dots\}$ be a subset of an R -algebra A . Show the following are equivalent:

- (a) The set \mathcal{S} is algebraically independent.
 (b) The R -algebra homomorphism

$$\varphi : R[x_1, \dots, x_t, \dots] \longrightarrow A \quad x_i \mapsto y_i$$

is injective.

- (c) The R -subalgebra of A generated by \mathcal{S} is a polynomial ring over R (with “indeterminates” the elements of \mathcal{S}).
 (d) The collection of all monomials $y_1^{n_1} \dots y_t^{n_t}$ (where $y_i \in \mathcal{S}$ and $n_k \in \mathbb{Z}_{\geq 0}$) is **linearly independent** over R .

Solution: (a) \Rightarrow (b): This follows by definition. Choose $f \in \ker(\varphi)$. This implies that $f(y_1, \dots, y_t) = 0$. By definition of algebraic independence, $f = 0$, so φ has trivial kernel, and is injective.

(b) \Rightarrow (c): Since the above map is injective, it bijects onto its image. The image of φ is precisely the R -subalgebra generated by \mathcal{S} , and is of the form $R[\mathcal{S}]$, which is isomorphic to $R[x_1, \dots, x_t, \dots]$ where x_i are all indeterminates.

(c) \Rightarrow (d): We can consider these monomials in the R -subalgebra of A generated by \mathcal{S} . These monomials $y_1^{n_1} \dots y_t^{n_t}$ under the above bijection correspond to monomials $x_1^{n_1} \dots x_t^{n_t}$, where x_1, \dots, x_t are indeterminates. In the ring $R[x_1, \dots, x_t]$, these monomials are known to be linearly independent. Thus, $y_1^{n_1} \dots y_t^{n_t}$ are linearly independent in $R[\mathcal{S}]$, and are thus linearly independent over R .

(d) \Rightarrow (a): Notice that polynomials are just finite sums of monomials. Because monomials are linearly independent, it follows that if a sum of monomials evaluated on $y_i \in \mathcal{S}$ is zero, then the coefficients are zero, implying that the initial polynomial is identically the zero polynomial. Thus, \mathcal{S} is algebraically independent.

Problem 2. Find two different polynomial sub-algebras over which each finitely generated \mathbb{Q} -algebra is module finite.

- (a) $\mathbb{Q}[x, y, z]$.
 (b) $\mathbb{Q}[x, y, z]/\langle z^2 - xy \rangle$.

- (c) $\mathbb{Q}[x, y]/\langle xy \rangle$. [Hint: Try a simple linear change of coordinates].

Solution:

- (a) $\mathbb{Q}[x, y, z], \mathbb{Q}[x^2, y, z]$
 (b) $\mathbb{Q}[x, y], \mathbb{Q}[x^2, y]$
 (c) $\mathbb{Q}[x + y], \mathbb{Q}[x^2 + 2xy + y^2]$

Problem 3. Let $A = R[x_0, x_1, \dots, x_n]$ be a polynomial ring over R . For any $g_1, \dots, g_n \in R[x_0]$, consider the map

$$R[x_0, \dots, x_n] \xrightarrow{\phi} R[x_0, \dots, x_n] \quad x_i \mapsto \begin{cases} x_0 & i = 0 \\ x_i + g_i(x_0) & i > 0 \end{cases}.$$

- (a) Show that ϕ is an R -algebra automorphism.
 (b) When each g_i is some power of x_0 , show that $\phi(x_0 \dots x_n)$ is **monic** as a polynomial in x_0 .
 (c) When each $g_i = x_0^{D^i}$ for some $D > 0$, show that $\phi(rx_0^{a_0}x_1^{a_1} \dots x_n^{a_n})$ has leading term

$$rx_0^{a_0 + a_1 D + a_2 D^2 + \dots + a_n D^n}$$

as a polynomial in x_0 .

- (d) Fix $f \in R[x_0, \dots, x_n]$, and choose D larger than any exponent a_i on any x_i appearing in f . With g_i as in (c), show that $\phi(f)$ has leading term of the form rx_0^M as a polynomial in x_0 for some $M \in \mathbb{N}$ and non-zero $r \in R$. [Hint: Use the uniqueness of the Base D expansion of a natural number to show there is no troublesome cancellation.]

Solution:

- (a) It is clear that this is a homomorphism, as we've defined it on generators. We now verify that this is an automorphism by constructing an inverse map. Let $\psi^{-1} : R[x_0, \dots, x_n] \rightarrow R[x_0, \dots, x_n]$ be defined as follows. $\psi^{-1} : x_0 \mapsto x_0$, and for $i > 1$, $\psi^{-1} : x_i \mapsto x_i - g_i(x_0)$. It is clear that $\psi^{-1} \circ \psi(x_0) = \psi \circ \psi^{-1}(x_0) = x_0$. Furthermore, for $i > 1$, $\psi^{-1} \circ \psi(x_i) = x_i - g_i(x_0) + g_i(x_0) = x_i$, and $\psi \circ \psi^{-1}(x_i) = x_i + g_i(x_0) - g_i(x_0) = x_i$. Thus, ψ is an automorphism.
 (b) If this is the case, then

$$\psi(x_0 \dots x_n) = x_0 \prod_{i=1}^n (x_i + g_i(x_0)) = x_0 \prod_{i=1}^n (x_i + x_0^{D^i})$$

The leading term as a polynomial over x_0 has an exponent which is just the sum of all the n_i terms, and 1, since this is the maximal possible degree that x_0 term can attain in this product. Thus,

$$x_0 \prod_{i=1}^n (x_i + x_0^{n_i}) = x_0^{1+n_1+\dots+n_n} + \{\text{lower order } x_0 \text{ terms}\}$$

Implying that $\psi(x_0 \dots x_n)$ is a monic in x_0 .

(c) as before,

$$\phi(rx_0^{a_0} x_1^{a_1} \dots x_n^{a_n}) = rx_0^{a_0} \prod_{i=1}^n (x_i + g_i(x_0))^{a_i} = rx_0^{a_0} \prod_{i=1}^n (x_i + x_0^{D^i})^{a_i}$$

The leading term contains the product of all the $x_0^{D^i}$ terms, as this is the maximal possible degree that any x_0 term attains in the product. Thus, we have:

$$rx_0^{a_0} \prod_{i=1}^n (x_i + x_0^{D^i})^{a_i} = rx_0^{a_0+a_1D^1+a_2D^2+\dots+a_nD^n} + \{\text{lower order } x_0 \text{ terms}\}$$

(d) Suppose

$$f = \sum_{j=1}^m r_{(j)} x_0^{a_0^{(j)}} x_1^{a_1^{(j)}} \dots x_n^{a_n^{(j)}}$$

Then,

$$\psi(f) = \psi \left(\sum_{j=1}^m r_{(j)} x_0^{a_0^{(j)}} x_1^{a_1^{(j)}} \dots x_n^{a_n^{(j)}} \right) = \sum_{j=1}^m \psi \left(r_{(j)} x_0^{a_0^{(j)}} x_1^{a_1^{(j)}} \dots x_n^{a_n^{(j)}} \right)$$

Citing the previous part, we know that

$$\psi(f) = \sum_{j=1}^m r_{(j)} x_0^{a_0^{(j)}+a_1^{(j)}D^1+a_2^{(j)}D^2+\dots+a_n^{(j)}D^n} + \{\text{lower order } x_0 \text{ terms}\}$$

Thus, assuming there are no cancellations, the leading term is of the form $r_{(j)} x_0^{a_0^{(j)}+a_1^{(j)}D^1+a_2^{(j)}D^2+\dots+a_n^{(j)}D^n}$ for some maximal j . To show there are no cancellations, suppose that $a_0^{(j)} + a_1^{(j)}D^1 + a_2^{(j)}D^2 + \dots + a_n^{(j)}D^n = a_0^{(i)} + a_1^{(i)}D^1 + a_2^{(i)}D^2 + \dots + a_n^{(i)}D^n$ (i.e. two leading terms from different monomials have the same power). If this is the case, by uniqueness of the base D expansion of a natural number, $a_k^{(i)} = a_k^{(j)}$ for each $k \leq n$, implying that they were from the same monomial after all. Thus, we cannot have any troublesome cancellations, as each leading term from different monomials has different powers. In particular, there are finitely many monomials, so we can choose a unique maximal leading term. Thus, the leading term of $\psi(f)$ is of the form $r_0 x^M$ for some $M \in \mathbb{N}$.

Problem 4. PROOF OF NOETHER NORMALIZATION: Let R be an arbitrary domain, and let A be a finitely generated R -algebra. We will prove the general Noether Normalization statement by induction on the number of generators for the R -algebra A .

- (a) Check the case where the empty set generates A over R .
- (b) Assume we have proven the statement for R -algebras generated by n elements. Let A be generated over R by $\theta_0, \dots, \theta_n$. Explain why the statement holds for A if the θ_i are algebraically independent.
- (c) If $\{\theta_0, \dots, \theta_n\}$ satisfy some polynomial in $R[x_0, \dots, x_n]$, use (3) to show how to replace the θ_i by some other set of generators for A over R which satisfy a *monic* polynomial in θ_0 with coefficients in $R[\frac{1}{c}][\theta_1, \dots, \theta_n]$ for some non-zero c .
- (d) Use induction to complete the proof of the general statement of Noether Normalization.
- (e) Deduce Noether Normalization over a field.

Solution:

- (a) In this case, notice that $R[\emptyset] = R$, thus there exists module finite extension $R \rightarrow A$, where R is a (trivial) polynomial ring over R .
- (b) First, recall that the composition of module finite extensions is module finite. If A is generated over R by $\theta_0, \dots, \theta_n$, it follows that we have a surjection $R[x_0, \dots, x_n] \rightarrow A$ where $x_i \mapsto \theta_i$. If θ_i are all linearly independent, then $R[x_1, \dots, x_n] \cong R[\theta_1, \dots, \theta_n] \rightarrow A$ is an extension, and thus a module finite extension, and we can conclude.
- (c) Suppose θ_i satisfy f . By passing our terms through the automorphism in the previous part, we can replace $\theta_i \mapsto \theta_i + \theta_0^{M_i}$ for sufficiently large M such that $f(\theta_0, \dots, \theta_n) = r\theta_0^M + \{\text{lower order } \theta_0 \text{ terms}\}$. While this is not monic over $R[\theta_1, \dots, \theta_n]$, this is in fact monic in θ_0 over $R[1/r][\theta_1, \dots, \theta_n]$.
- (d) Part (c) showed that $R[1/r][\theta_1, \dots, \theta_n] \rightarrow A[1/r][\theta_0, \dots, \theta_n] = A[1/r]$ is an integral (and in fact, a module finite) extension. By the inductive step, since $R[1/r][\theta_1, \dots, \theta_n]$ is generated by n -generators, there exists a map $R[1/r][x_1, \dots, x_\ell] \rightarrow R[1/r][\theta_1, \dots, \theta_n]$ such that it defines a module finite extension. This yields the sequence of module finite extensions

$$R[1/r][x_1, \dots, x_\ell] \rightarrow R[1/r][\theta_1, \dots, \theta_n] \rightarrow A[1/r]$$

Since the composition of module finite extensions is module finite, it follows that we've constructed a module finite extension from a polynomial ring to an algebra on $n + 1$ generators, completing the inductive step.

- (e) Over a field, $r \neq 0$ is already unit, so there is no need to localize. In other words, if R is a field, $R[1/r][x_1, \dots, x_n] = R[x_1, \dots, x_n]$. The field case of Noether Normalization follows.

Problem 5. Let L be a field extension of K , and assume L is finitely generated as an K -algebra. Use Noether Normalization to prove that L is a finite algebraic extension of K . [Hint: Recall that you've found a lower bound on the dimension of a polynomial ring over a field.]

Solution: There exists Noether normalization of L such that $K \subset K[y_1, \dots, y_d] \subset L$. We already know that $K[y_1, \dots, y_d]$ has dimension $\geq d$, but it follows that since L is a field, L has dimension 0. In particular, $K[y_1, \dots, y_d] \rightarrow L$ is an integral extension, so $\dim(L) = \dim(K[y_1, \dots, y_d])$. Since $0 = \dim(L) \geq d$, it follows that $d = 0$. Thus, L is a (finite) algebraic extension of K .

Problem 6. THE NULLSTELLENSATZ, REVISITED. Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over an algebraically closed field K . Let m be a maximal ideal of R . Prove that m can be generated by elements of the form $x_i - \lambda_i$ for $\lambda_i \in K$. [Hint: Consider the extension of fields $K \hookrightarrow R \twoheadrightarrow R/m$; Use (5)]

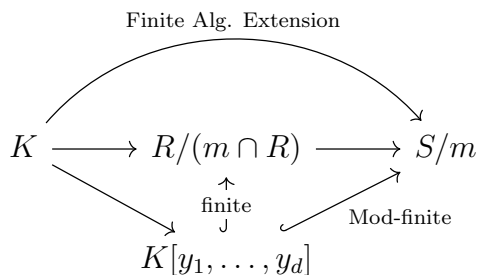
Solution: We follow the hint. Considering the extension of fields $K \hookrightarrow R \twoheadrightarrow R/m$, we can conclude from the previous problem that R/m is an algebraic extension of K . Since K is algebraically closed, $R/m \cong K$. It follows that m must be of the form $m = (x_1 - \lambda_1, \dots, x_n - \lambda_n)$.

Problem 7. Let R and S be finitely generated K -algebras, and let $R \rightarrow S$ be a K -algebra homomorphism.

- (a) Prove that every maximal ideal of $\text{Spec } S$ contracts to a maximal ideal of R . That is, the map of Spectra restricts to a map of maxSpec . [Hint: Consider $K \hookrightarrow R/(m \cap R) \hookrightarrow S/m$]
- (b) Give an example of a ring map for which some maximal ideal contracts to a non-maximal ideal.

Solution:

- (a) Choose $m \in \text{MaxSpec}(S)$. We follow the hint and consider $K \hookrightarrow R/(m \cap R) \hookrightarrow S/m$. It is sufficient to verify that $R/(m \cap R)$ is a field. We know that $K \rightarrow S/m$ is a finite algebraic extension from problem 5. Furthermore, taking the Noether Normalization yields the following diagram:



It follows that $R/(m \cap R) \rightarrow S/m$ is a module finite extension. In particular, they are integral. Thus, $\dim(S/m) = \dim(R/(m \cap R)) = 0$. Since $R/(m \cap R)$ is a domain, we can conclude that $R/(m \cap R)$ is a field, and that $m \cap R$ is a maximal ideal in R .

- (b) Consider the inclusion $\mathbb{Z} \rightarrow \mathbb{Q}$. \mathbb{Q} has 0 as its maximal ideal, but $(0) \cap \mathbb{Z} = (0)$ is not a maximal ideal of \mathbb{Z} .

Math 614: Commutative Algebra

IBL Lab Notebook

Class 11: October 10th

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Collaborators: Will Garland

Problem 1. Using the theorems above, compute the following:

- (a) Dimension of $\mathbb{Q}[x, y, \sqrt{x^2 + y^5}]$ (a subring of $\overline{\mathbb{Q}(x, y, z)}$)
- (b) Dimension of the localization $\mathbb{Q}[x, y, \frac{1}{f}]$, where f is some non-zero polynomial in $\mathbb{Q}[x, y]$.
[Hint: Use Theorem 3.]
- (c) Dimension of $\mathbb{R}[x, y, z]/\langle x^3 + yz^3 \rangle$.
- (d) Dimension of $\mathbb{C}[x, y, z]_{m_p}$ where m_p is the ideal of polynomial functions vanishing at the point $(1, 1, 1) \in \mathbb{C}^3$.
- (e) Dimension of $\mathbb{F}_{49}[x, y, z]_P$ where P is a height two prime.
- (f) The height of $\langle x_1, \dots, x_i \rangle$ in $K[x_1, \dots, x_d]$. [Hint: Use Theorem 3.]

Solution:

- (a) 2. This is an integral extension of $\mathbb{Q}[x, y]$, as $\sqrt{x^2 + y^5}$ satisfies the polynomial $z^2 - x^2 - y^5 \in \mathbb{Q}[x, y, z]$.
- (b) $0 \subset (x - 1) \subset (x - 1, y - 1)$ terminates in a maximal ideal of height 2, so this has dimension 2.
- (c) Integral extension of $\mathbb{R}[x, y]$, so has dimension 2.
- (d) Has unique maximal ideal $m_p \mathbb{C}[x, y, z]_{m_p}$ of height 3, so has dimension 3.
- (e) Has unique maximal ideal of height 2, so has height 2.
- (f) Height i , since taking a chain from this to (x_1, \dots, x_d) will have $d - i$ inclusion, and that is an ideal of height d .

Problem 2. PROOF OF THEOREM 1. Let R be a UFD.

- (a) Show that every non-zero prime ideal of R contains an irreducible element.
- (b) Prove that every height one prime is principal. [Hint: Remember irreducible elements are prime!]
- (c) Prove conversely that non-zero prime ideal that is principal must have height one.

Solution:

- (a) Choose $r \in P \subset R$ prime. Decompose $r = r_1^{n_1} \dots r_m^{n_m}$. This is finite and unique, since R is a UFD. Since P is prime, $\exists r_i \in P$, where r_i is irreducible.
- (b) Suppose that P is a prime ideal of height 1. From above, we know there exists an irreducible $r_i \in P$. Thus, $0 \subsetneq (r_i) \subset P$. (r_i) is prime, since r_i is irreducible. Since P is height 1, it follows that $(r_i) = P$.
- (c) Choose (p) a principal prime ideal, which is generated by an irreducible element p . Suppose that (p) is of height ≥ 1 . Then, $\exists A \in \text{Spec}(R)$ such that $0 \subsetneq A \subsetneq (p)$. From part (a), we can choose (q) in A irreducible, implying that $\subsetneq (q) \subset A \subsetneq (p)$. This would imply that $q|p$, a contradiction to the irreducibility of p . Thus, (p) has height 1.

Problem 3. PROOF OF THEOREM 2. Let $A = K[x_1, \dots, x_d] \subset R$ be a module finite extension, where A is a polynomial ring.

- (a) Show that $\dim A \geq d$ by exhibiting an explicit chain of prime ideals. [Hint: You did this before.]
- (b) Explain why, to prove Theorem 2, it suffices to show that $\dim A \leq d$.
- (c) Explain why, to prove Theorem 2, it suffices to show that for any prime ideal P_1 of height one in A , $\dim A/P_1$ has dimension $d-1$. [Hint: What happens if $P_0 \subset P_1 \subset \dots \subset P_{d+1}$ is a chain of primes of length more than d in A ?]
- (d) For $P_1 \in \text{Spec} A$ of height one, explain why after a change of variables, P_1 can be assumed to be generated by some polynomial f which is monic in x_d with coefficients in $k[x_1, \dots, x_{d-1}]$. [Hint: Use a lemma from last time.]
- (e) Use (d) to show that A/P_1 is integral over a polynomial ring in $d-1$ variables.
- (f) Complete the proof of Theorem 2 by induction.

Solution:

- (a) $(x_1) \subsetneq (x_1, x_2) \subsetneq \dots \subsetneq (x_1, \dots, x_d)$
- (b) Because R is an integral extension of A , $\dim(A) = \dim(R)$. Since we already know that $\dim(A) \geq d$, if we show $\dim(A) \leq d$, then $\dim(A) = d$, so $\dim(R) = d$.
- (c) Recall that $\text{Spec}(A/P_1) = \{Q \in \text{Spec}(A) \mid P_1 \subset Q\}$. If $\text{Spec}(A/P_1)$ has dimension $d-1$, then saturated chains in $\text{Spec}(A)$ which contain P_1 must be of length $d-1$. Since P_1 is of height 1, it follows that any saturated chain with P_1 in the chain must be of length d . Since this is true for any P_1 of height 1, it is sufficient to verify that any saturated chain must contain an ideal of height 1 in the chain.

First, recall that since K is a field, it is a UFD, so $K[x_1, \dots, x_d] = A$ is a UFD. Thus, any prime ideal P contains an irreducible element r . If P is not height 1, then $0 \subsetneq (r) \subsetneq (P)$ is a valid chain to P . If a chain $0 \subsetneq P_1 \subsetneq \dots \subsetneq P_t$ is a saturated chain of prime ideals that does not contain a height 1 ideal, notice that we can extend it to the chain $0 \subsetneq (r) \subsetneq P_1 \subsetneq \dots \subsetneq P_t$, violating maximality. Thus, any saturated chain must contain a height 1 ideal, implying from above that any saturated chain must be of length $\leq d$.

- (d) As we proved in the previous part, A is a UFD. In particular, $P_1 = (f)$ for some irreducible $f \in K[x_1, \dots, x_d]$. In the previous worksheet, when we proved Noether Normalization, we've deduced an automorphism (which is just a change of variables) that takes a polynomial and returns a polynomial with leading term rx_d^M , for some $r \in K, M \in \mathbb{N}$. Since K is a field, it follows that this polynomial is monic. Thus, we can rewrite P_1 as $P_1 = (\psi(f))$, where f is monic in x_d with coefficients in $K[x_1, \dots, x_{d-1}]$.
- (e) It follows from above that A/P_1 is integral over $K[\bar{x}_1, \dots, \bar{x}_{d-1}]$, where \bar{x}_i is the projection of x_i in A/P_1 .
- (f) We induct on the number of variables d . When $d = 1$, we know $k[x]$ is a PID of height 1. For the inductive step, we've shown that A/P_1 is of the same dimension as a $d - 1$ variable polynomial ring, which we know via the inductive hypothesis to have dimension $d - 1$. Thus, A/P_1 has dimension $d - 1$, completing the inductive step by part (c).

Problem 4. Let $R = K[x, y, z]/\langle xz, yz \rangle$. Find two saturated chains in $\text{Spec } R$ of different lengths. Why does this not contradict Theorem 3?

Solution: Notice that the chains $(z) \subsetneq (z, x-1) \subsetneq (z, x-1, y-1)$ and $(x, y) \subsetneq (x, y, z-1)$ are both saturated chains (the first one is the "normal" way to construct a saturated chain, and the second one follows from the fact that $(x), (y)$ are not prime in R). This does not violate theorem 3, since R is NOT a domain. In particular, $x \neq 0, z \neq 0$ but $xz = 0$.

Problem 5. Let $R = K[x, y, z, w]/\langle xz - yw \rangle$. Let P be the ideal generated by the classes of x and y .

- (a) Show that P is prime of height one. [Hint: Use Theorem 2 on the ring $K[x, y, z, w]$.]
- (b) (*) Show that P not principal. [Hint: If principal, lift to a statement in the UFD $K[x, y, z, w]$ and use the grading.¹]
- (c) Why does this not violate Theorem 1? Demonstrate explicitly an instance of the failure of unique factorization in R .

¹If $\langle x, y \rangle = \langle f \rangle$ for some f , note that lifting f to the polynomial ring, it can not have a non-zero constant term and it must have a non-zero linear term. Using the UFD property, what is this linear term?

Solution:

- (a) P is prime because it is generated by irreducible elements. Notice that the height of P is the same as the dimension of R_P .

$$R_P = (K[x, y, z, w]/(xz - yw))_{(x,y)} = (K(z, w)[x, y])/\overline{(xz - yw)}$$

Where $\overline{(xz - yw)}$ denotes the projection into the localization. This equality holds because localization commutes with quotients. Notice that $\overline{(xz - yw)} = \overline{(x - y(w/z))}$, since z is unit. This implies that $(K(z, w)[x, y])/\overline{(xz - yw)}$ is an integral extension of $K(z, w)[y]$, which is of dimension 1, as it is a 1 variable polynomial ring over a field. Thus, $\dim(R_P) = 1$, implying that P is of height 1.

- (b) (*)

- (c) R is not a UFD, so this does not contradict theorem 1.

Problem 6. DIMENSION AND TRANSCENDENCE DEGREE. Recall that if L/K is a field extension, then a **transcendence basis** for L over K is a maximal set of elements in L that are algebraically independent over K . These always exist (by Zorn's lemma) and have the same cardinality, which is called the **transcendence degree of L/K** .

Use Noether Normalization to prove that if R is a domain finitely generated over a field K , then $\dim R$ is equal to the transcendence degree of L/K where L is the fraction field of R .

Solution: Let $K[x_1, \dots, x_d] \rightarrow R$ be the Noether Normalization. We wish to show that $L = \text{Frac}(R)$ has transcendence degree d . Let $L' = K(x_1, \dots, x_d) = \text{Frac}(K[x_1, \dots, x_d])$. It follows that L is an algebraic extension of L' since R is a module finite extension of $K[x_1, \dots, x_d]$. We also know $\text{trdeg}(L') = d$. Since extending algebraically does not change transcendence degree, it follows that $d = \text{trdeg}(L') = \text{trdeg}(L)$.

Problem 7. PROOF OF THEOREM 3. Let $A = K[x_1, \dots, x_d] \subset R$ be a Noether normalization for a domain R finitely generated over K .

- (a) Fix any maximal ideal m in R and consider a *saturated* chain of prime ideals $\langle 0 \rangle = Q_0 \subset Q_1 \subset \dots \subset Q_k = m$ in R . Explain why $k \leq d$ and why, to prove Theorem 3, it suffices to show that $d = k$.
- (b) Explain how we know Q_1 has height one, and why to prove Theorem 3, it suffices to show that the induced chain $\langle 0 \rangle = Q_1/Q_1 \subset Q_2/Q_1 \subset \dots \subset Q_k/Q_1$ (in the quotient ring R/Q_1) has length $d - 1$.
- (c) Let $P_1 = Q_1 \cap A$. Show that P_1 is height one. [Hint: Recall (a corollary of) Going Down.]
- (d) Use the same method as in the proof of Theorem 2 to show that after changing variables in A , we have module finite extensions $K[x_1, \dots, x_{d-1}] \subset A/P_1 \subset R/Q_1$, and complete the proof of Theorem 3 by induction.

Solution:

- (a) We already know that $\dim(R) = d$, so any chain in R must have length at most d . In particular the chain $\langle 0 \rangle = Q_0 \subset Q_1 \subset \dots \subset Q_k = m$ must be of length $\leq d$, so $k \leq d$. If we show that $d = k$, since we've picked the maximal ideal arbitrarily, we know that all maximal ideals have a saturated chain of length d . In particular, all maximal ideals have saturated chains of the same length.
- (b) In our proof for problem 3(b), we've shown that every saturated chain must contain an ideal of height 1. In particular, if we show that the chain to Q_k/Q_1 has length $d-1$, then this must lift to a chain $Q_1 \subset Q_2 \subset \dots \subset Q_k$, where Q_i/Q_1 lifts to Q_i for $i > 1$, and 0 lifts to Q_1 . This is because $Q_i/Q_1 \in \text{Spec}(R/Q_1)$, which corresponds to all the prime ideals containing Q_1 . Thus, if $\langle 0 \rangle = Q_1/Q_1 \subset Q_2/Q_1 \subset \dots \subset Q_k/Q_1$ has length $d-1$, the lift into R must have length $(d-1) + 1 = d$.
- (c) $A \rightarrow R$ is module finite (and in particular, an integral) extension where $A = K[x_1, \dots, x_d]$ is a UFD and is thus normal. Thus, from a corollary of Going Down we proved, we know that $\forall Q \in \text{Spec}(R)$, Q and P have the same height for $P = Q \cap R$. Since Q_1 has height 1, we know that P_1 has height 1.
- (d) Because P_1 is a height 1 ideal in a UFD, it is principal. Suppose $P_1 = (f)$ for some $f \in K[x_1, \dots, x_d]$. As before, we can do a change of variables such that f is a monic polynomial in x_d with coefficients in $K[x_1, \dots, x_{d-1}]$. It follows that we have the module finite extensions $K[x_1, \dots, x_{d-1}] \subset A/P_1 \subset R/Q_1$.

We now proceed with induction: When $d = 1$, $K[x]$ is a PID, where all nonzero prime ideals are maximal. Since all prime ideals in PIDs have height 1, it follows that all maximal ideals have height 1, and are the same length.

For the inductive step, suppose that all maximal ideals in dimension $d-1$ algebras have height $d-1$. If this is the case, we've shown above that the chain $\langle 0 \rangle = Q_1/Q_1 \subset Q_2/Q_1 \subset \dots \subset Q_k/Q_1$ is in a dimension $d-1$ algebra, and thus has height $d-1$. From (b), we can then conclude that $0 = Q_0 \subset \dots \subset Q_k = m$ has length d for any maximal ideal m , completing the inductive step.

Problem 8. Let $R = \mathbb{Z}_{(p)}[t]$ where p is a prime integer. Observe that R is a UFD.

- (a) Prove that $\langle pt - 1 \rangle$ is a maximal ideal of height one.
- (b) Prove that $\langle p, t \rangle$ is a maximal ideal of height at least two.
- (c) Why does this not contradict Theorem 3?

Solution:

- (a) We verify that this ideal is maximal. Notice that

$$\mathbb{Z}_{(p)}[t]/\langle pt - 1 \rangle \cong \mathbb{Z}_{(p)}[1/p] \cong \mathbb{Q}$$

Since quotienting by $(pt - 1)$ yields a field, $(pt - 1)$ is maximal. In particular, this implies that $(pt - 1)$ is prime. We've shown that any non-zero principal prime ideal must have height 1. Thus, $(pt - 1)$ is a height 1 maximal ideal.

(b) $\mathbb{Z}_p[t]/(p, t) \cong \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$, and since $p\mathbb{Z}_{(p)}$ is the unique maximal ideal of $\mathbb{Z}_{(p)}$, it follows this is a field, implying that (p, t) is maximal. Furthermore, this ideal has a height of at least 2, since we have a chain $0 \subsetneq (t) \subsetneq (p, t)$.

(c) R is not finitely generated over a field, and thus does not contradict Theorem 3.

Math 614: Commutative Algebra

IBL Lab Notebook

Class 12: October 17th

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Problem 1. Prove corollary 1.

Solution: Let $K[x_1, \dots, x_d] \rightarrow R$ be the Noether Normalization of R . We wish to show that $L = \text{Frac}(R)$ has transcendence degree d . Let $L' = K(x_1, \dots, x_d) = \text{Frac}(K[x_1, \dots, x_d])$. It follows that L is an algebraic extension of L' since R is a module finite extension of $K[x_1, \dots, x_d]$. We also know $\text{trdeg}(L') = d$. Since extending algebraically does not change transcendence degree, it follows that $d = \text{trdeg}(L') = \text{trdeg}(L)$.

Problem 2. Consider the ring $R = K[X, t]$ and the prime ideal $P = (X)$. Using the results above:

- Compute the dimension of R , height of P , and dimension of R_P .
- Compute the maximal cardinality of a K -algebraically independent set in R_P .
- Why does this not contradict corollary 1?
- Consider the polynomial ring $K(t)[x]$. What is the maximal cardinality of an algebraically independent set in R over $K(t)$? Over K ?

Solution:

- $\dim(R) = 2$, P is principal prime ideal in a UFD and thus height 1. $\dim(R_P)$ is the same as the height of P , so $\dim(R_P) = 1$.
- By Problem 1, since $\dim(R_P) = 1$, the maximal cardinality must be 1.
- With corollary 1, since $\text{Frac}(R_P) = \text{Frac}(R)$, we would expect the answer to be 2. However, R_P is not finitely generated over K , so we cannot apply the corollary.
- Using the corollary, we know that the maximal cardinality over $K(t)$ is 1. Over K , the maximal cardinality is infinite, since $\{1/t^p\}_{p \in \mathbb{N} \text{ prime}}$ are algebraically independent.

Problem 3. Find the dimension of the following K -subalgebras of $K(x, y, z)$:

- $K[x, xy, xz]$

- (b) $K[xy, 1/xy]$
 (c) $K[x^4, x^2y^{11}, xz^{13}, 1/xyz]$
 (d) $K[x^2y^2, xy^2z, y^2z^2]$

Solution:

- (a) 3, as x, xy, yz are algebraically independent.
 (b) 1, as the fraction field is just $K(xy)$, which has transcendence degree 1.
 (c)

$$\begin{aligned} \text{trdeg}(K(x^4, x^2y^{11}, xz^{13}, 1/xyz)) &= \text{trdeg}(K(x, x^2y^{11}, xz^{13}, 1/xyz)) \\ &= \text{trdeg}(K(x, y^{11}, z^{13}, 1/xyz)) \\ &= \text{trdeg}(K(x, y, z)) = 3 \end{aligned}$$

Thus, the dimension is 3.

- (d)
- $$\text{trdeg}(K(x^2y^2, xy^2z, y^2z^2)) = \text{trdeg}(K(xy, yz)) = 2$$

Thus, the dimension is 2.

Problem 4. Proof of corollary 2. Let R be a finitely generated K -algebra.

- (a) Fix prime ideals $P \subset Q$ in R . Given a saturated chain of primes from P to Q of length c , prove there is a saturated chain of primes in the domain R/P of length c from (0) to Q/P .
 (b) Prove corollary 2 follows from the theorem above proved last time.
 (c) Assume R is a domain. For $P \in \text{Spec}(R)$, Prove that the height $P = \dim(R) - \dim(R/P)$.
 (d) Find a counterexample to (c) when R is not a domain.

Solution:

- (a) Let $P = P_1 \subset \dots \subset P_c = Q$ be the chain in question. Recall that $\text{Spec}(R/P) = \{Q \in \text{Spec}(R) \mid P \subset Q\}$. Thus, since every prime ideal in the chain contains P , we can lift the chain to R/P . $P_1 = P$ lifts to zero, and the rest lift to nonzero ideals $0 \subsetneq P_2/P_1 \subsetneq \dots \subsetneq P_c/P_1 = Q/P$.

- (b) Suppose $P \subset Q \subset M$ for M a maximal ideal, and P, Q prime. We choose a saturated chain of length c from P to Q , and a saturated chain of length d from Q to M . The length d chain corresponds to a length d chain from 0 to M/Q in $\text{Spec}(R/Q)$. The theorem from last time told us that this implies that all saturated chains from 0 to M/Q , and thus from Q to M , have length d . Similarly, the saturated chain from P to M has length $c + d$. This corresponds to a saturated chain of length $c + d$ in $\text{Spec}(R/P)$ from 0 to M . This implies that all saturated chains from 0 to M in $\text{Spec}(R/P)$ are of length $c + d$. Thus, if all saturated chains $P \subset M$ is of length $c + d$, and all saturated chains $Q \subset M$ are of length d , it follows that all saturated chains $P \subset Q$ are of length c .
- (c) Consider any saturated chain $0 \subsetneq \dots \subsetneq P \subsetneq \dots \subsetneq M$ for M a maximal ideal. Any saturated chain $0 \subset M$ has length $\dim(R)$, as we've shown before. Any saturated chain $P \subset M$ has length $\dim(R/P)$ from corollary. Thus, it follows that any saturated chain $0 \subset P$ has length $\dim(R) - \dim(R/P)$, so the height of P is $\dim(R) - \dim(R/P)$.

Problem 5. Consider the following examples. Why do they not contradict Theorem 1?

- (a) Let $R = K[x]/(x^3)$. Show that the ideal (x^2) is NOT the intersection of all maximal ideals containing it.
- (b) Let $R = K[x, y]_m$, where m is an arbitrary maximal ideal of $K[x, y]$. Show that most radical ideals are not the intersection of maximal ideals containing them. Indeed, only one ideal has that property: which one?

Solution:

- (a) Note that each maximal ideal containing (x^2) would also need to contain (x) , since $(x^2) \subset (x)$. However, if this were the case, (x) would be contained in the intersection of all the maximal ideals containing (x^2) . If (x^2) was this intersection, then $x \in (x^2)$, which is a contradiction. This does not contradict Theorem 1 because (x^2) is not radical.
- (b) $K[x, y]_m$ is a local ring with unique maximal ideal $mK[x, y]_m$. Thus, for an ideal to be equal to the intersection of the maximal ideals containing it, it must be precisely $mK[x, y]_m$, since there is only 1 maximal ideal in the (trivial) intersection.

Problem 6. Polynomials as functions. **Lemma:** Assume K is infinite. For any nonzero polynomial in $K[x_1, \dots, x_d]$, the function $K^d \rightarrow K$ sending $p \mapsto f(p)$ is a nonzero function of K^d .

- (a) Prove the lemma when $d = 1$.
- (b) use induction to prove the lemma in general. [Hint: write $f \in K[x_1, \dots, x_d]$ as a polynomial in 1 variable with coefficients as polynomials in the others.]

- (c) For K finite, find a non-zero polynomial in $K[x]$ which determines the zero function on K .

Solution:

- (a) We want to show that the map $K \rightarrow K$ sending $p \mapsto f(p)$ for $f \in K[x]$ is nonzero for any choice of (nonzero) f . Well, suppose $p \mapsto f(p)$ was the zero map. If this was the case, then $f(x) = 0 \forall x \in K$, implying that f is the zero map, a contradiction.
- (b) Note that our proof above did not use the fact that K is a field. Thus, we write $f \in K[x_1, \dots, x_{d-1}][x_d]$. Notice that the map $K \rightarrow K[x_1, \dots, x_{d-1}]$ which maps $p \mapsto f'(p)$ is nonzero for any choice of f , where $f'(p) = f(x_1, \dots, x_{d-1}, p)$. Now, notice that $f'(p)$ is a polynomial in $K[x_1, \dots, x_{d-1}]$. This then induces a map $K^{d-1} \rightarrow K$ which sends $q = (q_1, \dots, q_{d-1}) \mapsto f'(q)$. By the inductive hypothesis, this map is nonzero, so the map $K^d \rightarrow K$ which sends $(q_1, \dots, q_{d-1}, p) \mapsto f(q_1, \dots, q_{d-1}, p)$ is nonzero. Since p, q were chosen arbitrarily, as was f , we can conclude.
- (c) If K is finite, then $|K| = q$, for $q = p^n$ for some power of p . Consider the polynomial $x^q - x$. This is a nonzero polynomial in $\mathbb{F}_q[x]$ which is identically zero under the map $p \mapsto f(p)$, since $x^q \equiv x \pmod{p}$.

Problem 8. Let R be a Noetherian ring.

- (a) Show that the arbitrary intersection of radical ideals is radical.
- (b) Prove the corollary to Theorem 1. [Hint: Use the fact that $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$.]

Solution:

- (a) Let $\{I_\lambda\}_{\lambda \in \Lambda}$ be a collection of radical ideals.

$$g = f^n \in \bigcap_{\lambda \in \Lambda} I_\lambda \Rightarrow f^n \in I_\lambda, \forall \lambda \in \Lambda \Rightarrow f \in I_\lambda, \forall \lambda \in \Lambda \Rightarrow f \in \bigcap_{\lambda \in \Lambda} I_\lambda$$

- (b) We already know that the radical \sqrt{I} is the intersection of all maximal ideals containing it. Recall that $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$. This implies that the prime ideals that contain I are precisely those that contain \sqrt{I} . Since maximal ideals are prime, we know they contain \sqrt{I} if and only if they also contain I . Thus,

$$\sqrt{I} = \bigcap_{m \in \mathbb{V}(\sqrt{I})} m = \bigcap_{m \in \mathbb{V}(I)} m$$

Math 614: Commutative Algebra**IBL Lab Notebook****Class 13: October 22nd***Name: Vignesh Jagathese**Collaborators: Malavika Mukundan, Will Garland*

Problem 1. Review the following facts about the Zariski Topology on $\text{Spec}(R)$:

- (a) The closed sets are $\mathbb{V}(\{f_\lambda\}_{\lambda \in \Lambda}) = \mathbb{V}(\langle \{f_\lambda\}_{\lambda \in \Lambda} \rangle) = \mathbb{V}(\sqrt{\langle \{f_\lambda\}_{\lambda \in \Lambda} \rangle})$.
- (b) For any subsets $I \subset J$, $\mathbb{V}(I) \supset \mathbb{V}(J)$.
- (c) $\bigcap_{\lambda \in \Lambda} \mathbb{V}(I_\lambda) = \mathbb{V}(\bigcup_{\lambda \in \Lambda} I_\lambda)$.
- (d) $\bigcap_{i=1}^t \mathbb{V}(I_i) = \mathbb{V}(I_1 \cap \dots \cap I_t) = \mathbb{V}(I_1 \dots I_t)$.
- (e) If $N \subset R$ denotes the Nilradical of R , then $\mathbb{V}(N) = \text{Spec}(R)$.
- (f) For any ideal I , contraction for the quotient map $R \rightarrow R/I$ induces a homeomorphism $\text{Spec}(R/I) \cong \mathbb{V}(I)$, where $\mathbb{V}(I) \subset \text{Spec}(R)$ has the subspace topology.

Solution:

- (a) Since $S \subset \langle S \rangle$, by the next part we have that $\mathbb{V}(\langle S \rangle) \subset \mathbb{V}(S)$. Now, we choose $P \in \mathbb{V}(S)$. Thus, $S \subset P$. Since P is an ideal, it follows that if any elements are in P , then their products and sums are in P . Thus, $\langle S \rangle \subset P$, so $P \in \mathbb{V}(\langle S \rangle)$.

$I \subset \sqrt{I}$, so $\mathbb{V}(\sqrt{I}) \subset \mathbb{V}(I)$. For the converse, choose $P \in \mathbb{V}(I)$. Thus, $I \subset P$. We wish to show that $\sqrt{I} \subset P$. Choose $f \in \sqrt{I}$. $f^n \in I \subset P$ for some n . Well, since P is prime, and because $f^n = f^{n-1}f$, either $f \in P$ or $f^{n-1} \in P$. If $f^{n-1} \in P$, repeat the process with f, f^{n-2} . It follows from this logic that $f \in P$, so $\sqrt{I} \subset P$, so $P \in \mathbb{V}(\sqrt{I})$, so $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$.

- (b) Choose $P \in \mathbb{V}(J)$. Then $J \subset P$, implying that $I \subset J \subset P$, so $P \in \mathbb{V}(I)$.
- (c)

$$P \in \bigcap_{\lambda \in \Lambda} \mathbb{V}(I_\lambda) \iff I_\lambda \subset P, \forall \lambda \in \Lambda \iff \bigcup_{\lambda \in \Lambda} I_\lambda \subset P \iff P \in \mathbb{V}\left(\bigcup_{\lambda \in \Lambda} I_\lambda\right)$$

- (d)

$$P \in \mathbb{V}(S_1 \dots S_t) \iff S_1 \dots S_t \subset P \iff \exists i, S_i \subset P \iff P \in \mathbb{V}(S_1) \cup \dots \cup \mathbb{V}(S_t)$$

The claim $S_1 \dots S_t \subset P \iff \exists i, S_i \subset P$ follows from P being prime.

(e) Recall that the nilradical is the intersection of all prime ideals in R . In other words,

$$\mathbb{V}(N) = \mathbb{V}\left(\bigcap_{P \in \text{Spec}(R)} P\right) = \bigcup_{P \in \text{Spec}(R)} \mathbb{V}(P) = \text{Spec}(R)$$

(f) For $\pi : R \rightarrow R/I$, and $P \in \text{Spec}(R/I)$, this map is defined as $\pi^* : P \mapsto \pi^{-1}(P)$, and is injective because if $P \in \ker(\pi^*)$, then $\pi^{-1}(P) = 0 \Rightarrow P = \pi(0) \Rightarrow P = 0$. Thus, π^* has trivial kernel. In particular, π^* defines an isomorphism onto the image. The image is of the form

$$\text{Spec}(R/I) \cong \text{im}(\pi^*) = \{P \in \text{Spec}(R) \mid I \subset P\} = \mathbb{V}(I)$$

Problem 2. Using the theorems above, which of the following are irreducible topological spaces? Count the number of components for those that are not.

- (a) $\text{Spec}(\mathbb{Z})$
- (b) $\text{Spec}(\mathbb{Z}/24\mathbb{Z})$
- (c) $\text{Spec}(\mathbb{Z}/25\mathbb{Z})$
- (d) $\text{Spec}(K[x, y, z, w]/(xy^5 + z^3y^2 + wx))$ [Hint: Eisenstein with the prime (z, w) .]
- (e) $\text{Spec}(K[x, y, z, w]/(xy, xz, xw))$
- (f) $\text{Spec}(L_1 \times L_2)$, L_i fields.
- (g) $\text{Spec}(R)$, where R is the product over all prime integers p , of $\mathbb{Z}/p\mathbb{Z}$.

Solution:

(a) There are infinitely many components. $(2), (3), \dots, (p) \forall$ primes p all define components of $\text{Spec}(\mathbb{Z})$.

(b) Notice that $24 = 2 \times 2 \times 3$. Furthermore,

$$\text{Spec}(24\mathbb{Z}) = \mathbb{V}(24) = \{(2), (3)\}$$

Thus, we have 2 components.

(c) Similarly, $\text{Spec}(25\mathbb{Z}) = \mathbb{V}(25) = \{(5)\}$. Thus this has just 1 component, and is thus irreducible.

(d) Via Eisenstein, we know that $xy^5 + z^3y^2 + wx$ is irreducible. In particular, it generates a prime ideal, and quotienting by it yields a domain. By Theorem 1, this implies that $\text{Spec}(K[x, y, z, w]/(xy^5 + z^3y^2 + wx))$ is irreducible.

- (e) $\text{Spec}(K[x, y, z, w]/(xy, xz, xw)) = \mathbb{V}(x) \cup \mathbb{V}(y, z, w)$. Thus, this has 2 components.
- (f) $\text{Spec}(L_1 \times L_2) = \text{Spec}(L_1) \sqcup \text{Spec}(L_2)$. L_1, L_2 are fields, so they both have a spectra of a single point. Thus, their product spectra is just the disjoint union of two points, implying it has 2 irreducible components as a topological space.
- (g) We can decompose this like before at first:

$$\text{Spec} \left(\prod_p \mathbb{Z}/p\mathbb{Z} \right) = \bigsqcup_p \text{Spec}(\mathbb{Z}/p\mathbb{Z})$$

After this, to be honest, I wasn't sure how to conclude. I believe this decomposes into finitely many parts, but I forgot how to do it and can't seem to figure it out now, either.

Problem 3. Proof of Theorem 1. Let R be arbitrary.

- (a) Prove that if $\text{Spec}(R) = \mathbb{V}(x)$, then x is nilpotent.
- (b) Suppose that R is reduced. Show that if $\text{Spec}(R)$ is irreducible, then R is a domain. [Hint: recall that $\mathbb{V}(xy) = \mathbb{V}(x) \cup \mathbb{V}(y)$]
- (c) Assume R is a domain. Prove that $\text{Spec}(R)$ is irreducible. [Hint: Say $\text{Spec}(R) = \mathbb{V}(I) \cup \mathbb{V}(J)$. Find $x \in I \setminus J, y \in J \setminus I$ so that $\text{Spec}(R) = \mathbb{V}(x) \cup \mathbb{V}(y)$.]
- (d) Show that every irreducible set of $\text{Spec}(R)$ has the form $\mathbb{V}(P)$ where P is a minimal prime of R .
- (e) Prove that $\text{Spec}(R) = \bigcup_{P \in \text{minSpec}(R)} \mathbb{V}(P)$.

Solution:

- (a) If $\mathbb{V}(x) = \text{Spec}(R)$, then $(x) \subset P \forall P \in \text{Spec}(R)$. If x is contained in every prime ideal, it is contained in the intersection of every prime ideal, which we know to be the nilradical. Thus, x is nilpotent.
- (b) Suppose that R was not a domain. Then, we can choose $x, y \neq 0$ such that $xy = 0$. Now, notice that

$$\text{Spec}(R) = \mathbb{V}(0) = \mathbb{V}(xy) = \mathbb{V}(x) \cup \mathbb{V}(y)$$

But $\mathbb{V}(x), \mathbb{V}(y) \not\subset \text{Spec}(R)$, since R is reduced, which is a contradiction.

- (c) We verify that if I is prime, then $\mathbb{V}(I)$ is irreducible, a more general claim. If R is a domain, then 0 is a prime ideal, implying that if we replace I with zero, we get that $\mathbb{V}(0) = \text{Spec}(R)$ is irreducible.

Suppose that $V(I) = X_1 \cup X_2$, for X_1, X_2 closed sets. Then, since $X_1, X_2 \subsetneq V(I)$, it follows that $I(X_1) \supsetneq I$, and $I(X_2) \supsetneq I$. Choose $f_1 \in I(X_1), f_2 \in I(X_2)$, but $f_1, f_2 \notin I$. However, $f_1 f_2 \in I$, violating the condition that I is prime.

- (d) We know irreducible sets are closed, so if X is irreducible, then $X = \mathbb{V}(I)$ for some ideal I . We wish to show that I is prime. We want to check that if $ab \in I$, then $a \in I$ or $b \in I$. Observe that

$$V((I, a)) \cup V((I, b)) = V((I, a)(I, b)) = V(I)$$

Thus, one of $a, b \in I$, implying that I is prime. We now verify that I is minimal. If not, then there exists some prime ideal $P \subset I$. However, this would imply that $\mathbb{V}(I) \subset \mathbb{V}(P)$, contradicting that $\mathbb{V}(I)$ is a component (i.e. a maximal irreducible set).

- (e) $\mathbb{V}(P) \subset \text{Spec}(R)$, so $\bigcup_{P \in \text{minSpec}(R)} \mathbb{V}(P) \subset \text{Spec}(R)$. In addition, Because $\mathbb{V}(P)$ is minimal with respect to prime ideals, $\mathbb{V}(P)$ is maximal, since $\mathbb{V}(-)$ is order reversing. Since $P \in \mathbb{V}(P)$, it follows that

$$\text{Spec}(R) \subset \bigcup_{P \in \text{Spec}(R)} \mathbb{V}(P) \subset \bigcup_{P \in \text{minSpec}(R)} \mathbb{V}(P)$$

Thus, $\text{Spec}(R) = \bigcup_{P \in \text{minSpec}(R)} \mathbb{V}(P)$.

Problem 4. Proof of Theorem 2. We will use Noetherian induction.

- Explain why Theorem 2 is equivalent to the statement that every Noetherian ring has finitely many minimal primes.
- To prove Theorem 2, fix a Noetherian ring R . Consider the set of all ideals in R that have infinitely many minimal primes. Show that this set has a maximal element if it is nonempty.
- To prove Theorem 2, show that it suffices to prove it for R with the property that every proper quotient has finitely many minimal primes.
- Explain why, if R in (c) is a domain, the proof of Theorem 2 is complete.
- With R as in (c), suppose that x, y are nonzero and $xy = 0$. Show that every minimal prime in R contains either x, y , or both.
- With R as in (c), Show that (x) and (y) have only finitely many minimal primes. [Hint: Use the Noetherian Induction hypothesis (c).]
- Prove Theorem 2 and its corollary.

Solution:

- Minimal primes of I in R are equivalent to the minimal primes of the (Noetherian) ring R/I . Thus, if we show every Noetherian ring has finitely many minimal primes, it follows that any ideal will have finitely many minimal primes.

- (b) We can put a poset structure on this set with respect to inclusion. Due the ascending chain condition on Noetherian rings, every ascending chain of ideals (elements in this set) is bounded. Thus, if the set is nonempty, there exists a maximal element by Zorn's Lemma.
- (c) Let J denote the maximal element of the set in the previous part. Notice that every proper quotient of R/J , by construction, has finitely many minimal primes. Thus, if we prove the statements for rings with the above property, we can conclude.
- (d) If R is a domain, (0) is a prime ideal. In particular, it is the only minimal prime ideal, since any prime ideal contains 0 . Thus there is only 1 minimal prime ideal. In particular, the number is finite.
- (e) Notice that $0 \in P$ for any minimal prime P . Thus, $xy \in P$. Since P is prime, this implies $x \in P, y \in P$, or both.
- (f) If every proper quotient of R has finitely many minimal primes, it follows that $R/(x)$ and $R/(y)$ have finitely many minimal primes, implying that (x) and (y) have finitely many minimal primes.
- (g) When R is a domain, we've already proven theorem 2. When R is not a domain, we've shown that ideals generated by zero divisors have finitely many minimal primes. In particular, we know that any minimal prime P must contain x or y , and (x) and (y) both have finitely many minimal primes. This implies the whole ring only has finitely many minimal primes, as all minimal primes are minimal primes of (x) or (y) , and there are only finitely many of those. This completes the proof of Theorem 2.

Recall that we've previously shown that irreducible components of $\text{Spec}(R)$ correspond precisely to $\mathbb{V}(P)$, where P is a minimal prime. Since there are only finitely many such P , there are only finitely many such $\mathbb{V}(P)$, implying that $\text{Spec}(R)$ has finitely many irreducible components, proving the corollary.

Problem 5. Fix a vector space over an infinite field. Let $\{W_1, \dots, W_n\}$ denote a finite collection of vector subspaces. Prove that if V is a subspace contained in $\bigcup_{i=1}^n W_i$, then $V \subset W_i$ for some i .

Solution: Let V be a vector space over an (infinite) field F . We wish to show that V cannot be written as a finite union of proper subspaces. We proceed by contradiction. Suppose that $V = \bigcup_{i=1}^n W_i$, where each W_i is a proper subspace of V . Without loss of generality, we stipulate that $W_i \not\subset W_j$ for $i \neq j$, since if this were the case, we could just omit W_i from the union.

Choose $w \in W_i$, and $v \in V \setminus W_i$. Since F has infinitely many elements, the set $\{w + cv \mid c \in F\}$ also has infinitely many elements in V . Thus, there must exist W_j such that $\{w + cv \mid c \in F\} \cap W_j$ has infinitely many elements, since W_j is a member of a finite decomposition of V . Observe that $j \neq i$, since $\{w + cv \mid c \in F\} \cap W_i = \emptyset$ by construction. Choose $w + c_1v, w + c_2v \in W_j$ distinct. Since W_j is a vector space,

$w + c_1v - (w + c_2v) = (c_1 - c_2)v \in W_j$. Since $w + c_1v, w + c_2v \in W_j$ are distinct, $c_1 - c_2 \neq 0$, so $v \in W_j$.

Since $v \in W_j$, it follows that $w \in W_j$. Thus, we've shown that an arbitrary $w \in W_i$ is contained in W_j for $j \neq i$, implying that $W_i \subset W_j$, a contradiction. Thus, we can conclude that every vector space over an infinite field is not a finite union of proper subspaces.

Problem 6. In the ring $\mathbb{C}[x, y, z]$, let $U = \mathbb{C}[x, y, z] \setminus ((x) \cup (y, z))$. Let $R = U^{-1}\mathbb{C}[x, y, z]$. Describe $\text{Spec}(R)$: What are the maximal and minimal primes? How many components? What is the dimension? What are the heights of the different maximal ideals? How does this look as a poset? as a subset of $\text{Spec}(\mathbb{C}[x, y, z])$? What is its closure in $\text{Spec}(\mathbb{C}[x, y, z])$?

Solution: This is like localizing by "2 primes". The maximal ideals are precisely $(x)R$ and $(y, z)R$, which have heights 1 and 2 respectively.

$$\text{Spec}(R) = \{P \in \text{Spec}(\mathbb{C}[x, y, z]) \mid P \cap U = \emptyset\} = \{P \in \text{Spec}(\mathbb{C}[x, y, z]) \mid P \subset (x) \cup (y, z)\}$$

Minimal primes should just be the same as in R . R is a dimension 2 ring, since $0 \subset (y) \subset (y, z)$ forms a valid saturated chain, but no chains of length 3 can be made. R is a domain, and thus $\text{Spec}(R)$ is irreducible.

Problem 7. Prime Avoidance Lemma. Let R be any ring, and let I_1, \dots, I_t be ideals of R . Suppose that $J = I_1 \cup \dots \cup I_t$.

- (a) If R is an algebra over an infinite field, Prove that $J \subset I_k$ for some index k .
- (b) More generally, assume that at most 2 ideals (without loss of generality, I_{t-1}, I_t) are not prime. Prove that $J \subset I_k$ for some index k .

Solution:

- (a) First, observe that this is (not so secretly) a statement about vector spaces, as an algebra over a field is a vector space equipped with a bilinear product. Ideals of algebras over a field are precisely vector subspaces.

We prove this by contradiction. Suppose that $J \not\subset I_j$ for $1 \leq j \leq t$. This implies that $J \cap I_j \subsetneq I_j$. However, since $J \subset I_1 \cup \dots \cup I_t$, it follows that

$$J = \bigcup_{j=1}^t (J \cap I_j)$$

Where each $J \cap I_j$ is a proper subspace of J . However, any vector space over an infinite field is not a finite union of proper subspaces (a proof of this will be given at the end of the homework), which is a contradiction. Thus, it follows that $J \cap I_j = J$ for some $1 \leq j \leq t$, implying that $J \subset I_j$.

(b) We prove the contrapositive. Suppose $J \not\subset I_j \forall j \leq t$. We wish to show that $J \not\subset \bigcup_{j=1}^t I_j$. To do this, we construct an element $r \in J$ such that $r \notin I_j$ for each $j \leq t$.

We prove such an element exists by inducting on t . The $t = 1$ case is immediate, so we assume the statement holds for $t - 1$ ideals, then show it holds for t ideals. From the inductive hypothesis, we know that for any $i \leq t$,

$$J_i := J \setminus \left(\left(\bigcup_{j \neq i} I_j \right) \cap J \right)$$

is nonempty. This is because J is not contained in any I_j (in particular, all the I_j where $j \neq i$). Since the collection of ideals $\{I_j\}_{j \neq i}$ has $t - 1$ ideals, by the inductive hypothesis it follows that $J \not\subset \bigcup_{j \neq i} I_j$, so J_i is nonempty.

Choose $r_i \in J_i \subset J$ for each $i \leq t$. Define

$$r = r_1 + r_2 r_3 \dots r_t$$

Since each $r_i \in J$, it follows that $r \in J$. I claim that $r \notin I_j \forall j \leq t$. Suppose $r \in I_j$ for $j \neq 1$. Notice that $r_2 r_3 \dots r_t \in I_j$. It follows then that $r - r_2 r_3 \dots r_t = r_1 \in I_j$, a contradiction.

Now, we just need to verify that $r \notin I_1$. Suppose it is. Then since $r_1 \in I_1$, it follows that $r - r_1 = r_2 r_3 \dots r_t \in I_1$. If $t = 2$, this implies that $r_2 \in I_1$, a contradiction. If $t > 2$, then we know I_1 is prime. Thus, $r_2 r_3 \dots r_t \in I_1$ implies that one of $r_2, r_3, \dots, r_t \in I_1$, which is a contradiction in every case.

Math 614: Commutative Algebra

IBL Lab Notebook

Class 14: October 24th

Name: Vignesh Jagathese

Collaborators: Don't Remember

Problem 1. Show that $\mathbb{Z}_{(p)}$ is a valuation ring.

Solution: Note that $\mathbb{Z}_{(p)} \subset \mathbb{Q}$. Choose $r/s \in \mathbb{Q}$ such that r/s is in lowest terms. Because of this, we have 3 cases. If $p|r$, then $r/s \in \mathbb{Z}_p$. If $p|s$, then $(r/s)^{-1} = s/r \in \mathbb{Z}_p$. If p does not divide r or s , then r/s and s/r are in \mathbb{Z}_p . p cannot divide both r and s since r/s is written lowest terms. Thus, we can conclude that $\mathbb{Z}_{(p)}$ is a valuation ring of \mathbb{Q} .

Problem 2. Which of the following sets is a valuation ring of $\mathbb{C}(x, y)$:

- (a) $R = \mathbb{C}[x, y]_{(y)}$
- (b) $\mathbb{C}[x, y]_{(x, y)}$
- (c) Let S be the subset of rational functions $f(x, y)/g(x, y)$ such that the order of f is \geq the order of g . (order is the degree of the lowest degree term)

Solution:

- (a) This is a valuation ring, using the same argument as above. Writing $f(x, y)/g(x, y) \in \mathbb{C}(x, y)$ in lowest terms, we split into three cases as in the problem 1 case.
- (b) This is not a valuation ring, because both x/y and y/x are not contained in $\mathbb{C}[x, y]_{(x, y)}$.
- (c) Choose $f(x, y)/g(x, y) \in \mathbb{C}(x, y)$. If the order of f is \geq the order of g , then $f/g \in S$. If the order of g is \geq the order of f , then $g/f \in S$. Thus, S is a valuation ring.

Problem 3. Prove that every ordered abelian group is torsion free.

Solution: Choose x in our ordered abelian group, which is an ordered \mathbb{Z} -modules. Choose $n \in \mathbb{Z}$ positive. We have two cases. If $x > 0$, then $nx = x + \cdots + x > 0$, so $nx \neq 0$. If $x < 0$, then $nx = x + \cdots + x < 0$, so $nx \neq 0$. Now suppose $n < 0$. Then, if $x > 0$, $nx = -x + \cdots + (-x) < 0$; so $nx \neq 0$. If $x < 0$, then $nx = -x + \cdots + (-x) > 0$, so $nx \neq 0$.

Thus, x has no torsion, and our ordered abelian group is torsion free.

Problem 4. Let ν be an arbitrary valuation on K . Say $f, g \in K^\times$.

- (a) Show that $\nu(1/f) = -\nu(f)$.
- (b) Show that $\nu(f) = \nu(-f)$.
- (c) Show that if $\nu(f) > \nu(g)$, then $\nu(f + g) = \nu(g)$. [Hint: $(f + g) + (-f) = g$.]

Solution:

- (a) A valuation is a group homomorphism, so it sends inverses to inverses.
- (b) $\nu(-f) = \nu(-1) + \nu(f)$. We just need to verify that $\nu(-1) = 0$. Well, $\nu(1) = 0$, since ν is a group homomorphism, and $0 = \nu(1) = \nu((-1)(-1)) = \nu(-1) + \nu(-1)$. We've just proven that ordered abelian groups are torsion free, so if $2\nu(-1) = 0$, then $\nu(-1) = 0$, and we can conclude.
- (c) Following the hint,

$$\nu(g) \geq \min(\nu(f + g), \nu(-f)) = \min(\nu(f + g), \nu(f))$$

This is $\nu(f + g)$, since if it was $\nu(f)$, then that would contradict $\nu(f) > \nu(g)$. This implies that $\nu(g) \geq \nu(f + g)$, Conversely,

$$\nu(f + g) \geq \min(\nu(f), \nu(g)) = \nu(g)$$

Thus, $\nu(f + g) \geq \nu(g)$, so $\nu(f + g) = \nu(g)$.

Problem 6. Let ν be an arbitrary valuation on K .

- (a) Prove that the set $V \subset K$ of elements f such that $\nu(f) \geq 0$ is a valuation ring on K . We call this the valuation ring of ν .
- (b) Show that f is a unit in V if and only if $\nu(f) = 0$.
- (c) Show that V is local with maximal ideal $\{f \in K \mid \nu(f) > 0\}$.
- (d) Show that $\forall \gamma \in \Gamma$, the sets $\{f \in K \mid \nu(f) > \gamma\}$ and $\{f \in K \mid \nu(f) \geq \gamma\}$ are ideals of V .
- (e) Show that if $k \subset K$ is a subfield such that $\nu|_{k^\times} = 0$, then V is a k -algebra.

Solution:

- (a) From 4(a), we know that $\nu(f^{-1}) = -\nu(f)$. Thus, either $\nu(f) \geq 0$, in which case $f \in V$, or $\nu(f) \leq 0$, so $\nu(f^{-1}) \geq 0$, so $f^{-1} \in V$. Thus, if V is in fact a ring, it is a valuation ring. $1 \in V$, since $\nu(1) = 0$. By convention $\nu(0) = \infty$, so $0 \in V$.

To verify the existence of additive inverses, notice that $\nu(f) = \nu(-f)$, so $\nu(f) \geq 0 \iff \nu(-f) \geq 0$. For closure, choose $f, g \in V$.

$$\nu(f + g) \geq \min(\nu(f), \nu(g)) \geq 0$$

$$\nu(fg) = \nu(f) + \nu(g) \geq 0 + 0 = 0$$

Thus, V is a valuation ring.

- (b) Choose $f \in V$. $\nu(1/f) = -\nu(f)$, so it follows that $\nu(1/f) \leq 0$. Thus,

$$1/f \in V \iff \nu(1/f) \geq 0 \iff \nu(1/f) = 0 \iff \nu(f) = 0$$

- (c) By the previous part, we know that every element outside of the ideal is a unit. Thus, the ring must be a local ring, with the unique maximal ideal being the one described above.
- (d) These are abelian groups, from similar logic to part (a) (notice that 0 is contained in both of these, since $\nu(0) = \infty$ by convention). To show these are ideals, choose $v \in V, f \in K$ such that $\nu(f) > \gamma$.

$$\nu(vf) = \nu(v) + \nu(f) > 0 + \gamma = \gamma$$

Thus, $\{f \in K \mid \nu(f) > \gamma\}$ is an ideal. The case where $\nu(f) \geq \gamma$ is similar.

- (e) It is sufficient to verify that there exists a ring homomorphism $k \rightarrow V$. Well, k is a subfield of K , and in particular, it is a subring. Furthermore, $k \subset V$, implying that k is a subring of V . Thus, we can take the natural inclusion map $k \rightarrow V$ as the ring homomorphism.

Problem 7. Fix a prime number p .

- (a) Using the UFD property of \mathbb{Z} , explain why every non-zero element a/b of \mathbb{Q} can be written (up to sign and reordering) as $p_1^{a_1} \dots p_t^{a_t}$ where the p_i are distinct primes and $a_i \in \mathbb{Z}$.
- (b) For non-zero $a/b \in \mathbb{Q}$, define $\nu_p(a/b) = a_p$, the exponent of p where a/b is written as in (a). Show that ν_p is a discrete valuation on \mathbb{Q} .
- (c) Show that the valuation ring of ν_p is $\mathbb{Z}_{(p)}$.

Solution:

- (a) We can decompose a, b into irreducibles, since $a, b \in \mathbb{Z}$. Thus,

$$\frac{a}{b} = \frac{p_1^{a_1} \cdots p_t^{a_t}}{q_1^{b_1} \cdots q_s^{b_s}} = p_1^{a_1} \cdots p_t^{a_t} q_1^{-b_1} \cdots q_s^{-b_s}$$

Where $a_i, b_i \in \mathbb{N}$, implying that $a_i, -b_i \in \mathbb{Z}$. Thus, we have a decomposition into prime ideals, with powers in \mathbb{Z} . If any of these primes are the same, then they can be absorbed into the same exponent term, i.e. if $p_i = q_j$, then $p_i^{a_i} q_j^{b_j} = p_i^{a_i+b_j}$ for $a_i + b_j \in \mathbb{Z}$.

- (b) We first check this is a group homomorphism.

$$\nu_p(a/b \cdot c/d) = \nu_p(p^{\nu_p(a/b)}(a/b) \cdot p^{\nu_p(c/d)}(c/d)) = \nu_p(p^{\nu_p(a/b)+\nu_p(c/d)}(a/b \cdot c/d))$$

Where this is just $= \nu_p(a/b) + \nu_p(c/d)$, as we've factored p out of both $a/b, c/d$ completely. Furthermore, $\nu(1) = \nu(p^0) = 0$.

Our value group is clearly \mathbb{Z} from part(a), so it is sufficient to verify that valuation axiom.

$$\nu_p(a/b + c/d) = \nu_p(p^{\nu_p(a/b)}(a/b) + p^{\nu_p(c/d)}(c/d))$$

If $\nu_p(a/b) \leq \nu_p(c/d)$, then $\nu_p(a/b)$ can factor out of both terms, and $\nu_p(a/b + c/d) = \nu_p(a/b)$. If $\nu_p(a/b) \geq \nu_p(c/d)$, then $\nu_p(c/d)$ factors out of both terms, and $\nu_p(a/b + c/d) = \nu_p(c/d)$. The valuation axiom follows.

- (c) Choose $n/m \in \mathbb{Z}_{(p)}$. Since p does not divide m , $\nu_p(n/m) = \nu(n) \geq 0$. Similar logic shows that if we pick $a/b \in \mathbb{Q}$ in reduced form, then if $\nu_p(a/b) \geq 0$, then $a/b \in \mathbb{Z}_{(p)}$. Thus, $\mathbb{Z}_{(p)}$ is the valuation ring of ν_p .

Problem 8. Let V be a valuation ring on K .

- (a) Show that for any nonzero elements $x, y \in V$, $x|y$ or $y|x$.
- (b) Show that every finitely generated ideal of V is principal.
- (c) Prove that the ideals of V form a totally ordered set. That is, for any two ideals I, J in V , either $I \subset J$ or $J \subset I$. [Hint: if $x \in I, x \notin J$, consider whether $y \in J$ can divide x .]
- (d) Prove that valuation rings are local.

Solution:

- (a) $y = x(x^{-1}y)$, $x = y(y^{-1}x)$. Notice that $(x^{-1}y)^{-1} = y^{-1}x$, implying one of $y^{-1}x, x^{-1}y \in V$, implying that one of the equalities $y = x(x^{-1}y)$, $x = y(y^{-1}x)$ holds in V . Thus, $x|y$ or $y|x$.

- (b) We induct on the number of generators. A finitely generated ideal with 1 generator is principal, so we continue with the inductive step. Suppose that $I = (f_1, \dots, f_n)$. Notice that either $f_1|f_n$ or $f_n|f_1$, implying that $I = (f_1, \dots, f_{n-1})$ or $I = (f_2, \dots, f_n)$. In either case, I is generated by $n - 1$ generators, and by the inductive hypothesis, any ideal on $n - 1$ generators is principal, so I is principal.
- (c) Suppose that I and J are two distinct ideals. Choose $x \in I$, and $y \in J$. Since I and J are distinct, either $x \notin J$ or $y \notin I$. In the former case, this implies that y does not divide x , so x divides y , so $y \in I$, so $J \subset I$. In the latter case, this implies that x does not divide y , so y divides x , so $x \in J$, so $I \subset J$.
- (d) All rings have a maximal ideal. If V had 2 maximal ideals, m_1, m_2 where $m_1 \neq m_2$, then by the previous part, $m_1 \subset m_2$ or $m_2 \subset m_1$, both of which would be a contradiction. Thus, V has only 1 maximal ideal, and is local.

Problem 9. Explain why a Noetherian valuation ring is a local PID. Compute its Krull dimension. [Use (8).]

Solution: If a Noetherian ring R is a valuation ring, then all of its ideals are finitely generated. By 8(b), we know that all of its ideals are principal, so R is a PID. 8(d) shows us that R is local, so R is a local PID.

Since R is a PID, it has Krull Dimension 1.

Problem 10. Let $\nu : K^\times \rightarrow \mathbb{Z}$ be a discrete valuation on K , and let (V, m) be the corresponding discrete valuation ring.

- (a) Let $I \subset V$ be any nonzero ideal, explain why we can choose an element $f \in I$ such that $\nu(f)$ is minimal (among elements in I).
- (b) Prove that every ideal of V is principal. In particular, a discrete valuation ring is Noetherian.
- (c) Let t generate m . Show that all other ideals V are generated by t^n for some $n \in \mathbb{N}$.

Solution:

- (a) Recall that the valuation ring corresponds to the set of elements with valuation ≥ 0 . Thus, we can choose an element $f \in I$ such that $\nu(f)$ has minimal valuation. We've shown that $\nu(f) = 0$ implies that f is unit, so assuming we are working in a proper ideal, the minimal valuation will be ≥ 1 .
- (b) I claim that $I = (f)$, for f minimal. To show this, it is sufficient to show that $f|x$ $\forall x \in I$. Well, either $x|f$ or $f|x$. If the former is true, then $xy = f$ for some y . Thus, $\nu(f) = \nu(x) + \nu(y)$. However, minimality of $\nu(f)$ tells us that $\nu(f) = \nu(x)$

or $\nu(f) = \nu(y)$. Well, $\nu(x) \neq 0$ since $x \in I$ and is thus non-unit, so it follows that $\nu(y) = 0$ and y is unit. This then implies that $y^{-1}f = x$, and $f|x$ as desired. Thus, $I = (f)$.

- (c) Since V is a PID, we can write $m = (t)$. Since m is the unique maximal ideal, $I = (f) \subset (m)$ for all ideals $I \subset V$. In particular, for all $f \in R$ $f = yt$ for some y . This implies that $\nu(f) = \nu(y) + \nu(t)$, and that $\nu(y) < \nu(f)$, as $\nu(t) > 0$. Now, notice that $y = y_2t$ for some y_2 , and by similar construction $\nu(y_2) < \nu(y)$. This would imply that $f = yt = y_2t^2$. Since this valuation ring consists of elements with positive valuation, with zero valuation elements being units, we can keep repeating this process, strictly decreasing valuation, until we hit a unit u where $\nu(u) = 0$. This implies that $f = uttttt \dots t = ut^n$. In particular, $I = (t^n)$.

Problem 13. Let V be a valuation ring on K .

- (a) Let Γ' be the set of nonzero principal ideals in V . Show that Γ' has the structure of a totally ordered abelian monoid (under multiplication). What is the identity element?
- (b) Explain how to extend Γ' to an ordered abelian group Γ whose elements are rank 1 free V -submodules of K .
- (c) Construct a valuation $\nu : K^\times \rightarrow \Gamma$ whose valuation ring covers V .
- (d) Show that if V is Noetherian, then the corresponding valuation is discrete- that is, $\Gamma \cong \mathbb{Z}$.

Solution:

- (a) The identity element here is V , with operation $(p) \cdot (q) = (pq)$. We've already shown that the ideals of V form a totally ordered set with respect to inclusion, so we can just restrict this ordering to the principal ideals. However, given our choice of identity element, we need to implement the ordering in reverse. In other words $(p) < (q) \iff (p) \supset (q)$.
- This allows us compatibility. In other words, with $(x) \subset (y), (z) \subset (w)$, $(x)(z) \subset (y)(z) \subset (y)(w)$.
- (b) Given that $V \subset K$ for K a field, we can define $(1/f)$ as the rank 1 free V -submodule $(1/f)V$. This is naturally an inverse of the ideal $(f) \subset V$.
- (c) The map $a \mapsto (a)V$ forms a valuation. It is by construction a group homomorphism, and if $(a) \subset (b)$, then $(a) + (b) \subset (b)$, and if $(b) \subset (a)$, then $(a) + (b) \subset (a)$. Thus $(a) + (b)$ is contained in the larger of $(a), (b)$.
- (d) While in problem (10) we assumed that V was a DVR, notice that if we restrict to making a statement on principal ideals, then the proof for (c) will hold in this case. Now, this implies that each principal ideal is of the form t^n . Furthermore, if $(f) = (t^n)$, then that ideal corresponds to the index n , and $(1/f)V$ corresponds

to the index $-n$. Thus, we have a bijective map $\mathbb{Z} \leftrightarrow \Gamma$ mapping $n \mapsto (t^n)$ for $n \geq 0$, and $n \mapsto (1/(t^{-n})V$ for $n \leq 0$. By this construction, notice that $0 \mapsto V$, so this sends the identity to the identity. This map is clearly bijective and is a group homomorphism, as $(t^m)(t^n) = (t^{m+n}) \mapsto m + n$, so we can conclude that $\Gamma \cong \mathbb{Z}$.

Math 614: Commutative Algebra

IBL Lab Notebook

Class 15: October 29th

Name: Vignesh Jagathese

Collaborators: Don't Remember

Problem 1. Let R be an A -graded ring.

- (a) Show that 1_R is homogeneous of degree $0 \in A$. [Hint: For $r \in R_b$, consider $1 \cdot r$. Use the fact that the multiplicative identity in a ring is unique.]
- (b) Show that R_0 is a subring of R , so that R is an R_0 -algebra.
- (c) Show that each homogeneous component R_a has a natural R_0 -module structure.

Solution:

- (a) Notice that $1_R \cdot R_a = R_a$. Thus it follows that $1_R \in R_0$.
- (b) By construction R_0 is an abelian group, so it is sufficient to verify that it is a ring. Well, notice that if $r, s \in R_0$, then

$$(rs) \cdot R_a = r(s \cdot R_a) = r(R_a) = R_a$$

thus, $rs \in R_0$, implying R_0 is a subring. the inclusion map $R_0 \rightarrow R$ makes R an R_0 algebra.

- (c) Since $R_0 \cdot R_a = R_a$, there is a natural R_0 action on R_a . The action is clearly distributive, since ring operations are distributive. It follows that R_a has a R_0 -module structure.

Problem 2. Let $R = K[x, y]$. Verify that each of the following uniquely defines a **graded ring structure** on R , and describe the homogeneous components R_a for arbitrary $a \in A$. Show that none are isomorphic as graded rings.

- (a) THE STANDARD GRADING BY \mathbb{N} . Here x, y both have degree $1 \in \mathbb{N}$.
- (b) A NON-STANDARD GRADING BY \mathbb{N} . Define $\deg x = 2$ and $\deg y = 3$.
- (c) ANOTHER NON-STANDARD GRADING BY \mathbb{N} . Define $\deg x = 0$ and $\deg y = 1$.
- (d) THE STANDARD MONOMIAL GRADING BY \mathbb{N}^2 . Define $\deg x = (1, 0)$ and $\deg y = (0, 1)$.
- (e) A NON-STANDARD GRADING BY \mathbb{N}^2 . Define $\deg x = (1, 0)$ and $\deg y = (0, 2)$.

Solution:

- (a) Recall that R_N for $N \in \mathbb{N}$ has a natural R_0 module structure. Here each graded component corresponds to the span of the subset of $K[x, y]$ consisting of degree N monomials. More concretely, $R_N = \text{span}_{R_0}(x^i y^j \mid i + j = N)$, for $R_0 = K$.
- (b) Utilizing the construction in the last part, under this grading we find that $R_N = \text{span}_{R_0}(x^i y^j \mid 2i + 3j = N)$, for $R_0 = K$.
- (c) Since all x terms have degree 0, and all y terms have nontrivial degree, it follows that $R_0 = K[x]$. Viewing our ring as $K[x, y] = K[x][y]$ allows us to default to the grading in a), so $R_N = \text{span}_{K[x]}(y^N)$.
- (d) In this case, $R_{(m,n)} = \text{span}_{R_0}(x^m y^n)$, for $R_0 = K$.
- (e) $R_0 = K$. For n even, $R_{(m,n)} = \text{span}_{R_0}(x^m y^{n/2})$. For n odd, it necessarily cannot contain any y components, so for n odd $R_{m,n} = \text{span}_{R_0}(x^m)$.

It follows that none of these rings are isomorphic as graded rings. (a)-(c) are in a different category than (d)-(e), and it is clear that (a)-(c) and (d)-(e) all have distinct graded components.

Problem 3. Let $R = K[x, y]/\langle y^7 - x^5(x + 1) \rangle$. Show that R has a natural $\mathbb{Z}/7\mathbb{Z}$ -grading whose zero-th graded piece is $K[x]$. [Hint: Note that R is a free $K[x]$ -module of rank 7 with basis classes of (certain) powers y .]

Solution: Set $\deg(y) = \bar{1}$, $\deg(x) = \bar{0}$. Notice that in R , $y^7 = x^5(x + 1)$. Thus, we can define $R_{\bar{n}} := \text{span}_{K[x]}(y^{\bar{n}})$ and have it be well defined, since any polynomial in R will have y degree < 7 due to the identification $y^7 = x^5(x + 1)$.

Problem 4. Let R be an A -graded ring, and let $I \subset R$ be any ideal. Show that the following are equivalent.

- (a) I can be generated by homogeneous elements of R .
- (b) As an abelian group, $I = \bigoplus_{a \in A} I \cap R_a$. [Hint: Note that an arbitrary R -linear combination of the homogeneous generators for I is a sum of homogeneous elements of I .]
- (c) I is an A -graded R -submodule of R —that is, the inclusion $I \hookrightarrow R$ is a map of graded R -modules.
- (d) Whenever $f = \sum_{a \in A} f_a$ is in I , where the f_a are homogeneous of degree a , it follows that all components $f_a \in I$.

Solution: First we verify that (a) implies (b). Since $I \cap R_a \subset I$, it is clear that $\bigoplus_{a \in A} I \cap R_a \subset I$. We just need to verify that opposite direction. Choose $r \in I$. Since I is generated by homogenous elements of R , we know that $r = \sum_{a_i \in A} r_i r_{a_i}$ where $r_{a_i} \in R_{a_i}$. Well, this implies that $r_{a_i} \in I \cap R_{a_i}$, and in particular that $r \in \bigoplus_{a_i \in A} I \cap R_{a_i} \subset \bigoplus_{a \in A} I \cap R_a$, completing this direction.

To show (b) implies (a), pick $(r_{a_i})_{i \in I_a}$ as generators for each R_a . From the decomposition given by (b), we know that $I = (r_{a_i})_{i \in I_a, a \in A}$, completing the proof. Furthermore, (b) implies (c) follows by definition of a graded ring. (c) also implies (d), since if I is a graded submodule of R , then $I = \bigoplus_a I_a$ for some graded components over $a \in A$. It follows that if $f = \sum_{a \in A} f_a$, then $f_a \in I$.

To conclude we verify that (d) implies (b). Choose $f \in I$. If $f = \sum_{a \in A} f_a$, for $f_a \in R_a$, by (d) we know that $f_a \in I$, so $f_a \in I \cap R_a$. Thus, $f \in \bigoplus_{a \in A} I \cap R_a$. Since any $f \in R \supset I$ can be written this way, it follows that $I \subset \bigoplus_{a \in A} I \cap R_a$. The other direction is immediate, so (b) is proven. Thus, (a) \iff (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (b), so all 4 statements are equivalent.

Problem 5. Show that if $I \subset R$ is a homogenous ideal, then there is a natural A -grading on the quotient ring R/I such that the quotient map $R \rightarrow R/I$ is a morphism in the category of A -graded rings.

Solution: Recall that $I = \bigoplus_{a \in A} I \cap R_a$. As abelian groups, we have that

$$R/I = \bigoplus_{a \in A} R_a / \bigoplus_{a \in A} I \cap R_a \cong \bigoplus_{a \in A} (R_a / (I \cap R_a))$$

This defines a natural A grading on R/I . It follows that the map $R \mapsto R/I$ is an A -graded isomorphism.

Problem 6. All of the following quotients of $K[x, y, z, w]$ have \mathbb{N} -gradings. Describe one non-trivial grading for each.

- (a) $K[x, y, z, w] / \langle xy - zw \rangle$
- (b) $K[x, y, z, w] / \langle x^2 + y^3 + z^5 + w^7 \rangle$
- (c) $K[x, y, z, w] / \langle xy - z^2, w^4 - xy \rangle$

Solution:

- (a) This ideal is homogenous, as it is generated by a polynomial comprised of only degree 2 monomials, i.e. $xy - zw \in R_2$. Thus, we can just endow it with the induced grading from the standard grading, as in problem 5.

- (b) Notice that $2 \cdot 3 \cdot 5 \cdot 7 = 210$. Further notice that in this ring, x^2 is identified with $-1(y^3 + z^5 + w^7)$, with similar identifications for y^3, z^5, w^7 . Thus, we impose a $\mathbb{Z}/210\mathbb{Z}$ grading, where $\deg(x) = 210/2 = 105$, $\deg(y) = 210/3 = 70$, $\deg(z) = 210/5 = 42$, $\deg(w) = 210/7 = 30$.
- (c) If we set $\deg(x) = \deg(y) = \deg(z) = 2$, we see that $xy - z \in R_4$. Setting $\deg(w) = 1$ gives us that $w^4 - xy \in R_4$ as well. Thus, both of these polynomials are homogenous of the same degree. This implies that we can use problem 5 and induce a grading on the quotient with respect to the initial grading given above.

Problem 7. For the rings in 6(a) and 6(c), find

- (a) An \mathbb{N}^2 -grading that has zero-th graded piece K .
- (b) An \mathbb{N} -grading that has zero-th graded piece $K[x]$.

Solution:

- (a) For both 6(a) and 6(c), let the new $(n, 0)$ graded components be the old n -graded components, and let $R_{n,m} = 0$ for $m \neq 0$.
- (b) We want to pick a grading where $\deg(x) = 0$, and the other terms have positive degree, but the quotiented ideal is still homogenous.
- In 6(a), let $\deg(y) = 2$, $\deg(x) = 0$, and $\deg(z) = \deg(w) = 1$. Thus, $xy - zw \in R_2$. In 6(c), Let $\deg(y) = 4$, $\deg(z) = 2$, $\deg(w) = 1$, and $\deg(x) = 0$. Then, $xy - z^2, w^4 - xy \in R_4$.

Problem 8. Let K be an algebraically closed field, and let S be the polynomial ring $K[x, y]$ with its standard grading.

- (a) Show that every homogeneous polynomial in S factors into homogeneous polynomials of degree one. [Hint: For $f(x, y)$ homogeneous of degree d , show $\frac{f(x, y)}{y^d} = f(\frac{x}{y}, 1)$.]
- (b) Describe the homogeneous prime ideals in S . There are three types. Describe their heights.
- (c) Define a bijection between the set of non-zero, non-maximal homogenous prime ideals in S and the set of one dimensional vector subspaces of K^2 .

Solution:

- (a) This follows from the hint, and the fact that K is algebraically closed. A polynomial of degree n will have precisely n roots, and will factor into those roots.

- (b) Prime ideals are of height 0, 1, or 2. If height 0, this is precisely the zero ideal. If height 1, then it must be generated by nonzero linear combinations of x, y , since $x - a, y - b$ are not homogenous polynomials, and polynomials of any higher powers of x or y are homogenously reducible by the previous part. The only height 2 ideals are (x, y) and (1) , by similar logic to the height 1 case.
- (c) The nonzero, non-maximal, homogenous prime ideals are precisely the height 1 prime ideals, which are the form $(\alpha x + \beta y)$, where one of α, β is nonzero. Thus, the bijection is of the form

$$(\alpha x + \beta y) \longleftrightarrow \text{span}_K(\alpha e_1 + \beta e_2)$$

Where e_1, e_2 are basis vectors of K^2 . The backwards direction is well defined because we can choose a singular basis element up to scaling, which corresponds to a generator of the ideal, which is also defined up to scaling.

Problem 9. Let S be an \mathbb{N} -graded ring.

- (a) Show that $S_+ = \bigoplus_{n>0} S_n$ is an ideal of S , called the **irrelevant ideal**.
- (b) Show that a homogeneous ideal $I \subset S$ is prime if and only if for all *homogeneous* $f, g \in S$, if $fg \in I$ then either f or g is in I . [Hint: If $fg \in I$, use induction on the total number of non-zero terms in f and g together.]
- (c) Show that S_+ is prime if and only if S_0 is a domain, and S_+ is maximal if and only if S_0 is a field.

Solution:

- (a) S_+ is the direct sum of abelian groups, and is thus an abelian group. For $r \in S, s \in S_+$, with $\deg(r) = m \geq 0, \deg(s) = n > 0$, notice that $\deg(rs) = m + n > 0$. Thus, $rs \in S_+$, and S_+ is an ideal.
- (b) The forward direction follows from the definition of a prime ideal. Thus, we tackle the reverse case. Choose $fg \in I$, for f, g not necessarily homogenous. However fg is a homogeneous polynomial. We follow the hint and induct on the total number of non-zero terms $f = \sum_{i \in \mathbb{N}} f_i$ and $g = \sum_{j \in \mathbb{N}} g_j$ share. If they share no graded components, then it is immediate that f, g are both homogeneous, so we can conclude by our hypothesis.

If f, g share n graded components, denoted $f_1, \dots, f_n, g_1, \dots, g_n$, notice that $(f - f_n)(g - g_n) = fg - f_n g - g_n f + f_n g_n = fg - f_n(f + g) + f_n^2$, implying that

$$fg = (f - f_n)(g - g_n) + f_n(f + g) - f_n^2$$

In particular, fg can be written as a sum of terms which each satisfy the inductive hypothesis, allowing us to conclude.

- (c) $S_0 \cong S/S_+$, so the result follows from what we already know about quotients and maximal/prime ideals.

Math 614: Commutative Algebra**IBL Lab Notebook**

Class 16: October 31st

Name: *Vignesh Jagathese*Collaborators: *Noah Luntzlar, Aniruddh Agrawal***Problem 1.** THE TOPOLOGY ON $\text{Proj } S$. Fix an \mathbb{N} -graded ring S .

- (a) Show that every closed set of $\text{Proj } S$ has the form $\mathbb{V}(I) = \{P \in \text{Proj } S \mid I \subset P\}$ for some *homogeneous* ideal I . [Hint: Recall that I is homogeneous if and only if, whenever $f \in I$, then all graded components of f are in I .]
- (b) Let f be homogeneous in S_+ . Show that the set $D_+(f) = \{P \in \text{Proj } S \mid f \notin P\}$ is open.
- (c) Show that $\text{Proj } S$ has a basis by sets of the form $D_+(f)$ where f is homogeneous.

Solution:

- (a) We know all closed sets of $\text{Spec}(S)$ are of the form $\mathbb{V}(I)$ for any I . Given we have restricted to homogeneous prime ideals, it follows that any ideal contained in a homogeneous prime ideal must be homogeneous. Thus, $\mathbb{V}(I) = \{P \in \text{Proj}(S) \mid I \subset P\}$ is closed only if I is itself a homogeneous ideal in $\text{Proj}(S)$.
- (b) $D_+(f) = S \setminus \mathbb{V}(f)$.
- (c) We know that sets of the form $D(f)$ form a basis for $\text{Spec}(S)$. It follows that when restricting to homogeneous ideals that the only elements of the basis remaining are of this form, implying that the basis of $\text{Proj}(S)$, which must be a subset of the basis of $\text{Spec}(S)$, consists of sets of the form $D_+(f)$ for f homogeneous.

Problem 2. THE COMPLEX PROJECTIVE LINE. Let $S = K[x, y]$ with its standard grading. Consider $\text{Proj } S$ with its Zariski topology, and let $\mathcal{X} \subset \text{Proj } S$ be the subspace of **closed points**.

- (a) Prove $\text{Proj } S$ has exactly one dense point—that is, a point η in every non-empty open set.
- (b) Show that $\mathcal{X} = \text{Proj } S \setminus \{\eta\}$.
- (c) Thinking about Krull dimension, what do you think should be the dimension of $\text{Proj } S$?
- (d) Show that every non-zero homogeneous polynomial in $\mathbb{C}[x, y]$ factors into *linear* polynomials. [Hint: For f of degree d , show that $\frac{f}{y^d} = f(\frac{x}{y}, 1)$.]
- (e) Assume $K = \mathbb{C}$ for the remaining problems. Show that if $P \in \mathcal{X}$, then $P = \langle ay - bx \rangle$ for some $a, b \in \mathbb{C}$. To what extent are a, b unique?

- (f) Show that \mathcal{X} can be naturally identified with the set of lines through the origin in \mathbb{C}^2 .
- (g) Show that there is a bijection between \mathcal{X} and $\mathbb{C} \cup \infty$ given by sending a line to its slope.
- (h) Show that the Zariski topology on \mathcal{X} is the finite-complement topology.

Solution:

- (a) Just as in $\text{Spec}(S)$, the dense point is at (0) .
- (b) Recall that all ideals are of height 0, 1, 2 in $\text{Proj}(S)$. The 0 height ideal is the dense point, leaving just the height 1 and 2 ideals. Height 2 ideals contain S_+ , leaving just height 1 ideals, which all lie in the subspace of closed points.
- (c) The dimension *should* be 1.
- (d) See problem 8 on the previous worksheet.
- (e) See problem 8 on the previous worksheet.
- (f) See problem 8 on the previous worksheet; a line in \mathbb{C}^2 is precisely a 1 dimensional vector subspace of \mathbb{C}^2 .
- (g) \mathcal{X} bijectively corresponds to 1-dimension vector subspaces of \mathbb{C}^2 , which are just lines through the origin. Lines through the origin are uniquely determined by their slope. The line corresponding to (x) has infinite slope, and the line corresponding to (y) has slope 0. This implies that \mathcal{X} bijectively corresponds to the set of possible slopes, which is precisely $\mathbb{C} \cup \infty$.
- (h)

Problem 3. EASY LEMMAS. Let S be an \mathbb{N} -graded ring. Take any homogeneous $f \in S$ of positive degree.

- (a) Show that $S[\frac{1}{f}]$ has a naturally induced \mathbb{Z} -grading.
- (b) Recall that $R = [S[\frac{1}{f}]]_0$ is subring of $S[\frac{1}{f}]$. Show that every ideal of R expands to a *homogeneous* ideal of $S[\frac{1}{f}]$. [Hint: The elements of R have the form $\frac{g}{f^t}$ where g is homogeneous.]
- (c) Show that in an arbitrary A -graded ring, if I is homogeneous, so is \sqrt{I} .

Solution:

- (a) Any element of $S[1/f]$ is of the form sf^z for $z \in \mathbb{Z}$. We define the grading to be with respect of f in the term.

- (b) Notice that $R = \{r \in S \mid f \text{ does not divide } r\}$. Thus, $r \cdot s$ for s homogeneous in degree n is still homogeneous of degree n . It follows that any ideal $I \subset R$ expanded over $S[1/f]$ retains the homogeneity of $S[1/f]$.
- (c) Choose $r^n \in \sqrt{I}$, where $r \in I$ is homogeneous of degree m . Notice that r^2 is then homogeneous of degree $2m$, as $r^2 = r \cdot r$. This logic clearly extends, implying that $r^n = r \cdot \dots \cdot r$ is homogeneous of degree nm . Applying this to each generator of I , it follows that \sqrt{I} is still a homogeneous ideal.

Problem 4. Consider $S = K[x, y, z]$ with its standard \mathbb{N} -grading.

- (a) Show that $[S[\frac{1}{z}]]_0 = K[\frac{x}{z}, \frac{y}{z}]$ (a polynomial ring in two variables $s = \frac{x}{z}$ and $t = \frac{y}{z}$.)
- (b) Given $f \in S_d$, show that $\frac{f}{z^d} \in [S[\frac{1}{z}]]_0$, and that $\frac{f}{z^d}$ is $f(s, t, 1) \in K[s, t]$.
- (c) For P in $\text{Proj } S$, prove that $PS[\frac{1}{z}] \cap K[\frac{x}{z}, \frac{y}{z}]$ is prime or $z \in P$.
- (d) For each P in $\text{Proj } S$ below, compute generators for $PS[\frac{1}{z}] \cap K[\frac{x}{z}, \frac{y}{z}]$ in the ring $K[s, t]$:
- (a) $P = \langle x - az, y - bz \rangle$
 - (b) $P = \langle x - y \rangle$
 - (c) $P = \langle x^2 + y^2 - z^2 \rangle$.
 - (d) $P = \langle x, z \rangle$
- (e) For each of your answers in (d), sketch the closure in $\text{Spec } K[s, t]$, or in $\text{MaxSpec } K[s, t]$, when $K = \overline{K}$. How is this different from sketching the point itself?

Solution:

- (a) We know that $[S[\frac{1}{z}]]_0 = \text{span}_K(x^i y^j z^k \mid i + j + k = 0)$. Notice that $i, j \geq 0$ while $k \in \mathbb{Z}$, since only z is inverted. In particular, if $x^i y^j z^k \in [S[\frac{1}{z}]]_0$, $i + j = -k$, and we can rewrite it as $(x/z)^i (y/z)^j$, grouping i of the $1/z$ terms together and the remaining j together. It follows that $[S[\frac{1}{z}]]_0 = K[\frac{x}{z}, \frac{y}{z}]$.
- (b) Without loss of generality, let f be a monomial of the form $x^i y^j z^k$. We know that since $f \in S_d$, $i + j + k = d$. Thus, f/z^d is of degree $i + j + k - d = 0$, and thus is contained in $[S[\frac{1}{z}]]_0$. Following the identification $[S[\frac{1}{z}]]_0 = K[\frac{x}{z}, \frac{y}{z}]$ in the previous part, it is clear that f/z^d is $f(s, t, 1) \in K[s, t]$, where $s = x/z, t = y/z$.
- (c) This is simply contraction and expansion along the inclusion $i : [S[\frac{1}{z}]]_0 \rightarrow S[1/z]$, given that $z \notin P$. If $z \in P$, then $PS[1/z] = (1)$.
- (d) (1) $(x/z - a, y/z - b)$
 (2) $PS[1/z] = (x - y)$, and intersection down gets $(x/z - y/z)$.

$$(3) ((x/z)^2 + (y/z)^2 - 1).$$

$$(4) PS[1/z] = (1), \text{ since } z \text{ is inverted, so the intersection is all of } K[x/z, y/z].$$

- (e) (1) is just the point (a, b) , (2) is the curve $x/z = y/z$, (3) is the cone centered at $(0, 0, 0)$ and going up in the z direction. (4) is the empty set, as it is the vanishing set of the whole plane.

These are the exact same as sketching the point itself, except the last one. (x, z) corresponds to a point, but in projective space it corresponds to the empty set.

Problem 5. PROJECTIVE SPACE \mathbb{P}_K^n . Let $S = K[x_0, x_1, \dots, x_n]$ with its standard grading. Assume $K = \overline{K}$.

- (a) Show that $\bigcup_{i=0}^n D_+(x_i) = \text{Proj } S$.
- (b) Show that each $D_+(x_i)$ is homeomorphic to $\text{Spec } K[\frac{x_0}{x_i}, \dots, \frac{\hat{x}_i}{x_i}, \dots, \frac{x_n}{x_i}]$ (the $\frac{\hat{x}_i}{x_i}$, means that the i -th term is omitted).
- (c) Show point P of $\text{Proj } S$ is closed if and only if its height is n . (You don't need $K = \overline{K}$).
- (d) For an arbitrary closed point $P \in \text{Proj } S$ lying in the open chart $D_+(x_0)$, describe P looking "locally" in an appropriate open neighborhood $D_+(x_i)$, identified with $\text{Spec } K[\frac{x_0}{x_i}, \dots, \frac{\hat{x}_i}{x_i}, \dots, \frac{x_n}{x_i}]$.
- (e) Describe the same closed point P as a homogeneous prime in S (by describing generators). Prove that the algebraic set $\mathbb{V}(P)$ in K^{n+1} is a line through the origin.
- (f) Prove that the subspace of closed points of \mathbb{P}_K^n can be identified with the set of lines through $\mathbf{0} \in K^{n+1}$. [Hint: Use 6d.]

Solution:

- (a) It is immediate that $\bigcup_{i=0}^n D_+(x_i) \subset \text{Proj}(S)$. It is sufficient to verify that, for any homogeneous polynomial f , $D_+(f) \subset \bigcup_{i=0}^n D_+(x_i)$, since these form a basis of S . Well, Choose $P \in \text{Proj}(S)$ such that $f \notin P$. We want to show that there exists some $x_i \notin P$. Well, suppose there was. Then, $x_0, \dots, x_n \in P$. In particular, any polynomial would be in P in variables x_0, \dots, x_n would be in P , including f , a contradiction.

Thus, we can conclude that $\bigcup_{i=0}^n D_+(x_i) = \text{Proj}(S)$.

- (b) We can extend the statement in problem 4(a) to the statement that $S[1/x_i]_0 = K[x_0/x_i, \dots, \hat{x}_i/x_i, \dots, x_n/x_i]$. Thus, it is sufficient to verify that $D_+(x_i) \cong \text{Spec}(S[1/x_i]_0)$. From 4(c) we know the primes of $S[1/x_i]_0$ are precisely the primes in S that do not contain x_i (since if they did, their expansion/contraction would yields the whole space).

- (c) A prime ideal cannot be of height $n + 1$, since then it would contain S_+ . Thus, since closed points correspond precisely to maximal ideals, and ideals are maximal if and only if their heights are maximal, it follows that a point is a closed point if and only if its height is n .
- (d) We consider P in the open neighborhood $\text{Spec}(S[1/x_i]_0)$. This looks like a point (a_0, \dots, a_n) up to scaling by the i th coordinate. In other words, it looks like $(a_0/a_i, \dots, \hat{a}_i/a_i, \dots, a_n/a_i)$.
- (e) P is generated by a linear combination of x_1, \dots, x_n of the form $a_1x_1 + \dots + a_nx_n$. Replacing x_1, \dots, x_n with e_1, \dots, e_n (basis vectors of K^{n+1}) and taking the span of the resultant yields a n -dimensional hyperplane. It follows then that its vanishing set is an n -dimensional hyperplane that passes through the origin.
- (f) We've characterized one point above, so all closed points of $\mathbb{P}_K^n = \text{Proj}(S)$ form the set of all lines/hyperplanes through the origin.

Problem 6. PROOF OF THE THEOREM. Let S be an \mathbb{N} -graded ring. Take any homogeneous $f \in S$ of positive degree. Set $R = [S[\frac{1}{f}]]_0$.

- (a) Find a continuous map $D_+(f) \rightarrow \text{Spec } R$. [Hint: Factor through $\text{Spec } S[\frac{1}{f}]$.]
- (b) Show that the map in (a) is surjective. [Hint: For $P \in \text{Spec } R$, use (3) to show that $\sqrt{PS[\frac{1}{f}]}$ is a homogeneous prime of $S[\frac{1}{f}]$ which contracts to P . You might want to first try the case where $\deg f = 1$.]
- (c) Show that the map in (a) is one-to-one. [Hint: If $g \in Q \setminus P$, consider $\frac{g^{\deg f}}{f^{\deg g}}$ in R .]
- (d) Show that the map in (a) is a homeomorphism. [Hint: Show $\text{Spec } R \rightarrow \text{Spec } S[\frac{1}{f}]$ sending $P \mapsto \sqrt{PS[\frac{1}{f}]}$ is continuous.]
- (e) Conclude that $\text{Proj } S$ is covered by open sets each of which is homeomorphic to a topological space of the form $\text{Spec } R$ (for different R). [Hint: Remember $P \in \text{Proj } S$ means $P \not\supset S_+$.]
- (f) If S is finitely generated as an S_0 -algebra, prove that $\text{Proj } S$ is covered by *finitely many* open sets of the form $\text{Spec } R$.

Solution:

- (a) Notice that we have a map $D_+(f) \rightarrow \text{Spec}(S[1/f])$ sending $P \mapsto PS[1/f]$. As we've seen before, we know that $PS[1/f]$ is prime. Since we have an inclusion map $S[1/f]_0 \rightarrow S[1/f]$, by contravariance of Spec we now have a map $D_+(f) \rightarrow \text{Spec}(S[1/f]) \rightarrow \text{Spec}(R)$.

- (b) We've already shown in 4(c) that $PS[1/f] \cap R$ is prime provided that P is prime and $f \notin P$. The hint combined with this result immediately gives us that (a) is surjective.
- (c) Suppose that $P, Q \in D_+(f)$ and map to the same element of $\text{Spec}(R)$. We wish to show that $P = Q$. Well, if not, then if we choose $g \in Q \setminus P$, say of degree m , then $\deg(g^{\deg(f)}/f^{\deg(g)}) = \deg((m \deg(f))/(m \deg(f))) = 0$, so $g^{\deg(f)}/f^{\deg(g)} \in R$. A slight adjustment of 4(b) forces that the fiber of g is in P , which would be a contradiction.
- (d) The fact this is a homeomorphism follows from similar logic to problem 4.
- (e) We already know that the sets $D_+(f)$ for f homogeneous form a basis for $\text{Proj}(S)$. We've just shown that $D_+(f)$ is homeomorphic to $\text{Spec}(R)$ for different R s (namely, $R = S[1/f]_0$). Thus, we can conclude that $\text{Proj } S$ is covered by open sets each of which is homeomorphic to a topological space of the form $\text{Spec } R$.
- (f) We know from problem 1 of the previous worksheet that any graded ring R has a natural R_0 -algebra structure. If S is finitely generated as an S_0 -algebra, this implies that S_+ is finitely generated, and in particular, only finitely many f are needed to cover the space. Thus, $\text{Proj}(S)$ is covered by finitely many sets of the form $D_+(f) \cong \text{Spec}(R)$.

Math 614: Commutative Algebra

IBL Lab Notebook

Class 17: November 5th

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Problem 1. TENSOR PRODUCTS OF VECTOR SPACES. Let M and N be finite dimensional vector spaces over a field K of dimensions m, n respectively.

- (a) Fix bases so that $M \cong K^m$ and $N \cong K^n$ are identified with spaces of column vectors in the usual way. Using the matrix multiplication map

$$K^m \times K^n \rightarrow K^{m \times n} \quad (\vec{v}, \vec{w}) \mapsto \vec{v} \cdot \vec{w}^{tr},$$

prove that $K^m \otimes_K K^n$ is naturally² isomorphic to the space $K^{m \times n}$ of $m \times n$ matrices.

- (b) Conclude that $M \otimes_K N$ has dimension mn , and describe an explicit basis in terms of bases for M and N .
- (c) Let $X \subset K^{m \times n}$ be the image of the bilinear map in (a). Explain why X consists of the matrices of rank at most 1, and why this is an algebraic set.
- (d) How likely is it that a randomly chosen element of $M \otimes_K N$ can be written as $m \otimes n$?

Solution:

- (a) It is sufficient to verify that $K^{m \times n}$ satisfies the universal property of tensor products. Using the map given above, notice that if I have any map $f : K^m \times K^n \rightarrow P$, for some K -vector space P , it factors through uniquely to a map $\bar{f} : K^{m \times n} \rightarrow P$ where $\bar{f}(e_{ij}) = f(e_i, e_j)$, for standard basis vectors e_{ij} of $K^{m \times n}$ and e_i, e_j standard basis vectors of K^m, K^n .
- (b) $\dim(M \otimes_K N) \cong \dim(K^{m \times n}) = mn$. The basis of $M \otimes N$ is of the form $e_i \otimes f_j$ for $i \leq m, j \leq n$ and e_i a basis of M and f_j a basis of N .
- (c) All matrices in X are outer products, and must have rank at most 1. In general, matrices of rank $\leq t$ are an algebraic set, since they are the vanishing set of all the $t \times t$ minor determinants (which are polynomials).
- (d) It almost certainly will not be.

Problem 2. Explain why the tensor product of finitely generated modules is finitely generated, and why the tensor product of free modules (of rank m and n respectively) is free (of rank mn).

²meaning that the isomorphism can be described without choosing a basis.

Solution: Suppose M, N are finitely generated modules, with generators m_1, \dots, m_t and n_1, \dots, n_s . It follows from the logic of problem 1 that $M \otimes N$ has generators $\{m_i \otimes n_j \mid i \leq t, j \leq s\}$, and this set clearly has cardinality mn , implying that $M \otimes N$ is finitely generated. Repeating that same string of logic but replacing "generators" with "free basis" yields the desired statement about free modules.

Problem 3. RIGHT EXACTNESS OF TENSOR. For fixed M , consider the covariant functor $M \otimes_R -$ from $R\text{-mod}$ to $R\text{-mod}$ given by $N \mapsto M \otimes_R N$.

- (a) Prove that $M \otimes_R -$ is right exact. [Hint: Find a surjection $M \otimes_R M_2 / \text{im}(M \otimes_R M_1) \rightarrow M \otimes_R M_3$. To show it is an isomorphism, construct a bilinear map $M \times M_3 \rightarrow M \otimes_R M_2 / \text{im}(M \otimes_R M_1)$. Be sure to check your bilinear map is well-defined.]
- (b) Prove that $M \otimes_R -$ is not left exact in general. [Hint: Take $M = R/I$ and consider a sequence $0 \rightarrow R \xrightarrow{f} R \rightarrow R/\langle f \rangle$ where $f \in I$ is a non-zero-divisor of R .]
- (c) Prove free modules are flat. Conclude that $V \otimes_K -$ is exact in the category $K\text{-Vector spaces}$, for any V .

Solution:

- (a) Consider the short exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \longrightarrow 0$$

We want to verify that

$$A \otimes M \xrightarrow{i \otimes \text{Id}} B \otimes M \xrightarrow{\pi \otimes \text{Id}} C \otimes M \longrightarrow 0$$

is exact. Well, it is clear that $\pi \otimes \text{Id}$ is surjective, since for any $\sum_i c_i \otimes m_i \in C \otimes M$, notice that each c_i has a preimage b_i such that $\pi(b_i) = c_i$, due to surjectivity of π . Thus, it follows that $\pi \otimes \text{Id}(\sum_i b_i \otimes m_i) = \sum_i c_i \otimes m_i$.

Now we just need to verify that $\text{im}(i \otimes \text{Id}) = \ker(\pi \otimes \text{Id})$. Well, I claim that as R -modules,

$$\ker(\pi \otimes \text{Id}) \cong \ker(\pi) \otimes M = \text{im}(i) \otimes M \cong \text{im}(i \otimes \text{Id})$$

We can verify these with universal properties. Notice that any map $f : \ker(\pi) \times M \rightarrow P$ factors through $\ker(\pi \otimes \text{Id})$ to get a map $\bar{f} : \ker(\pi \otimes \text{Id}) \rightarrow P$ such that $\bar{f}(\sum k_j \otimes m_j) = \sum f(k_j, m_j)$.

Similarly, any map $g : \text{im}(i) \times M \rightarrow P$ factors through $\text{im}(i \otimes \text{Id})$ via a unique map $\bar{g} : \text{im}(i \otimes \text{Id}) \rightarrow P$ where $\bar{g}(\sum x_j \otimes m_j) = \sum g(x_j, m_j)$. Thus, the result follows.

- (b) We take the short exact sequence as in the hint. Tensoring yields the sequence

$$0 \longrightarrow R \otimes R/I \xrightarrow{r \otimes \bar{s} \rightarrow rf \otimes \bar{s}} R \otimes R/I \xrightarrow{\pi \otimes \text{Id}} R/(f) \otimes R/I \longrightarrow 0$$

However, notice that for any $r \otimes \bar{s}$, $rf \otimes \bar{s} = f(r \otimes \bar{s}) = r \otimes \overline{fs} = r \otimes \bar{0} = 0$. Thus, this is the zero map, and is clearly not injective, implying the sequence is not exact.

- (c) Suppose M is a free module, i.e. $M \cong R^{\oplus m}$ for some $m \in \mathbb{N}$. Tensoring the short exact sequence in part (a) by M yields the following short exact sequence:

$$0 \longrightarrow A \otimes R^{\oplus n} \longrightarrow B \otimes R^{\oplus n} \longrightarrow C \otimes R^{\oplus n} \longrightarrow 0$$

However, from here we have the following short exact sequence isomorphisms:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A \otimes R^{\oplus n} & \longrightarrow & B \otimes R^{\oplus n} & \longrightarrow & C \otimes R^{\oplus n} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \bigoplus_{i=1}^n R \otimes A & \longrightarrow & \bigoplus_{i=1}^n R \otimes B & \longrightarrow & \bigoplus_{i=1}^n R \otimes C \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \bigoplus_{i=1}^n A & \longrightarrow & \bigoplus_{i=1}^n B & \longrightarrow & \bigoplus_{i=1}^n C \longrightarrow 0 \end{array}$$

Where the maps in the lowest sequence are precisely the initial i, π acting on each of the n components. It follows that this sequence is exact via the exactness of the initial sequence.

Problem 4. LOCALIZATION. Let $U \subset R$ be a multiplicative system. For an R -module M , define $U^{-1}M$ as the set of equivalence classes $\frac{m}{u}$ where $m \in M$ and $u \in U$ with $\frac{m}{u} \sim \frac{m'}{u'}$ if there exists $v \in U$ such that $v(u'm - um') = 0$.

- Explain why $M \mapsto U^{-1}M$ is a functor from $R\text{-mod}$ to $U^{-1}R\text{-mod}$.
- For $R = \mathbb{Z}$ and $U = \mathbb{Z} \setminus \{0\}$, show that the localization functor kills torsion modules.
- Prove there is a unique isomorphism $U^{-1}R \otimes_R M \cong U^{-1}M$. [Hint: Universal properties.]
- Prove that the localization functor is exact.
- Show that $U^{-1}R$ is a flat R -module.

Solution:

- We need to verify $M \mapsto U^{-1}M$ acts on morphisms. Notice that for any R -module homomorphism $f : M \rightarrow N$, passing it through the functor yields the $U^{-1}R$ -module homomorphism $\bar{f} : U^{-1}M \rightarrow U^{-1}N$ such that $m/u \mapsto f(m)/u$.

- (b) Suppose M is a torsion \mathbb{Z} -module, i.e. $\exists r \in \mathbb{Z} \setminus 0$ such that $rm = 0 \forall m \in M$. We wish to show that localizing by $\mathbb{Z} \setminus 0$ kills M . Well, choose $m/u \in U^{-1}M$, for $U = \mathbb{Z} \setminus 0$. Suppose $zm = 0$ for $z \in \mathbb{Z} \setminus 0$.

$$\frac{m}{u} = \frac{1}{u}m = \frac{1}{u} \frac{1}{z}zm = \frac{1}{uz}0 = 0$$

Thus, $U^{-1}M = 0$.

- (c) Any map $f : U^{-1}R \times M \rightarrow P$ factors uniquely through $U^{-1}M$ via the map $\bar{f} : U^{-1}M \rightarrow P$ where $\bar{f}(m/u) = f(1/u, m)$.
- (d) Take a short exact sequence as in 3(a). Localizing yields

$$0 \longrightarrow U^{-1}A \xrightarrow{a/u \rightarrow i(a)/u} U^{-1}B \xrightarrow{b/u \rightarrow \pi(b)/u} U^{-1}C \longrightarrow 0$$

We call these new maps i', π' . Notice that $\forall c/u \in U^{-1}C$, $\exists b/u$ such that $\pi'(b/u) = \pi(b)/u = c/u$, via surjectivity of π . Furthermore, notice that $\pi' \circ i'(a/u) = (\pi \circ i)(a)/u = 0$, so $\text{im}(i) \subset \ker(\pi)$. Furthermore, if $b/u \in \ker(\pi')$, then $b \in \ker(\pi) = \text{im}(i)$, so $b/u \in \text{im}(i')$. Thus, $\text{im}(i') = \ker(\pi')$.

Finally, notice that if $a/u \in \ker(i')$, then $i(a)/u = 0$, so $i(a) = 0$. Since i is injective, $a = 0$, implying that $\ker(i') = 0$, and i' is injective. This completes the proof of exactness.

- (e) Since we know, for any short exact sequence as in 3(a), that

$$0 \longrightarrow U^{-1}A \xrightarrow{a/u \rightarrow i(a)/u} U^{-1}B \xrightarrow{b/u \rightarrow \pi(b)/u} U^{-1}C \longrightarrow 0$$

is exact, since $U^{-1}A \cong U^{-1}R \otimes A$, and similarly for B, C , this yields an isomorphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & U^{-1}A & \xrightarrow{a/u \rightarrow i(a)/u} & U^{-1}B & \xrightarrow{b/u \rightarrow \pi(b)/u} & U^{-1}C \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & U^{-1}R \otimes A & \longrightarrow & U^{-1}R \otimes B & \longrightarrow & U^{-1}R \otimes C \longrightarrow 0 \end{array}$$

Since isomorphisms preserve exactness, it follows that $U^{-1}R \otimes -$ is an exact functor, so $U^{-1}R$ is exact.

Problem 5. BASE CHANGE. Let $\varphi : R \rightarrow S$ be a ring homomorphism.

- (a) Show that $M \mapsto S \otimes_R M$ is a right exact functor from $R\text{-mod}$ to $S\text{-mod}$.
- (b) In particular, show that $S \otimes_R R/I \cong S/IS$ for all ideals $I \subset R$ and $S \otimes_R R[x_1, \dots, x_n]/J \cong S[x_1, \dots, x_n]/JS[x_1, \dots, x_n]$ for all ideals $J \subset R[x_1, \dots, x_n]$.

- (c) Show that this functor is not left exact in general. [Hint: Use (5).]
- (d) For any multiplicative system $U \subset R$, show that the functor from $R\text{-mod}$ to $U^{-1}R\text{-mod}$ sending $M \mapsto U^{-1}R \otimes M$ is exact.
- (e) Show that if S is a flat R -algebra, then $S \otimes_R I \cong IS$ for all ideals $I \subset R$.

Solution:

- (a) This is immediate from the proof that $M \mapsto S \otimes_R M$ is a right exact functor from $R\text{-mod}$ to $R\text{-mod}$, as the functor behaves identically on morphisms. the R -action on S acts via the specified homomorphism, but this does nothing to change exactness.
- (b) Consider the exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

By right exactness, we know that

$$I \otimes S \longrightarrow R \otimes S \longrightarrow R/I \otimes S \longrightarrow 0$$

Where $R \otimes S \cong S$. Notice that the image of $I \otimes S \rightarrow S$ is precisely IS , implying that $\ker(S \rightarrow R/I \otimes S) \cong IS$. Notice that this map is surjective, implying via the first isomorphism theorem that $S/IS \cong R/I \otimes S$, and we can conclude.

- (c) We can let $S = R/I$ and let φ be the quotient map, and reduce to part 3(b), where the map that needs to be injective for exactness to hold is in fact the zero map (both as an R -module and as an S -module homomorphism).
- (d) This follows from $U^{-1}R$ being a flat R -module, so tensoring preserves exactness in the category of R -modules. Given the way the functor acts on morphisms, it is clear that similar logic holds when viewing this as a functor from $R\text{-mod}$ to $U^{-1}R\text{-mod}$.
- (e) Consider the sequence as in part (b). By flatness, we have the following short exact sequence:

$$0 \longrightarrow I \otimes S \longrightarrow R \otimes S \longrightarrow R/I \otimes S \longrightarrow 0$$

We know that the kernel of $(R \otimes S \rightarrow R/I \otimes S)$ is IS , implying via exactness that $I \otimes S$ can be identified with IS .

Problem 6. ADJOINTNESS OF TENSOR AND HOM. Let $R \rightarrow S$ be a ring homomorphism. Let M and N be S modules, and let Q be an R -module.

- (a) Discuss natural R -module structures on M and N .
- (b) Discuss a natural S -module structure on $\text{Hom}_R(N, Q)$.

(c) Given a R -bilinear map $M \times N \rightarrow Q$, describe a natural R -module map $M \rightarrow \text{Hom}_R(N, Q)$.

(d) Show that there is an R -module isomorphism

$$\text{Hom}_R(M \otimes_R N, Q) \cong \text{Hom}_R(M, \text{Hom}_R(N, Q)).$$

(e) Show that there is an S -module isomorphism

$$\text{Hom}_R(M \otimes_S N, Q) \cong \text{Hom}_S(M, \text{Hom}_R(N, Q)).$$

Solution:

(a) $r \cdot m = \varphi(r)m$, and similarly for N .

(b) Choose $f \in \text{Hom}_R(N, Q)$. $r \cdot f(n) = f(\varphi(r)n)$.

(c) Given a map $b : M \times N \rightarrow Q$, we have a map $M \rightarrow \text{Hom}_R(N, Q)$ such that $m \mapsto b(m, -)$.

(d) We know that any bilinear map $M \times N \rightarrow Q$ extends uniquely to a map $M \otimes N \rightarrow Q$. Thus we have the naturally inverse maps

$$\begin{array}{ccc} & \xrightarrow{b(m \otimes n) \mapsto (m \mapsto b(m \otimes -))} & \\ \text{Hom}_R(M \otimes N, Q) & & \text{Hom}_R(M, \text{Hom}_R(N, Q)) \\ & \xleftarrow{(m \mapsto f_m(n)) \mapsto (m \otimes n \mapsto f_m(n))} & \end{array}$$

It is easy to check that these are R -module homomorphisms.

(e) We have nearly identical maps:

$$\begin{array}{ccc} & \xrightarrow{b(m \otimes n) \mapsto (m \mapsto b(m \otimes -))} & \\ \text{Hom}_R(M \otimes N, Q) & & \text{Hom}_S(M, \text{Hom}_R(N, Q)) \\ & \xleftarrow{(m \mapsto f_m(n)) \mapsto (m \otimes n \mapsto f_m(n))} & \end{array}$$

And these are similarly easy to verify as S -module homomorphisms.

Problem 7. TENSOR PRODUCT OF ALGEBRAS. Let A and B be R -algebras.

(a) Show that $A \otimes_R B$ has the structure of an R -algebra.

- (b) Show that there are R -algebra maps $A \rightarrow A \otimes_R B$ and $B \rightarrow A \otimes_R B$ which make $A \otimes_R B$ into a **coproduct** in the category of R -algebras. [Meaning: given any R -algebra T to which both A and B map, $\exists!$ R -algebra map $A \otimes_R B \rightarrow T$ making the relevant diagrams commute.]

Solution:

- (a) Let $\varphi : R \rightarrow A$ and $\psi : R \rightarrow B$ denote the defining ring homomorphisms. Define the map $\varphi \otimes \psi : R \rightarrow A \otimes B$ such that $r \mapsto \varphi(r) \otimes \psi(r)$. This ring homomorphism makes $A \otimes_R B$ an algebra.
- (b) Suppose there exists R -algebra homomorphisms $f_1 : A \rightarrow Y, f_2 : B \rightarrow Y$, along with inclusions $A \rightarrow A \otimes_R B, B \rightarrow A \otimes_R B$ which map $a \mapsto a \otimes 1, b \mapsto 1 \otimes b$ respectively. We want to find a map f that makes the following diagram commute.

$$\begin{array}{ccccc}
 & & Y & & \\
 & f_1 \nearrow & \uparrow f & \nwarrow f_2 & \\
 A & \longrightarrow & A \otimes_R B & \longleftarrow & B
 \end{array}$$

Well, consider the map $f' : A \times B \rightarrow Y$ where $f'(a, b) = f_1(a)f_2(b)$. This is R -bilinear, and thus extends uniquely to a map $f : A \otimes_R B \rightarrow Y$ which makes the diagram commute.

Problem 8. Let K be an algebraically closed field, and let $V \subset K^n$ and $W \subset K^m$ be algebraic sets.

- (a) Show that $V \times W$ is an algebraic set in K^{n+m} .
- (b) * Show that its coordinate ring is isomorphic to $K[V] \otimes_K K[W]$.

Solution:

- (a) Let $V = \mathbb{V}(I), W = \mathbb{V}(J)$, for $I \subset K[x_1, \dots, x_n], J \subset K[y_1, \dots, y_m]$. Observe that
- $$V \times W = \{(a_1, \dots, a_n, b_1, \dots, b_m) \mid f(a_1, \dots, a_n) = 0, g(b_1, \dots, b_m) = 0 \forall f \in I, g \in J\}$$
- Notice that $f + g$ is a polynomial in $n + m$ variables. Thus, this becomes
- $$V \times W = \{(a_1, \dots, a_n, b_1, \dots, b_m) \mid (f+g)(a_1, \dots, a_n, b_1, \dots, b_m) = 0, \forall f \in I, g \in J\}$$
- It follows that $V + W = \mathbb{V}(I + J)$.

Math 614: Commutative Algebra**IBL Lab Notebook**

Class 18: November 7th

Name: *Vignesh Jagathese*Collaborators: *Aniruddh Agarwal*

Problem 1. LOCAL PROPERTIES. For any $P \in \text{Spec } R$, let $M_P = (R \setminus P)^{-1}M$ be the localization of M at $P \in \text{Spec } R$. Show the following:

- (a) There is a natural R -module map $M \rightarrow M_P$ sending $m \mapsto \frac{m}{1}$, the “localization at P ” map.
- (b) Localization at P gives an *exact* functor $M \mapsto M_P$ from R -modules to R_P -modules.
- (c) For any R -module map $M \xrightarrow{f} N$, the formation of the kernel, cokernel, and image of f commute with localization at all $P \in \text{Spec } R$ (equivalently, $\text{MaxSpec } R$). That is, $(\ker f)_P = \ker f_P$, $(\text{coker } f)_P = \text{coker } f_P$, and $(\text{im } f)_P = \text{im } f_P$.
- (d) For any $m \in M$, show that $\text{ann}_R m = \{r \in R \mid rm = 0\}$ is an ideal of R .
- (e) Show $\text{ann}_R m$ is proper if and only if $m \neq 0$.
- (f) Show that $\frac{m}{1} \in M_P$ is nonzero if and only if $\text{ann}_R m \subset P$.
- (g) Show m is zero if and only if $\frac{m}{1} \in M_P$ is zero $\forall P \in \text{Spec } R$ (equivalently, $\text{MaxSpec } R$).
- (h) Show $M = 0$ if and only if $M_P = 0 \forall P \in \text{Spec } R$ (equivalently, $\text{MaxSpec } R$).
- (i) For any R -module map $f : M \rightarrow N$, f is injective (respectively surjective) if and only if f_P is injective (respectively, surjective) for all $P \in \text{Spec } R$ (equivalently, $\text{MaxSpec } R$).
- (j) Two submodules N_1, N_2 of M satisfy $N_1 \subset N_2$ if and only if $(N_1)_P \subset (N_2)_P$ in M_P for all $P \in \text{Spec } R$ (equivalently, $\text{MaxSpec } R$). [Hint: $N_1 \subset N_2$ if and only if $(N_1 + N_2)/N_2 = 0$.]
- (k) Prove the theorem.

Solution:

- (a) $rm \mapsto rm/1 = r(m/1)$, so this is well defined.
- (b) Follows from 4(d) from the previous worksheet, just localize by $(R \setminus P)$.
- (c) Let $f : R \rightarrow S$ be a ring homomorphism. Notice that we have a short exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow R \xrightarrow{f} \text{im}(f) \longrightarrow 0$$

Localizing at P gives us

$$0 \longrightarrow \ker(f)_P \longrightarrow R_P \xrightarrow{f_P} \operatorname{im}(f)_P \longrightarrow 0$$

This sequence is still exact, and in particular, f_P is surjective, so $\operatorname{im}(f_P) = \operatorname{im}(f)_P$. Furthermore, exactness tells us that $\ker(f_P) = \operatorname{im}(\ker(f)_P \rightarrow R_P) = \ker(f)_P$. To verify the statement with cokernels, we can take a different short exact sequence:

$$0 \longrightarrow \operatorname{im}(f) \longrightarrow S \xrightarrow{\pi} \operatorname{coker}(f) \longrightarrow 0$$

that we can localize:

$$0 \longrightarrow \operatorname{im}(f)_P \longrightarrow S_P \xrightarrow{\pi_P} \operatorname{coker}(f)_P \longrightarrow 0$$

Notice that $\operatorname{coker}(f_P) = \operatorname{im}(\pi_P) = \operatorname{coker}(f)_P$ via exactness.

(d) $0m = 0$, so $0 \in \operatorname{ann}_R(m)$. Furthermore, if $r, s \in \operatorname{ann}_R(m)$, $(r + s)m = rm + sm = 0 + 0 = 0$, so $r + s \in \operatorname{ann}_R(m)$. Furthermore, $-r(m) = -(rm) = 0$, so $-r \in \operatorname{ann}_R(m)$. This tells us that $\operatorname{ann}_R(m)$ is an abelian group. To conclude, for $t \in R$, $(tr)m = t(rm) = t0 = 0$, so we can conclude that this is an ideal of R .

(e) $\operatorname{ann}_R(m)$ is proper if and only if $\exists r \in R$ such that $rm \neq 0$, if and only if $m \neq 0$.

(f) Suppose that $u \in R \setminus P$.

$$\begin{aligned} m/1 \neq 0 &\iff (um/1)(1/u) \neq 0 \\ &\iff (um/1) \neq 0 \\ &\iff um \neq 0 \\ &\iff \operatorname{ann}_R(m) \cap (R \setminus P) = \emptyset \\ &\iff \operatorname{ann}_R(m) \subset P \end{aligned}$$

(g) $m/1 = 0 \iff \operatorname{ann}_R(m) \not\subset P \forall P \in \operatorname{MaxSpec}(R)$. Well, since maximal ideals are prime, this implies that $\operatorname{ann}_R(m) = R$, thus

$$\begin{aligned} m/1 = 0 &\iff \operatorname{ann}_R(m) \not\subset P \forall P \in \operatorname{MaxSpec}(R) \\ &\iff \operatorname{ann}_R(m) = R \\ &\iff m = 0 \end{aligned}$$

(h) Follows immediately from the previous part.

(i)

$$\begin{aligned} f_P \text{ is injective } \forall P \in \operatorname{Spec}(R) &\iff \ker(f_P) = 0 \forall P \in \operatorname{Spec}(R) \\ &\iff \ker(f)_P = 0 \forall P \in \operatorname{Spec}(R) \\ &\iff \ker(f) = 0 \\ &\iff f \text{ is injective} \end{aligned}$$

$$\begin{aligned} f_P \text{ is surjective } \forall P \in \operatorname{Spec}(R) &\iff \operatorname{coker}(f_P) = 0 \forall P \in \operatorname{Spec}(R) \\ &\iff \operatorname{coker}(f)_P = 0 \forall P \in \operatorname{Spec}(R) \\ &\iff \operatorname{coker}(f) = 0 \\ &\iff f \text{ is surjective} \end{aligned}$$

(j) We use the hint:

$$\begin{aligned}
 N_1 \subset N_2 &\iff 0 \rightarrow N_2 \rightarrow N_1 + N_2 \rightarrow 0 \text{ exact} \\
 &\iff 0 \rightarrow (N_2)_P \rightarrow (N_1 + N_2)_P \rightarrow 0 \text{ exact } \forall P \in \text{Spec}(R) \\
 &\iff (N_1 + N_2)_P / (N_2)_P = 0 \forall P \in \text{Spec}(R) \\
 &\iff (N_1)_P \subset (N_2)_P \forall P \in \text{Spec}(R)
 \end{aligned}$$

(k) Localizing the sequence in the theorem yields

$$0 \longrightarrow (M_1)_P \xrightarrow{f_P} (M_2)_P \xrightarrow{g_P} (M_3)_P \longrightarrow 0$$

Well, notice that f_P is injective $\forall P \in \text{Spec}(R)$ if and only if f is injective, and g_P is surjective $\forall P \in \text{Spec}(R)$ if and only if g is surjective via part (i), so it is sufficient to verify that $\text{im}(f) = \ker(g)$ if and only if $\text{im}(f_P) = \ker(g_P) \forall P \in \text{Spec}(R)$. Here we use part (j).

$$\begin{aligned}
 \text{im}(f) = \ker(g) &\iff \text{im}(f) \subset \ker(g), \text{im}(f) \supset \ker(g) \\
 &\iff \text{im}(f)_P \subset \ker(g)_P, \text{im}(f)_P \supset \ker(g)_P \forall P \in \text{Spec}(R) \\
 &\iff \text{im}(f_P) \subset \ker(g_P), \text{im}(f_P) \supset \ker(g_P) \forall P \in \text{Spec}(R) \\
 &\iff \text{im}(f_P) = \ker(g_P) \forall P \in \text{Spec}(R)
 \end{aligned}$$

And now we can conclude.

Problem 2. Fix any R -module M . Prove the following exactness properties of the following functors from $R\text{-mod}$ to $R\text{-mod}$.

- (a) Show that the covariant functor $N \mapsto \text{Hom}_R(M, N)$ is left exact.
 (b) Show that the contravariant functor $M \mapsto \text{Hom}_R(M, N)$ is left exact.

Solution:

(a) Let

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \longrightarrow 0$$

be a short exact sequence. We need to show that

$$0 \longrightarrow \text{Hom}(M, A) \xrightarrow{i \circ -} \text{Hom}(M, B) \xrightarrow{\pi \circ -} \text{Hom}(M, C)$$

is exact. Choose $\varphi \in \ker(i \circ -)$. This implies that $i \circ \varphi = 0$, so $\text{im}(\varphi) \subset \ker(i)$. Well, $\ker(i) = 0$, so $\text{im}(\varphi) = 0$, so $\varphi = 0$. It follows that $i \circ -$ is injective.

Next, choose $\varphi \in \text{im}(i \circ -)$, i.e. $\varphi = i \circ \psi$ for some $\psi \in \text{Hom}(M, A)$. This implies that $\pi \circ \varphi = \pi \circ i \circ \psi = 0$, since $\pi \circ i = 0$. Thus, $\varphi \in \ker(\pi \circ -)$, implying that $\text{im}(i \circ -) \subset \ker(\pi \circ -)$.

Finally, choose $\varphi \in \ker(\pi \circ -)$. This implies that $\pi \circ \varphi = 0$, so $\text{im}(\varphi) \subset \ker(\pi) = \text{im}(i)$. It follows that one can construct an R -module homomorphism ψ that maps an elements lift via i to the lift via φ , implying that $i \circ \psi = \varphi$, so $\varphi \in \text{im}(i \circ -)$.

(b) Given the same short exact sequence above, we wish to show that

$$0 \longrightarrow \text{Hom}(C, M) \xrightarrow{-\circ\pi} \text{Hom}(B, M) \xrightarrow{-\circ i} \text{Hom}(A, M)$$

is exact. First, choose $\varphi \in \ker(- \circ \pi)$. This implies that $\varphi \circ \pi = 0$. Thus, via surjectivity of π , we know that $\varphi(c) = \varphi \circ \pi(b) = 0$, so $\varphi = 0$, implying that $- \circ \pi$ is injective.

Next, we verify that $\text{im}(- \circ \pi) \subset \ker(- \circ i)$. Choose $\psi \circ \pi \in \text{im}(- \circ \pi)$. Notice that $(\psi \circ \pi) \circ i = \psi \circ (\pi \circ i) = \psi(0) = 0$, so $\psi \circ \pi \in \ker(- \circ i)$ as desired.

Finally, choose $\varphi \in \ker(- \circ i)$. This implies that $\varphi \circ i = 0$. Thus, $\ker(\varphi) \subset \text{im}(i) = \ker(\pi)$. Via a similar extension argument to part (a), we can conclude that $\varphi = \psi \circ \pi$, so $\varphi \in \text{im}(- \circ \pi)$.

Problem 3. PROJECTIVE MODULES.

- (a) Show that P is projective if and only if for every surjection $g : M_2 \twoheadrightarrow M_3$ and morphism $\phi : P \rightarrow M_3$, there exists $\tilde{\phi} : P \rightarrow M_2$ such that $\phi = g \circ \tilde{\phi}$.
- (b) Show that every free R -module is projective. [Use (a) and the Universal property of free modules.]
- (c) Show that if P is projective, then every surjection $M \twoheadrightarrow P$ splits in the category of R -modules. [Hint: Use (a).]
- (d) Show that if P is projective, then P is a direct summand of a free module. [Hint: Map a free module surjectively onto P .]
- (e) Conversely, suppose that P be an R -module with the property that there exists an R -module Q such that $P \oplus Q$ is free. Show that P is projective. [One straightforward way uses (a) and (b).]

Solution:

(a) Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence.

P is projective $\iff \text{Hom}(P, -)$ is exact
 $\iff \text{Hom}(P, -)$ is right exact
 $\iff \text{Hom}(P, M) \rightarrow \text{Hom}(P, N)$ is surjective
 $\iff \forall f : P \rightarrow N \exists g : P \rightarrow M$ such that the following diagram commutes:

$$\begin{array}{ccc} & & P \\ & \swarrow g & \downarrow f \\ M & \longrightarrow & N \longrightarrow 0 \end{array}$$

The first \iff is definition, the second comes from the fact that Hom functors are left exact. The final \iff follows from the definition of that map being surjective (i.e. we can pick preimages).

(b) We wish to show that $R^{\oplus I}$ is projective. Suppose we have a short exact sequence as above, and let $\{e_i\}_{i \in I}$ denote a free basis of $R^{\oplus I}$. Recall that maps from free modules are completely determined by where the free-basis is sent.

Suppose we have a map $f : R^{\oplus I} \rightarrow N$. Let $f(e_i) = n_i \forall i \in I$. Furthermore, we can pick preimages m_i such that $m_i \rightarrow n_i$ via the map $M \rightarrow N$, since this map is surjective. Thus, we can define $g : R^{\oplus I} \rightarrow M$ such that $g(e_i) = m_i$. This is well defined, since maps are uniquely defined by where they send free bases, and this obvious allows the diagram to commute.

(c) Let $\sigma : M \rightarrow P$ denote the surjection. Notice that this gives us a short exact sequence

$$0 \longrightarrow \ker(\sigma) \longrightarrow M \xrightarrow{\sigma} P \longrightarrow 0$$

Via projectivity of P , we have the maps

$$\begin{array}{ccc} & & P \\ & \swarrow \psi & \downarrow \text{Id} \\ 0 \longrightarrow \ker(\sigma) \longrightarrow M & \xrightarrow{\sigma} & P \longrightarrow 0 \end{array}$$

This gives us a map ψ such that $\sigma \circ \psi = \text{Id}$. By the splitting lemma, this implies that σ splits in the category of R -modules, i.e. $M \cong P \oplus \ker(\sigma)$.

(d) Notice that we have a surjection $\sigma : R^{\oplus P} \rightarrow P$. From the previous part, we know that $R^{\oplus P} \cong P \oplus \ker(\sigma)$. Consider a short exact sequence as in (a). Since $P \oplus Q$ is free it is projective, and $P \rightarrow P \oplus Q$ defines a natural inclusion map. Next, notice that for any map $f : P \rightarrow N$, there exists a map $\bar{f} : P \oplus Q \rightarrow N$ such that $\bar{f}(p, q) = f(p)$, i.e. any map $P \rightarrow N$ factors through $P \oplus Q$. This implies that if we have a map

$$\begin{array}{ccccccc}
 & & & & P & & \\
 & & & & \downarrow f & & \\
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0
 \end{array}$$

It factors through $P \oplus Q$:

$$\begin{array}{ccccccc}
 & & & & P & & \\
 & & & & \downarrow & \searrow f & \\
 & & & & P \oplus Q & & \\
 & & & & \downarrow \bar{f} & \swarrow & \\
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0
 \end{array}$$

But via projectivity of $P \oplus Q$, \bar{f} factors through B :

$$\begin{array}{ccccccc}
 & & & & P & & \\
 & & & & \downarrow & \searrow f & \\
 & & & & P \oplus Q & & \\
 & & & \swarrow \bar{g} & \downarrow \bar{f} & \swarrow & \\
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0
 \end{array}$$

Composing \bar{g} with the inclusion map yields the desired map g that makes P projective:

$$\begin{array}{ccccccc}
 & & & & P & & \\
 & & & & \downarrow & \searrow f & \\
 & & & & P \oplus Q & & \\
 & & & \swarrow \bar{g} & \downarrow \bar{f} & \swarrow & \\
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0
 \end{array}$$

(Note: A dashed arrow labeled g goes from P to B in the above diagram.)

Problem 4. PROJECTIVITY CAN BE TESTED LOCALLY FOR FINITELY PRESENTED MODULES. Assume the Proposition above on commutation of Hom with flat base change.

- (a) Prove that a finitely presented R -module Q is projective if and only if Q_P is a projective R_P -module for all $P \in \text{Spec } R$ (or $\text{MaxSpec } R$).
- (b) Prove that if R is Noetherian, then this statement holds also for finitely generated R -modules. [Hint: Map a finitely generated free module onto M and consider the kernel.]

Solution:

- (a) We know that R_P is a flat R -algebra. thus, using the proposition, we see that

$$R_P \otimes \text{Hom}_R(M, N) \cong \text{Hom}_{R_P}(R_P \otimes M, R_P \otimes N) \cong \text{Hom}_{R_P}(M_P, N_P)$$

In particular, since the composition of exact functors is exact, we know that $\forall P \in \text{Spec}(R)$,

$$\begin{aligned} Q \text{ projective} &\iff \text{Hom}_R(Q, -) \text{ exact} \\ &\iff R_P \otimes \text{Hom}_R(Q, -) \text{ exact} \\ &\iff \text{Hom}_{R_P}(Q_P, -) \text{ exact} \\ &\iff Q_P \text{ projective} \end{aligned}$$

- (b) Over Noetherian rings, being finitely presented is equivalent to being finitely generated.

Problem 5. THE PROOF THAT HOM COMMUTES WITH FLAT BASE CHANGE. Let S be an R -algebra. Fix an R -module N .

- (a) For any R -module M , describe a natural S -module map

$$S \otimes \text{Hom}_R(M, N) \xrightarrow{\theta_M} \text{Hom}_S(S \otimes_R M, S \otimes_R N).$$

[Hint: First find an R -linear map.]

- (b) Fix N . Show that $\theta = \theta_M$ defines a natural transformation from the functor

$$S \otimes_R \text{Hom}_R(-, N) \quad \text{to} \quad \text{Hom}_S(S \otimes_R -, S \otimes_R N).$$

This means that, given an R -module map $M_1 \xrightarrow{g} M_2$, an appropriate diagram involving θ_{M_1} and θ_{M_2} commutes.

- (c) Show that θ_R is an isomorphism.
- (d) Show that $\theta_{M_1 \oplus M_2}$ can be identified with $\theta_{M_1} \oplus \theta_{M_2}$. In particular, show that θ_F is an isomorphism if F is a finitely generated free module.
- (e) Prove the Proposition. [Hint: Show the map in (a) is an isomorphism by using (2b) and right exactness of tensor to construct a commuting diagram from a presentation of M . The desired isomorphism will come from the kernels of isomorphic maps.]

Solution:

- (a) We first define a bilinear map

$$S \times \text{Hom}_R(M, N) \xrightarrow{\theta'_M} \text{Hom}_S(S \otimes M, S \otimes N)$$

$$(s, f) \longmapsto (s' \otimes m \mapsto ss' \otimes f(m))$$

This is clearly bilinear, and thus extends to the map θ_M above.

- (b) Given the morphism $g : M_1 \rightarrow M_2$, we want to verify that the following diagram commutes:

$$\begin{array}{ccc} S \otimes_R \text{Hom}_R(M_1, N) & \longrightarrow & \text{Hom}_S(S \otimes_R M_1, S \otimes_R N) \\ \downarrow & & \downarrow \\ S \otimes_R \text{Hom}_R(M_2, N) & \longrightarrow & \text{Hom}_S(S \otimes_R M_2, S \otimes_R N) \end{array}$$

Following along the top, we get $s \otimes f \mapsto s(\text{Id} \otimes f) \mapsto s(\text{Id} \otimes g \circ f)$. Going along the bottom, we get $s \otimes f \mapsto s \otimes g \circ f \mapsto s(\text{Id} \otimes g \circ f)$. Thus, this diagram commutes (we've only proven it for basic tensors, but for general tensors the proof is identical, just much harder to read).

- (c) We want to show that

$$S \otimes_R \text{Hom}_R(R, N) \cong \text{Hom}_S(S \otimes_R R, S \otimes_R N)$$

Well,

$$\begin{aligned} S \otimes_R \text{Hom}_R(R, N) &\cong S \otimes_R N \\ &\cong \text{Hom}_S(S, S \otimes_R N) \\ &\cong \text{Hom}_S(S \otimes_R R, S \otimes_R N) \end{aligned}$$

- (d) This follows because direct sum commutes with Hom, i.e. $\text{Hom}(M_1 \oplus M_2, N) \cong \text{Hom}(M_1, N) \oplus \text{Hom}(M_2, N)$. In particular, $\text{Hom}_S(S \otimes (M_1 \oplus M_2), S \otimes N) \cong \text{Hom}_S(S \otimes M_1, S \otimes N) \oplus \text{Hom}_S(S \otimes M_2, S \otimes N)$, so the maps $\theta_{M_1 \oplus M_2}$ and $\theta_{M_1} \oplus \theta_{M_2}$ can be identified with each other.

If F is a finitely generated free module, then $\theta_F = \theta_{R^{\oplus n}}$ can be identified with $\bigoplus^n \theta_R$, which we know to be an isomorphism. Thus, θ_F is just n copies of an isomorphism, and is thus an isomorphism itself.

- (e) Since M is finitely presented, we have a short exact sequence

$$R^{\oplus n} \longrightarrow R^{\oplus m} \longrightarrow M \longrightarrow 0$$

For $m, n \in \mathbb{N}$. We now pass it through hom functors and tensors in different orders.

$$0 \longrightarrow \text{Hom}_R(M, N) \longrightarrow \text{Hom}_R(R^{\oplus m}, N) \longrightarrow \text{Hom}_R(R^{\oplus n}, N)$$

$$0 \longrightarrow S \otimes \text{Hom}_R(M, N) \longrightarrow S \otimes \text{Hom}_R(R^{\oplus m}, N) \longrightarrow S \otimes \text{Hom}_R(R^{\oplus n}, N)$$

Here we pass the above short exact sequence first through $\text{Hom}_R(-, N)$, then $S \otimes -$. Exactness is preserved because Hom is left exact, and S is flat, so $S \otimes -$ is exact. Similarly, we can take the first short exact sequence and get the following short exact sequences:

$$S \otimes R^{\oplus n} \longrightarrow S \otimes R^{\oplus m} \longrightarrow S \otimes M \longrightarrow 0$$

$$0 \rightarrow \text{Hom}_S(S \otimes M, S \otimes N) \rightarrow \text{Hom}_S(S \otimes R^{\oplus m}, S \otimes N) \rightarrow \text{Hom}_S(S \otimes R^{\oplus n}, S \otimes N)$$

First passing through $S \otimes -$ and then through $\text{Hom}_S(-, S \otimes N)$. This still preserves exactness, since tensoring is right exact and Hom is left exact. This yields two short exact sequences with $\theta_{(-)}$ maps between them:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ S \otimes \text{Hom}_R(M, N) & \xrightarrow{\theta_M} & \text{Hom}_S(S \otimes M, S \otimes N) \\ \downarrow & & \downarrow \\ S \otimes \text{Hom}_R(R^{\oplus m}, N) & \xrightarrow{\theta_{R^{\oplus m}}} & \text{Hom}_S(S \otimes R^{\oplus m}, S \otimes N) \\ \downarrow & & \downarrow \\ S \otimes \text{Hom}_R(R^{\oplus n}, N) & \xrightarrow{\theta_{R^{\oplus n}}} & \text{Hom}_S(S \otimes R^{\oplus n}, S \otimes N) \end{array}$$

The latter two are isomorphism via the previous part, and by exactness, we can conclude that θ_M is also an isomorphism.

Problem 6. INJECTIVE MODULES.

- Show that J is injective if and only if for every injection $g : M_1 \hookrightarrow M_2$ and morphism $\phi : M_1 \rightarrow J$, there exists $\tilde{\phi} : M_2 \rightarrow J$ such that $\tilde{\phi} = g \circ \phi$.
- Show if J is injective, then every inclusion $J \hookrightarrow M$ splits in R -mod. [Hint: Use (a).]
- * Prove the converse of (b).

Solution:

(a) Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence.

- J is injective $\iff \text{Hom}(-, J)$ is exact
 $\iff \text{Hom}(-, J)$ is right exact
 $\iff \text{Hom}(M, J) \rightarrow \text{Hom}(L, J)$ is surjective
 $\iff \forall f : L \rightarrow J \exists g : M \rightarrow J$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & L & \hookrightarrow & M & \longrightarrow & N \longrightarrow 0 \\
 & & \downarrow f & & \swarrow g & & \\
 & & J & & & &
 \end{array}$$

This follows from similar logic to the projective case.

(b) This also follows from similar logic to the projective case. Let $\sigma : J \rightarrow M$ be an inclusion. Then we have a short exact sequence:

$$0 \longrightarrow J \xrightarrow{\sigma} M \longrightarrow \text{coker}(\sigma) \longrightarrow 0$$

Via injectivity of J , we have that

$$\begin{array}{ccccccc}
 0 & \longrightarrow & J & \xrightarrow{\sigma} & M & \longrightarrow & \text{coker}(\sigma) \longrightarrow 0 \\
 & & \downarrow \text{Id} & & \swarrow \psi & & \\
 & & J & & & &
 \end{array}$$

Implying that σ has a left inverse that is an R -module homomorphism, implying it splits in the category of R -modules.

Math 614: Commutative Algebra**IBL Lab Notebook****Class 19: November 12th***Name: Vignesh Jagathese**Collaborators: Aniruddh Agarwal***Problem 1.** Let M be a module over (R, m) .

- (a) Verify that M/mM really is a vector space over R/m .
- (b) Show that $M/mM \cong R/m \otimes_R M$.
- (c) Compute the R/m -dimension of M/mM in the following cases:
- (1) $R = K[x, y]_{\langle x, y \rangle}$ and $M = R^{\oplus 5}$.
 - (2) $R = K[x_1, x_2, \dots, x_n]_m$ where $m = \langle x_1, \dots, x_n \rangle$, and $M = m$.
 - (3) $R = \frac{K[[x, y, z]]}{\langle x^4 + y^4 + z^4 \rangle}$ and $M = m$.

Solution:

- (a) M/mM has an associated R/m action, implying it is an R/m -module. R/m is a field, and modules over fields are vector spaces. Thus, M/mM is a vector space over R/m .
- (b) Consider the short exact sequence

$$0 \longrightarrow m \longrightarrow R \longrightarrow R/m \longrightarrow 0$$

Tensoring via M is right exact, so we have a right exact sequence

$$m \otimes M \longrightarrow R \otimes M \longrightarrow R/m \otimes M \longrightarrow 0$$

Recall that $R \otimes M \cong M$. This would imply that the image of $m \otimes M \rightarrow M$ is precisely $mM \subset M$ under the above isomorphism. Via exactness, we can conclude that

$$R/m \otimes M \cong (R \otimes M)/mM \cong M/mM$$

Observe that this holds for any ideal $I \subset R$, not necessarily the maximal one.

- (c) (1) 5, since M is free.
 (2) n .
 (3) 3.

Problem 2. Let $R = K[[x]]$ and let M be the R -module $K[[x]][x^{-1}]$. Show that $mM = M$ for the ideal $m = \langle x \rangle$. Why doesn't this contradict Nakayama's Lemma?

Solution: Since x is unit in M , it follows that $mM = M$. This does not contradict Nakayama's lemma since M is not finitely generated.

Problem 3. PROOF OF NAKAYAMA'S LEMMA. Fix M finitely generated over (R, m) .

- (a) Prove Nakayama's Lemma in the case M is generated by one element.
[Hint: If x generates M , show there exists $r \in m$, such that $rx = x$; now observe that $r - 1$ is a unit.]
- (b) Show that if M is generated by x_1, \dots, x_n and $mM = M$, then $M' = M/Rx_n$ is generated by $n - 1$ elements and $M' = mM'$.
- (c) Prove Nakayama's Lemma by induction on the number of generators of M .
[Hint: For the inductive step, use (b) and then (a).]

Solution:

- (a) Suppose n generates M , i.e. $M = Rn$. If $mM = M = Rn$, then $mRn = mn = Rn$. It follows that $\exists x \in m$ such that $xn = n$, so $(x - 1)n = 0$. Notice that $x - 1 \notin m$, because then $1 \in m$. Thus, $x - 1$ is unit.

Furthermore, $x - 1 \in \text{Ann}_R(n)$. Thus, we have a unit in the annihilator, and $M = 0$.

- (b) It is clear that M' is generated by $n - 1$ elements. We just need to verify that $mM' = M'$. Since $M = mM$, we know that $\forall \sum r_i x_i \in M$, $\exists n \in m, s_i \in R$ such that

$$n \sum_{i=1}^n r_i x_i = \sum_{i=1}^n s_i x_i$$

Projecting down one side to mM/Rx_n and the other to M/Rx_n yields

$$\overline{n \sum_{i=1}^n r_i x_i} = \overline{\sum_{i=1}^n s_i x_i}$$

Notice that $\overline{n \sum_{i=1}^n r_i x_i} = n \overline{\sum_{i=1}^n r_i x_i}$, since $n \in m \subset R$ and is independent of a quotient by a module generator. It follows then that

$$\overline{n \sum_{i=1}^n r_i x_i} = \overline{\sum_{i=1}^n s_i x_i}$$

for any $\overline{\sum_{i=1}^n r_i x_i} \in M'$, so $mM' = M'$.

- (c) The induction is immediate from the previous steps. The base case is part (a) and the inductive step follows from (b).

Problem 4. THE SECOND VERSION OF NAKAYAMA'S LEMMA. Let M be finitely generated over (R, m) and fix $\{x_1, \dots, x_d\} \subset M$. Assume that $\{\bar{x}_1, \dots, \bar{x}_d\} \subset M/mM$ spans M/mM over R/m . Use the first version of Nakayama's Lemma to show that $\{x_1, \dots, x_d\}$ generate M .

[Hint: Let $N \subset M$ be generated by $\{x_1, \dots, x_d\}$. Show $M = N + mM$, and that this implies $M/N = m(M/N)$.]

Solution: We follow the hint. We know that $x_1, \dots, x_d \in M$, so we can take the submodule N generated by x_1, \dots, x_d . Notice that if $r \notin mM$, then $\bar{r} \in M/mM$ is nonzero, and can be written as a sum of generators $\bar{x}_1, \dots, \bar{x}_d$. This summation can then be lifted, implying that $r \in N$. It follows that $M = N \oplus mM$. Quotienting both sides by N yields $(M/N) = mM/N = m(M/N)$. By Nakayama's Lemma, since M/N is finitely generated over a local ring, we can conclude that $M/N = 0$, so $M = N$.

Problem 5. Show that every minimal set of generators for a finitely generated module over a local ring has the same cardinality. [Minimal generating set means no proper subset also generates.]

Solution: The minimal set of generators of a finitely generated module M over a local ring (R, m) , by the previous result, correspond to the minimal spanning set of M/mM as an R/m vector space. All minimal spanning sets of vector spaces are bases, and have a fixed cardinality. It follows that their lifts in M must also have the same cardinality.

Problem 6. Let (R, m) be a local domain that is not a field. Prove that the fraction field of R is not a finitely generated R -module. [Hint: What would Nakayama say about a minimal generating set?]

Solution: It is clear that $m\text{Frac}(R) = \text{Frac}(R)$. If $\text{Frac}(R)$ was finitely generated, then Nakayama's lemma says that $\text{Frac}(R) = 0$, which is false. Thus, $\text{Frac}(R)$ is not finitely generated.

Problem 7. Suppose $IM = M$ for some finitely generated R -module M and proper ideal I in a local ring. Prove that $M = 0$.

Solution: Since $I \subset m$, as R is a local ring, we can see that $IM \subset mM \subset M$. If $IM = M$, then it follows that $mM = M$, so $M = 0$ by Nakayama's Lemma.

Problem 8. Let $M \xrightarrow{g} N$ be a morphism of finitely generated modules over a local ring (R, m) .

- (a) Prove g is surjective if and only if the induced map of R/m vector spaces $M/mM \xrightarrow{g} N/mN$ is surjective. [Hint: Consider the right exact sequence $M \xrightarrow{g} N \rightarrow \text{coker } g \rightarrow 0$. Use (1b).]
- (b) Show that both implications of the corresponding statement about injectivity are false. [Hint: Let $R = \mathbb{Z}_{\langle p \rangle}$. Consider $R \xrightarrow{\times p} R$ and $R \rightarrow R/\langle p \rangle$.]

Solution:

- (a) Consider the sequence given in the hint. Tensoring the whole sequence preserves exactness by right exactness of tensor. This induces a map of sequences:

$$\begin{array}{ccccccc}
 R/m \otimes M & \xrightarrow{\text{Id} \otimes g} & R/m \otimes N & \longrightarrow & R/m \otimes \text{coker}(g) & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 M/mM & \xrightarrow{g'} & N/mN & \longrightarrow & \text{coker}(g') & \longrightarrow & 0
 \end{array}$$

Under this isomorphism, $\text{coker}(g')$ can be identified with $\text{coker}(g) \otimes R/m$. Since $R/m \neq 0$, it follows that $\text{coker}(g') = 0 \iff \text{coker}(g) = 0$, so g is surjective if and only if g' is surjective.

- (b) We follow the hint. Recall that \mathbb{Z}_p is a local ring with $p\mathbb{Z}_p$ as its unique maximal ideal. Notice that the first map $R \rightarrow R$ (the multiplication by p map) is injective, however, over $\mathbb{Z}_p/p\mathbb{Z}_p$ it is the zero map and is not injective. However, the projection map $\mathbb{Z}_p \rightarrow \mathbb{Z}_p/p\mathbb{Z}_p$ is most certainly not injective, but its projection to $\mathbb{Z}_p/p\mathbb{Z}_p$ is just the identity map, and is clearly injective.

Problem 9. PROJECTIVE VS FREE MODULES. Let M a finitely generated module over a local ring (R, m) .

- (a) Suppose that $x_1, \dots, x_n \in M$ is a minimal generating set for M . Explain why there is a surjection $R^{\oplus n} \xrightarrow{\pi} M$ sending $e_i \mapsto x_i$.
- (b) Assume M is projective. Show there is an R -linear map $M \xrightarrow{\sigma} R^{\oplus n}$ such that the composition $M \xrightarrow{\sigma} R^{\oplus n} \xrightarrow{\pi} M$ is the identity map.
- (c) Prove σ is an isomorphism. [Hint: Apply the functor $R/m \otimes_R -$ and use Nakayama's Lemma.]
- (d) Prove that a finitely generated module over a local ring is projective if and only if it is free.
- (e) Prove that a finitely presented module M over an arbitrary ring is projective if and only if M is locally free. [Hint: You proved last time that projectivity is local for finitely presented modules.]

Solution:

- (a) For any $m \in M$, $m = \sum_{i=1}^n r_i x_i$. Thus, under this map $\pi(r_1, \dots, r_n) = m$. Since this holds for all $m \in M$, it follows that π is surjective.
- (b) If M is projective, we have that

$$\begin{array}{ccccccc}
 & & & & M & & \\
 & & & & \downarrow \text{Id} & & \\
 & & \swarrow \sigma & & & & \\
 0 & \longrightarrow & \ker(\pi) & \longrightarrow & R^{\oplus n} & \xrightarrow{\pi} & M \longrightarrow 0
 \end{array}$$

By projectivity of M . Thus we have maps $M \xrightarrow{\sigma} R^{\oplus n} \xrightarrow{\pi} M$ that compose to the identity.

- (c) Applying the functor to $M \xrightarrow{\sigma} R^{\oplus n} \xrightarrow{\pi} M$ yields maps $M \otimes R/m \rightarrow R^{\oplus n} \otimes R/m \rightarrow M \otimes R/m$. This then yields map $M/mM \rightarrow (R/m)^{\oplus n} \rightarrow M/mM$. Recall that tensoring is right exact, and preserves surjections, so $(R/m)^{\oplus n} \rightarrow M/mM$ is surjective. Notice that $(R/m)^{\oplus n}$ is an n dimensional vector space surjecting onto a space with n generators.

Having n generators means that it has dimension $\leq n$, and being surjected to from an n dimensional vector space means it has dimension $\geq n$. Thus, M/mM is an n dimensional vector space, and since $(R/m)^{\oplus n} \rightarrow M/mM$ is surjective, it is an isomorphism. This implies that σ projected down is an isomorphism, and that the initial σ was also an isomorphism.

- (d) Free modules are projective, and we've just shown that finitely generated projective modules over local rings, are free, so the result follows.
- (e)

$$\begin{aligned}
 M \text{ is locally free} &\iff M_P \text{ is free } \forall P \in \text{Spec}(R) \\
 &\iff * M_P \text{ is projective } \forall P \in \text{Spec}(R) \\
 &\iff ** M \text{ is projective}
 \end{aligned}$$

* follows from the fact that M_P is an R_P -module, where R_P is a local ring. ** follows from the fact that projectivity can be tested locally on finitely presented modules.

Problem 11. FLAT VERSUS FREE MODULES. Let M be a module over an arbitrary ring S .

- (a) Show M is flat if and only if M_P is flat over S_P for all $P \in \text{Spec } S$ (or all $P \in \text{MaxSpec } S$). [Hint: Recall that exactness is local and note $U^{-1}R \otimes_R (M_1 \otimes_R M_2) \cong (U^{-1}M_1) \otimes_{U^{-1}R} (U^{-1}M_2)$.]
- (b) Show that if M is flat over (R, m) , then the map $m \otimes_R M \rightarrow M$ sending $r \otimes x \mapsto rx$ is injective.

- (c) * Show that a finitely presented flat module over a local ring (R, m) is free.
 [Hint: If M is minimally generated by n elements, we have an exact sequence $0 \rightarrow N \rightarrow R^n \rightarrow M \rightarrow 0$. Build a diagram from this by tensoring with each term in the sequence $0 \rightarrow m \rightarrow R \rightarrow R/m \rightarrow 0$.]
- (d) Prove the Theorem on the equivalence of flat, projective and locally free for f. p. modules.

Solution:

- (a) Recall that $U^{-1}S$ is flat. Notice that if $M \otimes -$ is an exact functor, then $U^{-1}S \otimes (M \otimes -)$ is an exact functor, since the composition of exact functors is exact. But $U^{-1}S \otimes M \cong U^{-1}M$, so $U^{-1}M \otimes -$ is an exact functor, so $U^{-1}M$ is flat. Letting $U = (S \setminus P)$ solves that half of the proof.

Next, suppose that M_P is flat $\forall P \in \text{Spec}(R)$. Recall that a sequence is a short exact sequence if and only if the localization is exact. Thus, consider a sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

We want to show this is exact under $M \otimes -$. Well, Notice that

$$0 \longrightarrow A_P \longrightarrow B_P \longrightarrow C_P \longrightarrow 0$$

is exact $\forall P \in \text{Spec}(R)$. Since M_P is flat, we know that

$$0 \longrightarrow M_P \otimes A_P \longrightarrow M_P \otimes B_P \longrightarrow M_P \otimes C_P \longrightarrow 0$$

is exact. From the hint, we know that this short exact sequence is isomorphic to

$$0 \longrightarrow (M \otimes A)_P \longrightarrow (M \otimes B)_P \longrightarrow (M \otimes C)_P \longrightarrow 0$$

Well, this is exact $\forall P \in \text{Spec}(R)$, so the following sequence is exact:

$$0 \longrightarrow M \otimes A \longrightarrow M \otimes B \longrightarrow M \otimes C \longrightarrow 0$$

Implying that M is flat.

- (b) The inclusion map $m \rightarrow R$ is obvious injective. Since M is flat, it preserves injectives. In particular since $m \rightarrow R$ is injective, $m \otimes M \rightarrow R \otimes M \cong M$ is injective.
- (c) (*)
- (d) We've already shown that flatness and projectivity can be locally tested, so without loss of generality we just need to show that flat, projective, and free are all equivalent over M_P , for any $P \in \text{Spec}(R)$. We already know that projective and free are equivalent, and we know that flat modules are projective. It is sufficient then to verify that projective modules are flat, when the module is finitely presented over a local ring, which is what M_P is. Well, this is the statement of (c), so we can conclude.

Problem 12. Show \mathbb{Q} is a flat \mathbb{Z} -module which is not locally free. Why doesn't this contradict the Theorem?

Solution: \mathbb{Q} is clearly not locally free, since any localizations of \mathbb{Q} are just \mathbb{Q} , and \mathbb{Q} is not free. \mathbb{Q} is a flat module because it is torsion free and over a PID. This does not contradict the theorem because \mathbb{Q} is not finitely presented.

Problem 13. NAKAYAMA'S LEMMA FOR ARBITRARY RINGS. Let S be an arbitrary ring, and let J be an ideal contained in every maximal ideal of S (that is, contained in the *Jacobson radical* $\bigcap_{m \in \text{MaxSpec} S} m$.)

- (a) Show that if $JM = M$ for some finitely generated S -module M , then $M = 0$.
- (b) Let $S = U^{-1}K[x, y, z]$ where $U = K[w, x, y, z] \setminus (P_1 \cup P_2)$ where $P_1 = \langle x, y \rangle$, $P_2 = \langle z, w \rangle$. Show that $J = P_1S \cap P_2S$ is contained in every maximal ideal of S .
- (c) If R is a finitely generated algebra over a field K , what ideals J have the property that $J \subset m$ for all $m \in \text{MaxSpec} R$? What is the Jacobson radical of R ?

Solution:

- (a) If $JM = M$, then $(JM)_P = M_P \forall P \in \text{Spec}(R)$. Let J' denote the projection of J to the localization. We know that $J'M_P = M_P$. From problem 7, since J' is a proper (by construction) and M is finitely generated, that $M_P = 0$. Since this holds for all $P \in \text{Spec}(R)$, we know that $M = 0$.
- (b) The maximal ideals in S are precisely P_1S and P_2S . J is thus necessarily contained in both.
- (c) The only ideals contained in every maximal ideal must necessarily be contained in the intersection of all maximal ideals, i.e. it must be contained in the Jacobson radical. The Jacobson radical is precisely what it is defined to be, $\bigcap_{m \in \text{MaxSpec} R} m$.

Math 614: Commutative Algebra**IBL Lab Notebook****Class 20: November 14th***Name: Vignesh Jagathese**Collaborators: Aniruddh Agarwal*

Problem 1. Let M be an arbitrary R -module over an arbitrary ring R .

- Show the support of M is empty if and only if $M = 0$. [Hint: Remember the worksheet on localization!]
- Show that if $P \in \text{Supp}(M)$, then $\mathbb{V}(P) \subset \text{Supp}(M)$. [Hint: If $Q \supset P$, describe a natural map $M_Q \rightarrow M_P$.]
- Show that the support of R/I is $\mathbb{V}(I) \subset \text{Spec } R$.

Solution:

- We know that $M_P = 0 \forall P \in \text{Spec}(R)$ if and only if $M = 0$, so this result follows.
- Since $Q \supset P$, $R \setminus Q \subset R \setminus P$. Thus, the natural inclusion map $M_Q \rightarrow M_P$ is well defined. In particular, M_Q has less inverted elements than M_P , so if $M_P \neq 0$, then M_Q cannot be zero, so $Q \in \text{Supp}(M)$.
- Recall that quotient commutes with localization. Thus, $(R/I)_P = R_P/IR_P$. This is nonzero if and only if $I \subset P$, since PR_P is the maximal ideal of R_P . It follows that the support is precisely $\mathbb{V}(I)$.

Problem 2. Consider the \mathbb{Z} -module $M = \bigoplus_{p \text{ odd prime}} \mathbb{Z}/p\mathbb{Z}$. Find the support of M and prove it is not closed in $\text{Spec } \mathbb{Z}$. Why doesn't this contradict the Proposition? [Hint: Remember \otimes distributes over \oplus .]

Solution: Suppose that $q\mathbb{Z} \in \text{Spec}(R)$. Notice that

$$M_Q = \left(\bigoplus_{p \text{ odd}} \mathbb{Z}/p\mathbb{Z} \right) \otimes \mathbb{Z}_q \cong \bigoplus_{p \text{ odd}} \mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Z}_q \cong \bigoplus_{p \text{ odd}} \mathbb{Z}_q/p\mathbb{Z}_q$$

If $q = 2$, then this is zero, since $(p) \not\subset (2)$ for any odd p . Furthermore, $(p) \not\subset (q)$ for any $p \neq q$. Thus, for q -odd, this direct sum reduces to $\mathbb{Z}_q/q\mathbb{Z}_q$, which in particular is nonzero. Also, notice that if $q = 0$, then $\mathbb{Z}_0 = \mathbb{Q}$, and all these quotients are zero. Thus, $\text{Supp}(M) = \text{Spec}(\mathbb{Z}) \setminus \{0, 2\mathbb{Z}\}$

Problem 3. PROOF OF THE PROPOSITION. Let M be a finitely generated module over an arbitrary R .

- (a) Show that $\text{ann}_R M$ is an ideal of R .
- (b) Show that if m_1, \dots, m_n generate M , then $\text{ann}_R(M) = \bigcap_{i=1}^n \text{ann}_R(m_i)$.
- (c) Prove $\text{Supp}(M) = \mathbb{V}(\text{ann}_R(M))$.

Solution:

- (a) We've already shown that $\text{ann}_R(m)$ was an ideal for $m \in M$. This is the same proof.
- (b) We already know that $\text{ann}_R(M) \subset \text{ann}_R(m_i)$, so it follows that $\text{ann}_R(M) \subset \bigcap_{i=1}^n \text{ann}_R(m_i)$. To show the converse, pick $r \in \bigcap_{i=1}^n \text{ann}_R(m_i)$. Now choose $m \in M$. In generator form, $m = \sum_{i=1}^m r_i m_i$. Next, notice that

$$rm = r \sum_{i=1}^m r_i m_i = \sum_{i=1}^m r_i (rm_i) = \sum_{i=1}^m 0 = 0$$

Thus, $r \in \text{ann}_R(M)$, and the claim is proven.

- (c) We've proven that $m/1 \in M_P$ is nonzero if and only if $\text{ann}_R(m) \not\subset P$. It follows that M_P is nonzero if and only if $\exists m/1 \in M_P$ nonzero if and only if $\text{ann}_R(m) \not\subset P$. It follows then that M_P is nonzero if and only if $\bigcap_{m \in M} \text{ann}_R(m) \not\subset P$. Well, $\bigcap_{m \in M} \text{ann}_R(m) \bigcap_{i=1}^n \text{ann}_R(m_i) = \text{ann}_R(M)$, so we know that

$$P \in \text{Supp}(M) \iff M_P \neq 0 \iff \text{ann}_R(M) \not\subset P \iff P \in \mathbb{V}(\text{ann}_R(M))$$

The result follows.

Problem 4. Let M be an arbitrary R -module over an arbitrary ring R . Fix $P \in \text{Spec } R$.

- (a) Show that $P \in \text{Ass}(M)$ if and only if $P = \text{ann}_R x$ for some non-zero $x \in M$.
- (b) Show that if R is a domain, the only associated prime of R is $\langle 0 \rangle$.
- (c) Let $R = K[x, y]$ and let $M = R/\langle xy, x^2 \rangle$. Show that $\{\langle x \rangle, \langle x, y \rangle\} \subset \text{Ass}(M)$. [Hint: Use (a). It might also be useful to remember that $K[x, y]$ is a UFD.]

Solution:

- (a) We prove the forward direction first. Suppose there exists an injection $f : R/P \rightarrow M$. Then $f(\bar{r}) = 0$ only if $\bar{r} = 0$, or $r \in P$. In particular, if $r \in P$, then $r \cdot f(\bar{1}) = f(\bar{r}) = 0$, and we know that $f(\bar{1}) \neq 0$ by injectivity. It follows that $P = \text{ann}_R(f(\bar{1}))$.

The converse follows similarly. Notice that R -module homomorphisms from $R/P \rightarrow M$ are determined by where they send 1. Thus, choose $f : R/P \rightarrow M$ such that $f(1) = x$. We verify that this is injective. Suppose that $f(\bar{r}) = 0$. Then $r \cdot f(\bar{1}) = 0$, so $r \cdot x = 0$, but then $r \in \text{ann}_R(x) = P$. Thus, $\bar{r} = 0$ and we can conclude that f has trivial kernel.

- (b) If R is a domain, (0) is indeed a prime, and an associated prime at that. Let $P = \text{ann}_R(x)$ be an associated prime, for $x \neq 0$. Well, then for any $r \in P$, $rx = 0$, so $r = 0$, since we are in a domain. Thus, $P = 0$.
- (c) Notice that $(x, y) = \text{Ann}_R(x)$, since $xy = 0, xx = x^2 = 0$ over M . $(x) = \text{Ann}_R(y)$, since $xy = 0$. These aren't any bigger because these are the only relations imposed.

Problem 5. Show that $\text{Ass}(M) \subset \text{Supp}M$ for any M . [Hint: Use the fact that R_P is a flat R -module, so it preserves injections.]

Solution: Choose $P \in \text{Ass}(M)$. Then, we have an injection $R/P \rightarrow M$. Recall that R_P is flat, and thus preserves injections. Thus, we have an injection $R/P \otimes R_P \rightarrow M \otimes R_P$, which is the same as an injection $R_P/PR_P \rightarrow M_P$. Thus, we have an injection $\kappa(P) \rightarrow M_P$. $\kappa(P) \neq 0$, so it follows that $M_P \neq 0$.

The reverse inclusion is not true in general.

Problem 6. Let $R = K[x, y]$. Fix any maximal ideal m such that $R/m \cong K$. Let $M = \text{Hom}_K(R, R/m)$.

- (a) Describe a natural R -module structure on M .
- (b) Show that the R -linear map $R \rightarrow M$ sending r to the composition $R \xrightarrow{\times r} R \twoheadrightarrow R/m$ induces an embedding $R/m \hookrightarrow M$.
- (c) Show that m is an associated prime of $\text{Hom}_K(R, K)$.

Solution:

- (a) $r \cdot f(s) = rf(s)$, $(f + g)(s) = f(s) + g(s)$.
- (b) Let the map $R \rightarrow M$ be denoted φ . The image of r is the map sends $x \mapsto \bar{rx}$ for \bar{rx} the projection of rx to R/m . This map is the zero map if and only if $rx \in m \forall x \in R$. In particular, this holds if and only if $r \in m$. Thus, $\varphi(m) = 0$, implying that the map $\varphi : R \rightarrow M$ uniquely extends to an embedding $R/m \rightarrow M$ by the universal property of quotients.
- (c) It follows from the above result that $m \in \text{Ass}(M)$ by definition.

Problem 7. Let $R = K[x_1, x_2, x_3, \dots]/J$ where $J = \langle x_i^{t+1} \mid t \in \mathbb{N} \rangle$. Prove that $\text{Spec } R$ consists of one point and that $\text{Ass } R$ is empty. Why doesn't this contradict Theorem 1? Is the reverse inclusion in Problem (5) true?

Solution: Notice that $\text{Spec}(K[x_1, \dots,]/J) \cong \mathbb{V}(J)$. $\mathbb{V}(J)$ is precisely $\{((x_t)_{t \in \mathbb{N}})\}$, which is just one point. We claim that $\text{Ass}(R) = \emptyset$. Since this is the only maximal ideal of R , it is sufficient to show that there is non nonzero element killed by $((x_t)_{t \in \mathbb{N}})$. Suppose that $f \neq 0$ and x_i does not occur in f (we can do this because polynomials have finitely many terms, and there are infinitely many variables). Then, $x_i f \neq 0$, and our claim is proven.

This does not contradict the theorem since this isn't a module over a Noetherian ring. (the infinite variable polynomial ring is clearly not noetherian).

Problem 8. A USEFUL LEMMA. Let R be Noetherian ring and M a non-zero R module. Show that the set of ideals $\{J \subset R \mid \exists m \in M \setminus \{0\} \text{ s.t. } J = \text{ann}_R(m)\}$ has a maximal element, and any such maximal element is prime. [Hint: If $xy \in \text{ann}_R m$, consider $\text{ann}_R(xm)$.]

Solution: We know that this set contains all associated primes since $J = \text{ann}_R(x)$ for some nonzero x . This set is nonempty, since M is nonzero, and every ascending chain (via Noetherianity) is bounded. Then, by Zorn's Lemma, there exists a maximal element of the set. Suppose $J = \text{ann}_R(m)$ is our maximal element. Pick $xy \in J$. $\text{ann}_R(xm) \supset \text{ann}_R(m)$, which implies that $\text{ann}_R(xm) = \text{ann}_R(m)$ or $xm = 0$. If $\text{ann}_R(xm) = \text{ann}_R(m)$, then it follows that $y \in J$, since $xym = 0$, so $x \in \text{ann}_R(xm) = \text{ann}_R(m)$. If $xm = 0$, $x \in J$ by construction. Thus, J is prime.

Problem 9. PROOF OF THEOREM 2. Let M be a finitely generated module over a Noetherian ring.

- Let $P \in \text{Ass}_R(M)$. Prove every element of P is a zero-divisor on M . [Hint: Use (4a).]
- Assume that $rm = 0$ for some non-zero $m \in M$. Show that there exists $s \in S$ such that $\text{ann}_R(sm)$ is prime and contains r . [Hint: Use ideas from (8).]
- Prove Theorem 2.

Solution:

- $P = \text{Ass}_R(x)$ for some $x \in M$. Thus, $\forall r \in P$, $rx = 0$, so r is a zero divisor.
- We know that $r \in \text{ann}_R(m)$. If this is prime, we are done. If not, notice that we can multiply m by s and get that $\text{ann}_R(m) \subset \text{ann}_R(sm)$. By (8), this process terminates, i.e. $\exists r_M$ such that $\text{ann}_R(r_M m) = \text{ann}_R(r' r_M m)$ for any $r' \in R$. This is the maximal element in (8), which we know to be prime. Letting $s = r_M$ allows us to conclude.

- (c) Part (a) tells all elements of associated primes are zero divisors, so the union of all associated primes is contained in the set of all zero divisors. Part (b) tells us that all zero divisors are contained in some associated prime. Thus, the union over all associated primes is necessarily the set of all zero-divisors of M .

Problem 10. PRIME CYCLIC FILTRATIONS. In this problem we show that every non-zero finitely generated module M over a Noetherian ring R admits a filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots \subset M_{n-1} \subset M_n = M$$

such that each subquotient $M_i/M_{i-1} \cong R/P_i$ for some $P_i \in \text{Spec } R$.

- (a) Use Noetherian Induction to reduce to the case that every quotient of M has a prime cyclic filtration. [Hint: Recall Noetherian Induction—if we have a counterexample M , mod out by a submodule N maximal with respect to the property that M/N is also a counterexample.]
- (b) Use (8) to find $x \in M$ such that $R/P \cong xR \subset M$ for some $P \in \text{Spec } R$.
- (c) Prove that every finitely generated module over a Noetherian ring has a prime cyclic filtration. [Hint: Splice together R/P and a filtration for M/xR .]

Solution:

- (a) Suppose we have a counter example M . Consider the set of all quotients M/N where M/N is a counter example. Then, via Noetherianity if this is nonempty, there exists some maximal element. Then, any quotient of M/N will have a prime cyclic filtration. In particular, a quotient where $(M/N)/L \cong R/P$ for some L , if it existed. Well, that would imply that M/N has a prime cyclic filtration, a contradiction. Thus, it is sufficient to verify that if every quotient of a module has a prime cyclic filtration, then so does it.
- (b) Choose $x \in M$ such that $P = \text{ann}_R(x)$ is maximal as in problem 8. It follows that $R/P \cong xR$, under the isomorphism $r \mapsto rx$. This is clearly surjective, and injective because $\text{ann}_R(x) = \text{ann}_R(rx)$ for any $r \in R$ by maximality. Thus, if $rx = 0$, then $r \in P$.
- (c) Suppose that every quotient of M has a prime cyclic filtration. Let $0 = M_1 \subset \dots \subset M_\ell = M/xR$ be a prime cyclic filtration for M/xR . This lifts to a filtration $xR = M'_1 \subset \dots \subset M'_\ell = M$, as $M'_{i+1}/M'_i \cong xRM_{i+1}/xRM_i \cong M_{i+1}/M_i \cong R/Q_i$ for some $Q_i \in \text{Spec}(R)$ by construction. Well, notice that $xR/0 \cong R/P$ for some P . thus, we have that

$$0 \subset xR = M'_1 \subset \dots \subset M'_\ell = M$$

Is a prime cyclic filtration of M .

Problem 11. PROOF OF THEOREM 1. Fix an arbitrary ring R .

- (a) Prove that if P is prime, then $\text{Ass}(R/P) = \{P\}$.
- (b) Show that if $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is an exact sequence of R -modules, then $\text{Ass}(M_2) \subset \text{Ass}(M_1) \cup \text{Ass}(M_3)$. [Hint: If $P = \text{ann}x$, consider two cases: either $Rx \cap M_1 = 0$ or if not. Use (a) for the second case.]
- (c) Suppose that $M_0 \subset M_1 \subset M_2 \subset \dots \subset M_{n-1} \subset M_n = M$. Show that $\text{Ass}(M) \subset \bigcup_{i=1}^n \text{Ass}(M_i/M_{i-1})$. [Hint: Use induction on n and (b).]
- (d) Prove that $\text{Ass}(M) \subset \{P_1, P_2, \dots, P_n\}$, the prime ideals appearing in a prime cyclic filtration of M .
- (e) Prove the Theorem on the finiteness of $\text{Ass}(M)$ for Noetherian M over Noetherian R .

Solution:

- (a) It is clear that $P \in \text{Ass}(R/P)$. We've shown that in a domain, the only associated prime is 0. Well, since P is prime, R/P is a domain, so the only associated prime is $(\bar{0})$, which corresponds to P .
- (b) We can assume without loss of generality that $M_1 \subset M_2$ and $M_3 \cong M_2/M_1$. Choose $x \in M_2$ such that $P = \text{ann}_R(x)$. If $Rx \cap M_1 \neq 0$, $\exists 0 \neq y \in Rx$ such that $y \in M_1$, and y has annihilator P . It follows that $P \in \text{Ass}(M_1)$. Otherwise, if $Rx \cap M_1 = 0$, then $Rx \cong R/P$ embeds into M_3 , so $P \in \text{Ass}(M_3)$, as desired.
- (c) We induct, where the $n = 1$ case is immediate. Because we have a short exact sequence $0 \rightarrow M_{n-1} \rightarrow M \rightarrow M/M_{n-1} \rightarrow 0$, it follows that $\text{Ass}(M) \subset \text{Ass}(M_{n-1}) \cup \text{Ass}(M/M_{n-1})$. Via the inductive hypothesis, $\text{Ass}(M_{n-1}) \subset \bigcup_{i=1}^{n-1} \text{Ass}(M_i/M_{i-1})$, so $\text{Ass}(M) \subset \bigcup_{i=1}^n \text{Ass}(M_i/M_{i-1})$.
- (d) Recall that $M_i/M_{i-1} \cong R/P_i$ for some $P_i \in \text{Spec}(R)$, so $\text{Ass}(M_i/M_{i-1}) = \{P_i\}$. Thus,

$$\text{Ass}(M) \subset \bigcup_{i=1}^n \text{Ass}(M_i/M_{i-1}) \subset \bigcup_{i=1}^n \{P_i\} = \{P_1, \dots, P_n\}$$

- (e) Notice that prime cyclic filtrations are of finite length, implying that the set above is of finite size. Since $\text{Ass}(M)$ is a subset of a finite set, it is itself a finite set. Furthermore, we know that it is nonempty, so we can conclude.

Math 614: Commutative Algebra**IBL Lab Notebook**

Class 21: November 19th

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- Problem 1.** (a) Show that if \mathfrak{p} is prime, then \mathfrak{p} is primary.
- (b) Show that if \mathfrak{q} is primary, then $\sqrt{\mathfrak{q}}$ is prime. So, a primary ideal is \mathfrak{p} -primary for some prime \mathfrak{p} .
- (c) Let $R = \mathbb{C}[x, y, z]$ and $\mathfrak{q} = \langle x^2, xy \rangle$. Show that $\sqrt{\mathfrak{q}}$ is prime but that \mathfrak{q} is not primary. Thus, the converse of (b) fails.

Solution:

- (a) If P is prime, then if $xy \in P$, $x \in P$ or $y \in P$. This is precisely the criterion for being primary, except there is one more possible case, which can be disregarded.
- (b) Suppose that P is primary. Then for all $xy \in P$, $x \in P$, $y \in P$, or $x^n \in P$. This implies that if $xy \in P \subset \sqrt{P}$, then $x \in P \subset \sqrt{P}$, $y \in P \subset \sqrt{P}$, or $x^n \in P$, implying that $x \in \sqrt{P}$. It then follows that one of $x, y \in \sqrt{P}$, so \sqrt{P} is prime.
- (c) $\sqrt{\mathfrak{q}} = (x, xy) = (x)$, which is prime. \mathfrak{q} is not primary because $xy \in \mathfrak{q}$, but $x, y \notin \mathfrak{q}$ and $y^n \notin \mathfrak{q}$ for any n .

- Problem 2.** Show that \mathfrak{q} is primary if and only if every zero-divisor of R/\mathfrak{q} is nilpotent.

Solution:

$$\begin{aligned} \mathfrak{q} \text{ primary} &\iff xy \in \mathfrak{q} \Rightarrow x \in \mathfrak{q}, y \in \mathfrak{q}, \text{ or } x^n \in \mathfrak{q} \\ &\iff \overline{xy} = 0, \overline{x} = 0, \overline{y} = 0, \overline{x}^n = 0 \quad (\in R/\mathfrak{q}) \\ &\iff x \text{ (a zero divisor) is nilpotent in } R/\mathfrak{q} \end{aligned}$$

Notice that in the reverse direction, since x is a zero divisor, it and y are necessarily not zero in R/\mathfrak{q} , so only the third option holds.

Problem 3. PRIMARY DECOMPOSITION IN PIDS.

- (a) Prove that an ideal \mathfrak{q} in a PID is primary if and only if it is generated by a power of an irreducible element.
- (b) Express the ideal $\langle 12 \rangle \cap \langle 18 \rangle$ of \mathbb{Z} as an intersection of primary ideals.

- (c) Is your intersection in (b) irredundant? Are the radicals of your primary components distinct? If not, remove redundant components and/or combine components with the same radical to get a minimal primary decomposition of $\langle 12 \rangle \cap \langle 18 \rangle$.
- (d) Find a *minimal* primary decomposition for the ideal $\langle x^4 - x^2 \rangle$ in $K[x]$.

Solution:

- (a) Notice that

$$(r) \text{ a primary ideal} \iff xy \in (r) \Rightarrow r|x, r|y, r|x^n$$

Continuing this process down to a product of irreducible elements (we can do this because PIDs are UFDs), we see that r must divide either an irreducible element (and thus r is the irreducible element) or r must divide a power of an irreducible element (in which case r is a [possibly smaller] power of an irreducible element. It follows that (r) is primary if and only if r is the power of some irreducible element.

- (b) Notice that $(12) = (2)(2)(3)$, $(18) = (2)(3)(3)$. It follows that $(12) \cap (18) = (2)^2 \cap (3)^2$.
- (c) Yes, it is irredundant.
- (d) $(x^4 - x^2) = (x^2(x^2 - 1)) = (x^2(x - 1)(x + 1)) = (x^2) \cap (x - 1) \cap (x + 1)$

Problem 4. CHARACTERIZATIONS OF PRIMARY. Fix a prime ideal \mathfrak{p} containing an ideal \mathfrak{q} . Show that the following are equivalent.

- (a) \mathfrak{q} is \mathfrak{p} -primary.
- (b) The nilradical of R/\mathfrak{q} is $\mathfrak{p}/\mathfrak{q}$ and every zero-divisor of R/\mathfrak{q} is nilpotent.
- (c) The radical of \mathfrak{q} is \mathfrak{p} and $\mathfrak{q} = \mathfrak{q}R_{\mathfrak{p}} \cap R$. [Hint: Show $x \in \mathfrak{q}R_{\mathfrak{p}} \cap R$ implies $\exists y \notin \mathfrak{p}$ s.t. $xy \in \mathfrak{q}$.]

Solution: We verify (a) \iff (b) first. From problem 2, this reduces to verifying that a primary ideal q is p primary provided that the nilradical of R/q is p/q . Well, we already know that the nilradical of R/q is \sqrt{q}/q , so it is sufficient to show that q is p -primary if and only if $\sqrt{q} = p$, which holds by definition.

We now verify that (a) implies (c). We've proven the hint on homework, so we assume it. The first part is immediate, so it is sufficient to verify that $q = qR_{\mathfrak{p}} \cap R$. We know that $q \subset qR_{\mathfrak{p}} \cap R$, so we just need to show \supset . Pick $x \in qR_{\mathfrak{p}} \cap R$. that means $\exists y \notin \mathfrak{p}$ such that $xy \in q$. Because q is primary, we know that $x \in q, y \in q$, or $y^n \in q$. Well, if $x \notin q$, then $y \in q$ or $y^n \in q$. If either of these are true, then $y \in \sqrt{q} = p$, a contradiction. Thus, $x \in q$, and $q = qR_{\mathfrak{p}} \cap R$, completing this case. Following this logic in reverse gives (c) implies (a) for free, so we conclude here.

Problem 5. Show that if \sqrt{q} is maximal, then q is primary. Is the converse true? [Hint: Note R/q is local.]

Solution: Suppose that $xy \in q$, and $x \notin \sqrt{q}$. By maximality of \sqrt{q} , and the fact that any prime ideal containing q must also contain \sqrt{q} , no maximal ideal contains both q and x . This implies that $(x) + q = R$, so $(y) = y(q + (x)) \subset q + (xy) = q$, implying that $y \in q$, and that q is primary.

Problem 6. Which of the following ideals in $K[x, y, z]$ is primary: $\langle x^2, y^3, z^5 \rangle, \langle xy - z^7 \rangle, \langle xy, yz, xz \rangle$.

Solution: The radical of $\langle x^2, y^3, z^5 \rangle$ is (x, y, z) , so it is primary. $\langle xy - z^7 \rangle$ is prime via Eisenstein's Criterion, so it is also primary. $xy \in \langle xy, yz, xz \rangle$ but $x, y, x^n, y^m \notin \langle xy, yz, xz \rangle$ for any n, m , so this is NOT primary.

Problem 7. Let $R = \mathbb{C}[x, y]$. Let $J = \langle x^2, xy \rangle$.

- (a) Show that $J = \langle x \rangle \cap \langle x^2, xy, y^n \rangle$ for any $n \geq 1$. [Hint: the UFD property might be useful.]
- (b) Show that $J = \langle x \rangle \cap \langle x^2, y - ax \rangle$ for any $a \in \mathbb{C}$.
- (c) Show that all decompositions in (a) and (b) are minimal primary decompositions.
- (d) Use Theorem 2 to compute $\text{Ass}(R/J)$.
- (e) Verify that this example comports with Theorem 3 by computing $JR_{\mathfrak{p}} \cap R$.

Solution:

- (a) It is clear that $J \subset (x), (x^2, xy, y^n)$, so $J \subset (x) \cap (x^2, xy, y^n)$. Next, choose r in the intersection. Then we can write $r = r_1x^2 + r_2xy + r_3y^n$, and similarly write $r = r_1x$. Since $r \in (x)$, $x|r$, and thus it divides all monomials of r . Since $x|r_3y^n$, it follows that $x|r_3$, and that $xy|r_3y^n$. Thus, this can be lumped in with the r_2xy term. Implying that $r = r_1x^2 + r'_2xy$, so $r \in J$.
- (b) We do the same process as before. It is clear that $J \subset (x) \cap (x^2, y - ax)$, so we prove the reverse case. Choose $r \in (x) \cap (x^2, y - ax)$. We can write $r = r_1x^2 + r_2(y - ax)$. Notice that $x|r$ since $r \in (x)$, implying that it divides each monomial, and in particular, divides $r_2(y - ax) = r_2y - r_2ax$. Thus, $x|r_2y$, so $x|r_2$. This means we can write $r_2 = r'_2x$, implying that $r = r_1x^2 + r'_2xy(y - ax) = r_1x^2 + (r'_2(y - ax))xy$, so $r \in J$.

- (c) In (a), the radicals corresponds to (x) and $(x, xy, y) = (x, y)$, which are distinct radicals. In (b), we have radicals (x) and $(x, y - ax)$, which are also distinct radicals. Thus, we have irredundancy.
- (d) $\text{Ass}(R/J)$ contains all these radicals, and is thus of the form $\{(x), (x, y), (x, y - ax) \mid a \in \mathbb{C}\}$.
- (e) Here, (x) is the only minimal associated prime of J . Notice that $JR_{(x)} \cap R = \{b \in R \mid \exists a \notin (x) \text{ such that } ab \in J\}$ If $ab \in J$, we know that $x \mid ab$. x does not divide a , so it divides b . Thus, $b \in (x)$. It follows that $(x) = JR_{(x)} \cap R$.

Problem 8. IRREDUCIBLE IDEALS. We say that a proper ideal is **irreducible** if it is not the intersection of two strictly larger ideals.

- (a) Define $(I : z) := \{y \in R \mid yz \in I\}$. Prove that $(I : z)$ is an ideal (proper iff $z \notin I$).
- (b) Prove that $(I : x^n)$ is proper for all $n \in \mathbb{N}$ if and only if $x \notin \sqrt{I}$.
- (c) Assume R is Noetherian. Fix $x \in R$. Prove there exists $N \in \mathbb{N}$ such that $(I : x^n) = (I : x^N)$ for all $n \geq N$.
- (d) Prove that for any ideal I in a Noetherian ring, and any $x \in R$, $(I + \langle x^n \rangle) \cap (I : x^n) = I$, for $n \gg 0$. [Hint: For $i \in I$, say $i + rx^n \in (I : x^n)$ for $n \geq N$. Use (c) to understand r .]
- (e) Prove that an irreducible ideal of a Noetherian ring is primary. [Hint: Say $xy \in I$ but $x \notin \sqrt{I}$.]

Solution:

- (a) If $yz, xz \in I$, then $yz + xz = z(y + x) \in I$, so $y + x \in (I : z)$. Furthermore, $0z = 0 \in I$, so $0 \in (I : z)$. Similar logic shows inverses are contained in $(I : z)$ is an abelian group. To show I is an ideal, pick $r \in R$, $y \in (I : z)$. Well, then $yz \in I$, so $ryz \in I$, so $ry \in (I : z)$. It follows that $(I : z)$ is an ideal.

Notice that $z \in I \iff 1z \in I \iff 1 \in (I : z) \iff (I : z)$ is proper.

- (b) $(I : x^n)$ is proper $\forall n \in \mathbb{N}$ if and only if $x^n \notin I \forall n \in \mathbb{N}$ if and only if $x \notin \sqrt{I}$.
- (c) Notice that if $zx^n \in I$, then $zx^{n+1} \in I$. This induces a chain of ideals

$$(I : x) \subset (I : x^2) \subset \dots$$

By the ascending chain condition, it follows that this chain terminates, yielding the desired result.

- (d) We know that $I \subset (I : x^n)$ and $I \subset I + \langle x^n \rangle$, so it is contained in their intersection. We thus need to prove the converse. Suppose that $u = i + rx^n$ is contained in the interesection, for $i \in I, r \in R$. Then, x^n multiplies into I , so $ux^n = ix^n + rx^{2n} \in I$, further implying that $rx^{2n} \in I$, so $r \in (I : x^{2n}) = (I : x^n)$ by part (c). But then $rx^n \in I$, so $u \in I$, as desired.

- (e) From (d), we know that any ideal I can be decomposed as $(I + \langle x^n \rangle) \cap (I : x^n) = I$. If I is irreducible, then either $I = (I : x^n)$ or $I = I + \langle x^n \rangle$ for any $x \in R$. In either case, if $xy \in I$, and $x \notin I$, this would imply that either $y \in I$ or $x^n \in I$, so I is primary.

Problem 9. Fix a prime ideal \mathfrak{p} . Prove that a finite intersection of \mathfrak{p} -primary ideals is \mathfrak{p} -primary.

Solution: Let q_1, \dots, q_t be \mathfrak{p} -primary ideals. First, we need to verify that $\sqrt{q_1 \cap \dots \cap q_t} = \sqrt{q_1} \cap \dots \cap \sqrt{q_t} = \mathfrak{p}$.

$$\begin{aligned} x \in \sqrt{q_1 \cap \dots \cap q_t} & \\ \iff x^n \in q_1 \cap \dots \cap q_t & \\ \iff x^n \in q_1, \dots, q_t & \\ \iff x \in \sqrt{q_1}, \dots, \sqrt{q_t} & \\ \iff x \in \sqrt{q_1} \cap \dots \cap \sqrt{q_t} & \end{aligned}$$

next, we verify that this is indeed primary. Via an inductive proof, we just need to check this in the case of $t = 2$. Choose $xy \in q_1 \cap q_2$. This implies that $xy \in q_1, q_2$. First, consider the case where $x \in q_1$. Since $\sqrt{q_1} = \sqrt{q_2}$, we know that since $x \in \sqrt{q_1} = \sqrt{q_2}$, $x^n \in q_2$ for some n , and we can conclude since $x^n \in q_1 \cap q_2$. The y case is similar, as is the case where $x^n \in q_1$ (some sufficiently high power of x will be contained in the intersection via an identical argument.)

Problem 10. EXISTENCE OF PRIMARY DECOMPOSITION. Assume that R is Noetherian.

- (a) Prove that every ideal of R is an intersection of finitely many irreducible ideals.
 (b) Prove Theorem 1. [Hint: Use (8) and (9).]

Solution:

- (a) Let \mathcal{X} denote the set of all ideals which do not lie in an intersection of finitely many irreducible ideals. We proceed by a proof by contradiction, and assume this is nonempty. Then, by Noetherianity of R , this family of ideals has some maximal element with respect to inclusion. Let m be the ideal maximal with respect to this property. If m is irreducible, then we have a contradiction. Thus, $m = I_1 \cap I_2$, for I_1, I_2 strictly larger, yet proper, ideals. By maximality of m , we know that I_1, I_2 must admit finite irreducible decompositions. Since m lies in their intersection, it must as well, a contradiction.

(b) Let I be any ideal of R . Notice that $I = I_1 \cap \dots \cap I_t$, for I_1, \dots, I_t irreducible, by the previous part. We know that irreducible ideals are primary, so we have a primary decomposition of I . Without loss of generality we can let this decomposition be irredundant. We just need to verify that these ideals have distinct radicals.

Suppose that I_i, I_j have the same radical J . It follows that I_i, I_j are J -primary, so by 8, $I_i \cap I_j$ is J -primary as well. Thus we can view the intersection here as one of the minimal prime components. Repeat this process of "collapsing" ideals with the same radical and no such ideals remain. We will end with at least 1 ideal, implying that a minimal primary decomposition has been achieved.

Math 614: Commutative Algebra

IBL Lab Notebook

Class 22: November 21st

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Problem 1. LEMMA 1. Prove that finite intersection commutes with taking radicals, localization, and computing colons. That is, for any finite set of ideals J_1, \dots, J_t in an arbitrary ring R , prove that

- (a) $\sqrt{(J_1 \cap J_2 \cdots \cap J_t)} = \sqrt{J_1} \cap \sqrt{J_2} \cdots \cap \sqrt{J_t}$;
- (b) $(J_1 \cap J_2 \cdots \cap J_t)U^{-1}R = J_1U^{-1}R \cap J_2U^{-1}R \cdots \cap J_tU^{-1}R$ for any multiplicative set $U \subset R$; and
- (c) $(J_1 \cap J_2 \cap \cdots \cap J_t) : x = (J_1 : x) \cap (J_2 : x) \cap \cdots \cap (J_t : x)$ for arbitrary $x \in R$.

Solution:

(a) This was used to solve problem 9 in the previous worksheet.

(b)

$$\begin{aligned} x/u \in (J_1 \cap \dots \cap J_t)U^{-1}R &\iff x \in J_1 \cap \dots \cap J_t \\ &\iff x \in J_1, \dots, J_t \\ &\iff x/u \in J_1U^{-1}R, \dots, J_tU^{-1}R \\ &\iff x/u \in J_1U^{-1}R \cap J_2U^{-1}R \cdots \cap J_tU^{-1}R \end{aligned}$$

(c)

$$\begin{aligned} y \in (J_1 \cap J_2 \cap \cdots \cap J_t) : x &\iff yx \in (J_1 \cap J_2 \cap \cdots \cap J_t) \\ &\iff yx \in J_1, \dots, J_t \\ &\iff y \in (J_1 : x), \dots, (J_t : x) \\ &\iff y \in (J_1 : x) \cap (J_2 : x) \cap \cdots \cap (J_t : x) \end{aligned}$$

Problem 2. LEMMA 2. Prove that in any ring, if $P_1 \cap P_2 \cap \dots \cap P_n$ is an intersection of mutually incomparable prime ideals, then minimal primes of this intersection are precisely the P_i . In particular, when is such an intersection prime? [Hint: $P_1P_2 \cdots P_n \subset P_1 \cap P_2 \cap \dots \cap P_n$.]

Solution: Let $P_1 \cap P_2 \cap \dots \cap P_n \subset Q$ for Q some prime ideal. We want to show that $P_i \subset Q$ for some i . Well, suppose not. Then choose $r_i \in P_i \setminus (Q \cap P_i)$ for each i . The product of these elements is contained in $P_1 \cap \dots \cap P_n$, but not in Q . This is a contradiction.

Problem 3. MINIMAL PRIMES. Let J be an ideal in an arbitrary ring which admits a primary decomposition $\mathfrak{q}_1 \cap \mathfrak{q}_2 \dots \cap \mathfrak{q}_t$.

- By grouping together primary ideals with the same radical, explain why we can assume the \mathfrak{q}_i have distinct radicals \mathfrak{p}_i . [Hint: A finite intersection of \mathfrak{p} -primary ideals is \mathfrak{p} -primary.]
- Prove that $\sqrt{J} = \bigcap_{i=1}^t \mathfrak{p}_i$.
- Prove the minimal primes among $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_t\}$ are precisely the min primes of J .
- Observe that (a) and (c) together establish part of the Theorem on Uniqueness of Primary decomposition.

Solution:

- This was proved as part of 10(b) on the previous worksheet. Here is the logic from there: "Suppose that I_i, I_j have the same radical J . It follows that I_i, I_j are J -primary, so by 8, $I_i \cap I_j$ is J -primary as well. Thus we can view the intersection here as one of the minimal prime components. Repeat this process of "collapsing" ideals with the same radical and no such ideals remain. We will end with at least 1 ideal, implying that a minimal primary decomposition has been achieved."
- $$\sqrt{J} = \sqrt{\mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_t} = \sqrt{\mathfrak{q}_1} \cap \dots \cap \sqrt{\mathfrak{q}_t} = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_t$$
- This follows from the previous part and problem 2. J is this intersection, and the minimal primes of the intersection are precisely the (minimal) primes in that intersection.
- We can assume the decomposition $\mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_t$ is minimal. We now know that the radicals of these are distinct, and that the minimal primes of those radicals are just the minimal primes of J . We just need to verify that the set of radicals is independent of choice of decomposition, and that the decomposition is uniquely determined via the formula $\mathfrak{q}_i = J R_{\mathfrak{p}_i} \cap R$.

Problem 4. LEMMA 3. Let \mathfrak{q} be any \mathfrak{p} -primary ideal in an arbitrary ring R .

- Prove that $\mathfrak{p} \subset \sqrt{(\mathfrak{q} : x)}$. [Hint: First show $\mathfrak{q} \subset (\mathfrak{q} : x)$.]
- Prove that

$$\sqrt{(\mathfrak{q} : x)} = \begin{cases} \mathfrak{p}, & \text{for } x \notin \mathfrak{q} \\ R & \text{for } x \in \mathfrak{q}. \end{cases}$$

- Prove that for $x \notin \mathfrak{p}$, we have $(\mathfrak{q} : x) = \mathfrak{q}$.

Solution:

- (a) It is clear that $q \subset (q : x)$, since if $r \in q$, then $rx \in q$, so $r \in (q : x)$. Taking the radical on both sides yields the desired result.
- (b) We know that $(q : x) = R$ for $x \in q$, so we just verify that $\sqrt{(q : x)} = p$ for $x \notin q$. We have one inclusion, so we just verify the other. Suppose that $u \notin p$. It is sufficient to verify that $u \notin \sqrt{(q : x)}$, or rather, that $u^t x \notin q$. However, since $u \notin p$, $u^t \notin p$ for any t , so $u^t \notin q$, so if $u^t x \in q$, then this implies that $x \in q$, which is a contradiction.
- (c) We know that $q \subset (q : x)$, so we just need to prove the \supset direction. Choose $r \in (q : x)$. thus, $rx \in q$. Since q is primary, we know that either $r \in q$, $x \in q$, or $x^n \in q$ for some large n . Since $x \notin p$, $x \notin q$, and $x^n \notin q$. Thus, $r \in q$, and we can conclude.

Problem 5. UNIQUENESS OF THE \mathfrak{p}_i . Let $J = \mathfrak{q}_1 \cap \mathfrak{q}_2 \dots \cap \mathfrak{q}_t$ be any primary decomposition of J in which the radicals \mathfrak{p}_i of \mathfrak{q}_i are distinct.

- (a) Use Lemmas 1 and 3 to show that for any $x \in R$,

$$\sqrt{J : x} = \bigcap_{x \notin \mathfrak{q}_i} \mathfrak{p}_i.$$

- (b) Fix i . Explain why, if the decomposition is irredundent, we can find x in every \mathfrak{q}_j except \mathfrak{q}_i .
- (c) With x as in (b), show that $\sqrt{J : x} = \mathfrak{p}_i$.
- (d) Show that if $\mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \dots \cap \mathfrak{q}_t$ and $\mathfrak{q}'_1 \cap \mathfrak{q}'_2 \cap \dots \cap \mathfrak{q}'_m$ are two different minimal primary decompositions of an ideal J , then

$$\{\sqrt{\mathfrak{q}_1}, \sqrt{\mathfrak{q}_2}, \dots, \sqrt{\mathfrak{q}_t}\} = \{\sqrt{\mathfrak{q}'_1}, \sqrt{\mathfrak{q}'_2}, \dots, \sqrt{\mathfrak{q}'_m}\}.$$

In particular $t = m$. [Hint: Use (a), (c) and Lemma 2.]

- (e) Conclude that part of the Theorem on Uniqueness of Primary decomposition is proven.

Solution:

- (a) If $x \in J$, then both sides are R , so we're done. Thus, assume that $x \notin J$

$$\sqrt{J : x} = \sqrt{(q_1 \cap \dots \cap q_t) : x} = \sqrt{(q_1 : x) \cap \dots \cap (q_t : x)} = \sqrt{(q_1 : x)} \cap \dots \cap \sqrt{(q_t : x)}$$

We know that $\sqrt{(q_1 : x)} = \mathfrak{p}_i$ if $x \notin q_i$, and if $x \in q_i$, then this is R . Since we're intersecting, we can ignore those cases. Thus, we have that

$$\sqrt{J : x} = \sqrt{(q_1 : x)} \cap \dots \cap \sqrt{(q_t : x)} = \bigcap_{x \notin q_i} \mathfrak{p}_i$$

- (b) If the decomposition were redundant, then $q_i \subset \bigcap_{i \neq j} q_j$, which it is not. $q_i \not\subset \bigcap_{i \neq j} q_j$, (and obviously $q_i \not\supset \bigcap_{i \neq j} q_j$), so we can choose an x in $\bigcap_{i \neq j} q_j$ but not q_i .
- (c) $\sqrt{J} : x = \bigcap_{x \notin q_i} p_i = p_i$.
- (d) Notice that $\sqrt{q_1 \cap \dots \cap q_t} = \sqrt{q'_1 \cap \dots \cap q'_m}$. Since taking radicals is compatible with intersections, we know that $\sqrt{q_1} \cap \dots \cap \sqrt{q_t} = \sqrt{q'_1} \cap \dots \cap \sqrt{q'_m}$. These are all prime ideals, implying that the minimal primes of J are precisely $\{\sqrt{q_1}, \dots, \sqrt{q_t}\}$ and precisely $\{\sqrt{q'_1}, \dots, \sqrt{q'_m}\}$, implying these sets are the same.
- (e) We now know all parts of the theorem except for the fact that, given a primary decomposition $J = q_1 \cap \dots \cap q_t$, that $q_i = JR_{p_i} \cap R$ (i.e. minimal decompositions are uniquely determined).

Problem 6. MINIMAL COMPONENTS. Let $J = q_1 \cap q_2 \cap \dots \cap q_t$ be a primary decomposition.

- (a) Let q be \mathfrak{p} -primary and P be any prime ideal such that $\mathfrak{p} \not\subset P$. Show that $qR_P = R_P$.
- (b) Suppose that \mathfrak{p}_i is minimal among $\{\sqrt{q_1}, \sqrt{q_2}, \dots, \sqrt{q_t}\}$. Prove that $JR_{\mathfrak{p}_i} \cap R = q_i$. [Hint: Use Lemma 1(b) and (a).]
- (c) Complete the proof of the Uniqueness Theorem for Primary Decomposition.

Solution:

- (a) If $\mathfrak{p} \not\subset P$, then $\exists r \in q$ such that r^n is unit in R_P . In particular, this implies that r is unit in R_P , so $r \in q$ is the unit ideal over R_P . Thus, $qR_P = R_P$.
- (b) First, notice that

$$JR_{\mathfrak{p}_i} \cap R = (q_1 \cap \dots \cap q_t)R_{\mathfrak{p}_i} \cap R = \bigcap_{j=1}^t q_j R_{\mathfrak{p}_i} \cap R$$

By in compatibility, we know that $q_j \not\subset q_i$ for $i \neq j$. This implies that $\sqrt{q_j} = p_j \not\subset \sqrt{q_i} = p_i$ for $i \neq j$. Thus, $q_j R_{\mathfrak{p}_i} = R_{\mathfrak{p}_i}$ for $i \neq j$. Thus, this intersection reduces to $q_i R_{\mathfrak{p}_i} \cap R$. Since q_i is p_i primary, we know that $q_i = q_i R_{\mathfrak{p}_i} \cap R$. It follows that

$$\bigcap_{j=1}^t q_j R_{\mathfrak{p}_i} \cap R = q_i R_{\mathfrak{p}_i} \cap R = q_i$$

Thus, $JR_{\mathfrak{p}_i} \cap R = q_i$.

- (c) This is all we had left to verify for the theorem, so it is proved.

Problem 7. Let R be the ring of all sequences of real numbers. Show that R has infinitely many minimal prime ideals, and therefore the zero ideal does not admit a primary decomposition.

Solution: Let $R = \mathbb{R}^\infty$. The minimal prime ideals of this ring are of the form

$$(0, \mathbb{R}, \mathbb{R}, \dots,)$$

$$(\mathbb{R}, 0, \mathbb{R}, \dots,)$$

$$(\mathbb{R}, \mathbb{R}, 0, \dots,)$$

.....

Any finite intersection of these is necessarily nonzero, as the intersection will kill only finitely many indices and each minimal prime contained all of \mathbb{R} in infinitely many indices. It follows that the zero ideal does not admit a primary decomposition.

Problem 8. LEMMAS ON ASSOCIATED PRIMES. Let M and N be arbitrary R -modules.

- Show $\text{Ass}(M \oplus N) = \text{Ass}(M) \cup \text{Ass}(N)$. [Hint: Consider $0 \rightarrow M \rightarrow M \oplus N \rightarrow N \rightarrow 0$.]
- Show that $\text{Ass}(R/J_1 \cap J_2) \subset \text{Ass}(R/J_1) \cup \text{Ass}(R/J_2)$. [Hint: Find $R/J \hookrightarrow R/J_1 \oplus R/J_2$.]
- For all multiplicative sets $U \subset R$, we have the following inclusion of sets in $\text{Spec } U^{-1}R$: $\{PU^{-1}R \mid P \in \text{Ass}(M) \text{ and } P \cap U = \emptyset\} \subset \text{Ass}(U^{-1}M)$.
- * Prove the converse to (c) when R is Noetherian.
- * For R Noetherian, show that every minimal prime of J is in $\text{Ass}(R/J)$.

Solution:

- We know from problem 11(b) that for a short exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$, $\text{Ass}(M_2) \subset \text{Ass}(M_1) \cup \text{Ass}(M_3)$. Using the exact sequence as in the hint, the result follows.
- From the previous problem, we know that $\text{Ass}(R/J_1) \cup \text{Ass}(R/J_2) = \text{Ass}(R/J_1 \oplus R/J_2)$. Now, consider the map $R/(J_1 \cap J_2) \rightarrow R/J_1 \oplus R/J_2$ such that $\bar{r} \mapsto (\bar{r}, \bar{r})$. This is injective, because if $(\bar{r}, \bar{r}) = (0, 0)$, it follows that $r \in J_1$ and $r \in J_2$, so it is contained in $J_1 \cap J_2$, implying that $\bar{r} = 0$ in $R/J_1 \cap J_2$. It follows that we have the short exact sequence

$$0 \longrightarrow R/(J_1 \cap J_2) \xhookrightarrow{i} R/J_1 \oplus R/J_2 \twoheadrightarrow \text{coker}(i) \longrightarrow 0$$

Implying that

$$\text{Ass}(R/J_1) \cup \text{Ass}(R/J_2) = \text{Ass}(R/J_1 \oplus R/J_2) = \text{Ass}(R/(J_1 \cap J_2)) \cup \text{Ass}(\text{coker}(i))$$

It follows that $\text{Ass}(R/(J_1 \cap J_2)) \subset \text{Ass}(R/J_1) \cup \text{Ass}(R/J_2)$.

(c) Suppose $P \in \text{Ass}(M)$. Then \exists an injection $R/P \rightarrow M$. Localizing is exact, and thus preserves injections. In particular, $U^{-1}(R/P) \rightarrow U^{-1}M$ is an injection. Since $U \cap P = \emptyset$, we can commute localizations with quotients and get something tangible, i.e. we get an injection $U^{-1}R/PU^{-1}R \rightarrow U^{-1}M$, so $PU^{-1}R \in \text{Ass}(U^{-1}M)$.

(d) (*)

(e) (*)

Math 614: Commutative Algebra

IBL Lab Notebook

Class 23: November 26th

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Problem 1. PROOF OF KRULL INTERSECTION THEOREM. Let (R, m) be a local Noetherian ring. Suppose that $\bigcap_{n \in \mathbb{N}} m^n = J$. We want to show that $J = 0$.

- Explain why (two words!) it suffices to show that $mJ = J$.
- Let $\mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \dots \cap \mathfrak{q}_t$ be a primary decomposition for mJ . Explain why it is enough to show that $J \subset \mathfrak{q}_i$ for all i .
- Show that if $\sqrt{\mathfrak{q}_i} \neq m$, then $J \subset \mathfrak{q}_i$. [Hint: Take $x \in m \setminus \mathfrak{p}_i$ and observe $xJ \subset mJ \subset \mathfrak{q}_i$.]
- Finally, show that if $\sqrt{\mathfrak{q}_i} = m$, then $J \subset \mathfrak{q}_i$. [Hint: Show there exists n such that $J \subset m^n \subset \mathfrak{q}_i$.]

Solution:

- Nakayama's Lemma.
- This proves that $J \subset mJ$, which implies that $J = mJ$.
- Choose $x \in m \setminus (\sqrt{\mathfrak{q}_i} \cap m)$. Then, $xJ \subset mJ \subset \mathfrak{q}_i$, but $x \notin \sqrt{\mathfrak{q}_i}$. Thus, $J \subset \mathfrak{q}_i$, by \mathfrak{q}_i being primary.
- Since m is finitely generated (as we are in a noetherian ring), we know that $m = (m_1, \dots, m_n)$. Because $\sqrt{\mathfrak{q}_i} = m$, we know that $m_i^{n_i} \in \mathfrak{q}_i$ for each i . Taking $N = \max(n_i)$ implies that $m^N \subset \mathfrak{q}_i$. But, we know that $J \subset m^N$ for any N , so $J \subset m^N \subset \mathfrak{q}_i$, so $J \subset \mathfrak{q}_i$.

Problem 2. NAGATA'S IDEALIZATION TRICK. Let R be any commutative ring (with 1) and M any R -module.

- Let $S = R \oplus M$ and define a multiplication by $(r \oplus m) \cdot (r' \oplus m') = rr' \oplus (rm' + r'm)$. Show that S is a commutative ring with 1.
- Show that $M = 0 \oplus M \subset S$ is an ideal of S , that $M^2 = 0$ and that $S/M \cong R$.
- Prove that every prime in S has the form $P \oplus M$ for some prime in R . In particular, $\text{Spec } R$ and $\text{Spec } S$ are homeomorphic.
- Prove that if R is local, so is S .
- Prove that if R and M are Noetherian, then so is S .

Solution:

- (a) this clearly has additive identity and multiplicative identity, and by the definition of the multiplication operation, it is multiplicatively closed. Notice that $r \oplus m$ has natural additive inverse $-r \oplus (-1)m$. Thus, this is a commutative ring with identity.
- (b) $0 \oplus M$ is clearly an abelian group. For any $r \oplus m \in S$, $(r \oplus m) \cdot (0 \oplus M) = (r0) \oplus (rM + 0m) = 0 \oplus M$, so this is an ideal. Furthermore, $(0 \oplus M) \cdot (0 \oplus M) = 0^2 \oplus (0M + M0) = 0 \oplus 0 = 0$. The identification $(R \oplus M)/M \cong R$ is clear, as the kernel of the (surjective) projection map $R \oplus M \rightarrow R$ is precisely $0 \oplus M = M$, so we conclude by the first isomorphism theorem.
- (c) For any ideal of S of the form $I \oplus N$, we want to show that if $rr' \oplus (rm' + r'm)$ is in $I \oplus N$, then $r \oplus m$ or $r' \oplus m'$ is. It follows that for this to be true, that I must be prime, since for the coordinates $rr' \oplus 0$, it follows that one of $r \oplus 0, r' \oplus 0$ must be in the ideal, implying that I is prime. Similar logic shows that we need $N = M$. Thus, all prime ideals are of the form $P \oplus M$ for $P \subset R$ prime.

This immediately implies that these have homeomorphic spectra, via the homeomorphism $P \oplus M \longleftrightarrow P$.

- (d) If R is local, then it has only one prime ideal. Since S and R have homeomorphic spectra, $\text{Spec}(R)$ and $\text{Spec}(S)$ have only one closed point, so S also only has one maximal ideal, so S is local.
- (e) Recall that rings are noetherian if and only if all their prime ideals are finitely generated. Since all prime ideals of S are of the form $P \oplus M$, where both P and M are finitely generated by hypothesis, it follows that all prime ideals of S are finitely generated, and S is Noetherian.

Problem 3. COROLLARY OF KRULL'S INTERSECTION THEOREM. Let M be a Noetherian module over a Noetherian local ring (R, m) . Prove that $\bigcap_{n \in \mathbb{N}} m^n M = 0$.
[Hint: Use Nagata's trick and apply the Krull intersection theorem to $(m \oplus M)$ in S .]

Solution: We know from the previous problem that $R \oplus M$ has the structure of a local noetherian ring. Its unique maximal ideal is $m \oplus M$, and $(m \oplus M)^{n+1} = m^{n+1} \oplus m^n M$. Any element of $\bigcap m^n M$ thus must be in every power of the maximal ideal of $R \oplus M$, so it therefore it must be 0 or some infinite, power, so therefore it is 0. Thus, $\bigcap_{n \in \mathbb{N}} m^n M = 0$.

Problem 4. Let J any ideal of R such that $\mathbb{V}(J)$ is a finite set $\{m_1, \dots, m_t\}$ of maximal ideals. Prove that J has a primary decomposition as follows:

- (a) Show that $\sqrt{JR_{m_1} \cap R} = \sqrt{JR_{m_1}} \cap R = (m_1 \cap m_2 \cap \dots \cap m_t)R_{m_1} \cap R = m_1 R_{m_1} \cap R = m_1$.
- (b) Set $\mathfrak{q}_i = JR_{m_i} \cap R$. Prove that \mathfrak{q}_i is m_i -primary.
- (c) Show that $J = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \dots \cap \mathfrak{q}_t$. [Hint: Check $\frac{\mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \dots \cap \mathfrak{q}_t}{J} = 0$ by checking it each maximal ideal.]

- (d) Explain (using a theorem you've proved!) why this decomposition is unique.
- (e) Use the Chinese Remainder theorem to show also that $R/J \cong R/\mathfrak{q}_1 \times R/\mathfrak{q}_2 \times \cdots \times R/\mathfrak{q}_t$.
[Hint: Check $\mathfrak{q}_i + \mathfrak{q}_j$ is the unit ideal by computing its radical.]

Solution:

- (a) This follows precisely from problem 6(b) in the previous worksheet.
- (b) m_i is maximal, and thus prime. We already know this to be an equivalent condition for being m_i primary, from problem 5 on the worksheet deriving the existence of primary decompositions.
- (c) We also know this from the uniqueness of primary decompositions. m_1, \dots, m_t are the minimal primes of J by construction, so the minimal primary decomposition of J is precisely $\bigcap_{i=1}^t JR_{m_i} \cap R = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_t$.
- (d) We know this is unique because minimal primary decompositions are unique (up to reordering).
- (e) Because this decomposition is unique and irredundant, we know that each of the q_i are comaximal. To verify this, we follow the hint: the radical of $q_i + q_j$ is strictly larger than m_i, m_j , so it is all of R . if $\sqrt{q_i + q_j} = R$, then $q_i + q_j = R$, so we have comaximality. Thus, via the Chinese Remainder Theorem we can conclude.

- Problem 5.** (a) Show that M is a simple R -module iff $M \cong R/m$ for some maximal ideal m of R .
- (b) Show that for any R module M , $\ell(M) = \ell(M_1) + \ell(M/M_1)$ for any submodule M_1 . Here we interpret $\infty + d = \infty$.
- (c) Prove that any module of finite length over R is both Noetherian and Artinian.
- (d) Show that if $R = K$ is a field, then for any K -module M , finite length, Noetherian and Artinian are all equivalent.
- (e) * A module M over a Noetherian ring has finite length if and only if M is finitely generated and $\text{Ass}(M)$ contains only maximal ideals.

Solution:

- (a) If $M \cong R/m$, R/m is a field, and since ideals of a ring are precisely its R -submodules, and since a field's only ideals are 0 and itself, it follows that M is simple.
- If M is a simple R -module, it admits a trivial prime cyclic filtration. i.e. $0 \subset M$ is the only possible prime cyclic filtration. Since these always exist, we know that $0 \subset M$ is a prime cyclic filtration, so $M \cong R/P$ for some prime ideal P . Well,

from the correspondence of ideals and submodules we discussed earlier, the ideals of R/P are just 0 and R/P if and only if P is maximal, so by simplicity of M , $M \cong R/P$ with P maximal.

(b) Notice that we can take a filtration of $0 \subset N_1 \subset \dots \subset N_{\ell(M/M_1)}M/M_1$ and lift it to a sequence $M_1 \subset N'_1 \subset \dots \subset N'_{\ell(M/M_1)} = M$. Thus, suspending a filtration of M_1 to this yields a filtration of M . Thus, $\ell(M/M_1) + \ell(M_1) = \ell(M)$.

(c) What the above statement tells us is that all submodules of finite length modules have finite length.

Suppose that M was not Noetherian. Then, there would exist a chain $M_1 \subset M_2 \subset \dots$ that does not terminate. Well, M_2/M_1 has finite length by the previous part, as does $M_{i+1}/M_i \forall i \in \mathbb{N}$. It follows that if all of these have finite length and this chain never ends, M must then have infinite length, as by appending these sequences to each other we've created a sequence of infinite length in M , a contradiction.

The proof for M being Artinian is similar. Suppose we have a descending chain $M \supset M_1 \supset M_2 \supset \dots$. M_1/M_2 has finite length, as does M_i/M_{i+1} for each i . If this was infinitely long, appending these all together gives us an infinite length sequence, which is a contradiction to the finite length of M .

(d) In this case, M is a vector space. Notice that M having finite length means that there exists a filtration $M_1 \subset \dots \subset M_n = M$ such that M_i/M_{i-1} is simple, but the only simple vector spaces are 1 dimensional. Thus, finite length, Noetherian, and Artinian are all equivalent to M being finite dimensional as a K vector space, and in particular, being finite length, Noetherian, and Artinian are all equivalent.

(e) (*)

Problem 6. PROOF OF THEOREM ON ZERO DIMENSIONAL RINGS.

(a) Prove that (i) implies (ii) by using (4) applied to the zero ideal of R .

(b) Assume (R, m) is a Noetherian local ring of dimension zero. Show that $m^n = 0$ for some n , and that $\ell(R) = \sum_{i=0}^n \dim_{R/m}(m^i/m^{i+1})$. [Hint: Use (5b) and the chain $R \supset m \supset m^2 \supset \dots$.]

(c) Conclude that (ii) implies (iii).

(d) Observe that (iii) implies (iv). Hint: This is quite general; see (5c).]

(e) Show that if R is Artinian, so is every quotient ring.

(f) Show that if R is Artinian, all primes are maximal. [Hint: If P is a non-maximal prime, take non-unit non-zero element $x \in R/P$. Use the stabilizer of $\langle x \rangle \supset \langle x^2 \rangle \supset \dots$ to show that x is a unit in R/P .]

(g) Show that if R is Artinian, then R has only finitely maximal ideals. [Hint: Look at $m_1 \supset m_1 \cap m_2 \supset \dots$.]

(h) Complete the proof of the theorem.

Solution:

- (a) If R is Krull dimension 0, then $\mathbb{V}(0) = \text{Spec}(R)$ consists solely of incomparable primes. Thus, the zero ideal admits a decomposition as in problem 4, implying that $R \cong R/0 \cong R/q_1 \times \dots \times R/q_t$. R/q_i has only m_i as a maximal ideal, since the only prime ideals in this ring are those that contain q_i , which is just m_i . Thus, since there is only one maximal ideal, R/q_i is local, and we can conclude.
- (b) Notice that $\sqrt{0} = m$ for m a maximal ideal, where $m = (f_1, \dots, f_t)$, for f_i all nilpotent by construction. Choose $r = \sum_i a_i f_i \in m$. Since each f_i is nilpotent, there exists an N sufficiently large that, independent of the a_i , annihilates the entire sum, i.e. $r^N = (\sum_i a_i f_i)^N = 0$. It follows that $m^N = 0$.
- Following the hint immediately tells us that $\ell(R) = \ell(m) + \ell(R/m) = \ell(m) + \ell(m/m^2) + \ell(R/m^2) = \dots$. Continuing this process until we hit N such that $m^N = 0$, we find that $\ell(R) = \sum_{i=0}^n \ell_{R/m}(m^i/m^{i+1}) = \sum_{i=0}^n \dim_{R/m}(m^i/m^{i+1})$
- (c) we know that if R was a local ring of dimension 0, that it would have finite length over itself. Since it is the finite product of local rings of dimension 0, we can quotient one local ring off at a time and still guarantee finite length, allowing us to conclude that (ii) implies (iii).
- (d) We know that finite length modules are Noetherian and Artinian, and in particular they are Artinian. It follows that (iii) implies (iv).
- (e) If R is Artinian over itself, any descending chain $R \supset R_1 \supset R_2 \supset \dots$ terminates. In particular, for any ideal J , the chain $R/J \supset R_1/(J \cap R_1) \supset R_2/(J \cap R_2) \supset \dots$ also must terminate. This implies that any quotient of R is also Artinian.
- (f) If R is Artinian, suppose we can pick a non-maximal prime P . Pick $x \in R/P$ nonzero and non-unit (we can pick it non-unit because P is not maximal). We follow the hint. Since the descending chain $\langle x \rangle \supset \langle x^2 \rangle \supset \dots$ halts, we know that $(x^n) = (x^{n+1})$ for sufficiently large n . It follows that $x^n \in (x^{n+1})$, which does not make sense unless this is the unit ideal, implying that x is unit, a contradiction. It follows that all primes are maximal.
- (g) Suppose that R was Artinian, but had infinitely many maximal ideals. Following the hint, there is a point in which that descending chain stabilizes. This implies that $m_{n+1} \cap \dots \cap m_1 = m_n \cap \dots \cap m_1$, implying that $m_{n+1} \subset m_1 \cap \dots \cap m_n$, contradicting maximality of m_{n+1} . It follows that R has finitely many maximal ideals.
- (h) Since R is Artinian, it has finitely many (incomparable) maximal ideals, implying it has finitely many prime ideals which are all incomparable. Thus, R is of Krull dimension 0, and since all primes are maximal, it is also Noetherian, giving us (iv) implies (i) and completing the proof of the theorem.

Problem 7. Consider the $\mathbb{Z}_{(p)}$ module $\mathbb{Q}/\mathbb{Z}_{(p)}$. Show that the classes of $\frac{1}{p^n}$ for all $n \in \mathbb{N}$ generate $\mathbb{Q}/\mathbb{Z}_{(p)}$, and that $\mathbb{Q}/\mathbb{Z}_{(p)}$ has DCC but not finite length. Why doesn't this contradict the Theorem?

Solution: Choose $r/s \in \mathbb{Q}/\mathbb{Z}_p$. Notice that this equals $r \cdot (1/s)$ for $r \in \mathbb{Z}_p$, so we just need to generate terms of the form $1/s$. Well, notice that s is unit in \mathbb{Z}_p unless $s \in (p)$, implying that $s = p^n$ for some n . It follows that the classes of $1/p^n$ generate \mathbb{Q}/\mathbb{Z}_p . This clearly does not have finite length, as it is not finitely generated and we can take a chain where we append each generator one by one. It remains to verify that \mathbb{Q}/\mathbb{Z}_p has DCC.

Well, notice that any descending chain is of the form $(1/r_1) \supset (1/r_2) \supset \dots$ where r_1, r_2, \dots are all (without loss of generality) non-negative integers. This relation implies that $r_1 > r_2$ as non-negative integers. Since there exists a lower bound to non-negative integers (namely, 0), any descending chain must terminate.

\mathbb{Q}/\mathbb{Z}_p is an Artinian module over itself, but it has finite length as a \mathbb{Z}_p module, not as an \mathbb{Q}/\mathbb{Z}_p module, so it does not contradict the theorem.

Problem 8. PROOF OF KRULL PRINCIPAL IDEAL THEOREM.

- (a) Show that if a counterexample exists to the Krull principal ideal theorem, then there is a counterexample in which (R, m) is a local Noetherian domain of dimension at least two and $\langle f \rangle$ is a principal ideal which has m as its unique minimal prime.
- (b) With notation as in (a), prove that $\bar{R} = R/\langle f \rangle$ is zero dimensional.
- (c) Let $Q \subset R$ be a non-maximal prime ideal. Show that the symbolic powers

$$Q\bar{R} \supset Q^{(2)}\bar{R} \supset Q^{(3)}\bar{R} \supset \dots$$

eventually stabilize; say $Q^{(m)}\bar{R} = Q^{(n)}\bar{R}$ for all $m \geq n$.

- (d) Prove that $Q^{(n)} = Q^{(m)} + fQ^{(n)}$ in R . [Hint: You will need to use the fact that $Q^{(n)}$ is Q -primary.]
- (e) Show that $Q^{(m)} = Q^{(n)}$ for all $m \geq n$ by applying NAK to the quotient $Q^{(n)}/Q^{(m)}$.
- (f) Show that if $Q \neq 0$, then $\bigcap_{m \in \mathbb{N}} Q^{(m)} \neq 0$. [Hint: Show that if $y \in Q$, then $y^n \in \bigcap_{m \in \mathbb{N}} Q^{(m)}$.]
- (g) Use Krull's Intersection Theorem to show $Q = 0$. [Hint: $Q^{(m)} \subset Q^m R_Q$.]
- (h) Complete the proof of Krull's Principal Ideal Theorem.

Solution:

- (a) If we have a counter example, it is still a counter example after localizing by a prime, so without loss of generality the counterexample is local. We know that the dimension is at least 2, since our counter example has an ideal of height 2. If we have a chain $Q_2 \supset Q_1 \supset Q_0$, we can quotient R to get R/Q_0 , and get a chain $Q_2/Q_0 \supset Q_1/Q_0 \supset (0)$, implying that we can restrict to the case of R quotiented by a prime ideal (i.e. in the case of the domain).

To be a counterexample, R is by hypothesis Noetherian. Thus, our counter example is a local Noetherian domain of height at least 2.

- (b) The ring $R/(f)$ has only one prime ideal, in particular, $m/(f)$. Therefore, it is a zero dimensional local ring.
- (c) From our theorem, we know that zero dimensional local Noetherian rings (i.e. \overline{R}) are Artinian, i.e. their ideals satisfy DCC. Thus, the descending chain of ideals given in the question must stabilize.
- (d) lifting $Q^{(m)}\overline{R} = Q^{(n)}\overline{R}$ to R gives us that $Q^{(n)} + (f) = Q^{(n+1)} + (f)$, implying that $Q^{(n)} \subset Q^{(n+1)} + (f)$. Then, $u = q + fr$, for $q \in Q^{(n+1)}$. Thus, $fr = u - q \in Q^{(n)}$, but since $f \notin Q$, as (f) has minimal prime m as the only minimal prime and Q is a non-maximal prime, and since $Q^{(n)}$ is Q -primary, it follows that $r \in Q^{(n)}$. It follows that $Q^{(n)} = Q^{(n+1)} + fQ^{(n)}$, so $Q^{(n)} = Q^{(n+1)} + fQ^{(n)} = Q^{(m)} + fQ^{(n)}$ for $m \geq n$.
- (e) This implies that $(Q^{(n)}/Q^{(m)}) = f(Q^{(n)}/Q^{(m)})$, By Nakayama's Lemma, $(Q^{(n)}/Q^{(m)}) = 0$, implying that $Q^{(m)} = Q^{(n)}$.
- (f) Fix n as above. If $Q \neq 0$, then $\bigcap_{m \in \mathbb{N}} Q^{(m)} = Q^{(n)} \neq 0$. This is because $Q^{(m)} \supset Q^{(m+1)}$ for any m , and this descending chain stabilizes as we saw from the previous part.
- (g) Since $Q^{(m)} \subset Q^m R_Q$, and QR_Q is a maximal ideal in R_Q , where R_Q is a local Noetherian ring, we have that

$$\bigcap_{m \in \mathbb{N}} Q^{(m)} \subset \bigcap_{m \in \mathbb{N}} Q^m R_Q = 0$$

By Krull's Intersection Theorem. It follows that $\bigcap_{m \in \mathbb{N}} Q^{(m)} = 0$, so by contrapositive of the previous statement $Q = 0$.

- (h) We've just shown that all non-maximal prime ideals in R are 0, implying that R is of height 1. This contradicts R having height at least 2, so we have a contradiction. Thus, there can exist no counter example to Krull's Principal Ideal Theorem, so it must be true.

Math 614: Commutative Algebra

IBL Lab Notebook

Class 24: December 3rd

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Problem 1. NORMALIZATION COMMUTES WITH LOCALIZATION. Let R be a domain and $U \subset R$ any multiplicative subset.

- (a) Be sure you understand why R and $U^{-1}R$ have the same fraction field. Call it K .
- (b) Prove that if $\frac{a}{b} \in K$ is integral over R , then $\frac{a}{b}$ is integral over $U^{-1}R$.
- (c) Conclude that $\widetilde{R} \subset \widetilde{U^{-1}R}$ and also $U^{-1}\widetilde{R} \subset \widetilde{U^{-1}R}$ as subsets of K .
- (d) Show that if $\frac{a}{b} \in K$ satisfies an equation of integral dependence

$$X^n + \frac{r_1}{u_1}X^{n-1} + \cdots + \frac{r_{n-1}}{u_{n-1}}X + \frac{r_n}{u_n} \in U^{-1}R[X]$$

over $U^{-1}R$, then $\frac{ua}{b} \in K$ is integral over R where $u = \prod_{i=1}^n u_i$.

- (e) Prove that $U^{-1}\widetilde{R} = \widetilde{U^{-1}R}$. That is, *normalization commutes with localization*.

Solution:

- (a) The fraction field is taken by localizing by the "biggest" multiplicative set, which is $R \setminus \{0\}$, which corresponds to inverting all nonzero elements. Inverting all nonzero elements is the same as inverting some nonzero elements, then inverting the rest. Thus, $R, U^{-1}R$ have the same fraction field.
- (b) a/b being integral over R means it satisfies a polynomial over R . a/b is also integral over $U^{-1}R$ since $R \subset U^{-1}R$, so we can just take the same polynomial that annihilates a/b over R .
- (c) This gives us that $\widetilde{R} \subset \widetilde{U^{-1}R}$. Notice also that $U^{-1} \subset \widetilde{U^{-1}R}$, so their product is contained in $\widetilde{U^{-1}R}$, i.e. $U^{-1}\widetilde{R} \subset \widetilde{U^{-1}R}$.
- (d) This works because we can just clear denominators by multiplying by $u/u = (\prod_{i=1}^n u_i)/(\prod_{i=1}^n u_i)$. This yields a polynomial of ua/b that is still zero.
- (e) We've just shown that satisfying integral dependence over $U^{-1}R$ implies satisfying an integral dependence over R . This implies that $U^{-1}\widetilde{R} \supset \widetilde{U^{-1}R}$, and we've already shown \subset , so it follows that $U^{-1}\widetilde{R} = \widetilde{U^{-1}R}$.

Problem 2. NORMALITY IS A LOCAL PROPERTY. Prove the following are equivalent for a domain R .

- (i) R is normal.
- (ii) $U^{-1}R$ is normal for all multiplicative sets $U \subset R$.
- (iii) R_P is normal for all $P \in \text{Spec } R$.
- (iv) R_m is normal for all $m \in \text{MaxSpec } R$.

[Hint: For (iv) implies (i), check the triviality of the R -module \tilde{R}/R locally.]

Solution: We verify (i) implies (ii) first. Notice that if R is normal, then $R = \tilde{R}$, so $U^{-1}R = U^{-1}\tilde{R} = \widetilde{U^{-1}R}$, so $U^{-1}R$ is normal for all multiplicative sets $U \subset R$.

(iii) is a special case of (ii), and (iv) is a special case of (iii), so it follows that (ii) implies (iii) implies (iv). Thus, it is sufficient to verify that (iv) implies (i).

Notice that \tilde{R}/R is an R -module. Recall that $M = 0$ if and only if $M_m = 0 \forall m \in \text{MaxSpec}(R)$. Thus, it follows that $\tilde{R}/R = 0 \iff (\tilde{R}/R)_m = \tilde{R}_m/R_m = 0 \forall m \in \text{MaxSpec}(R)$. In other words, $\tilde{R} = R \iff \tilde{R}_m = R_m \forall m \in \text{MaxSpec}(R)$. This gives us (iv) \iff (i), so we can conclude.

Problem 3. Prove that a UFD is normal.

Solution: Let R be a UFD. Choose $u/v \in \text{Frac}(R)$ in lowest terms (which we can do, as R is UFD). Suppose it satisfied some integral dependence over R .

$$\begin{aligned} (u/v)^n + a_{n-1}(u/v)^{n-1} \cdots + a_0 &= 0 \\ \Rightarrow u^n + a_{n-1}u^{n-1}v + \cdots + a_0v^n &= 0 \\ \Rightarrow u^n &= -v(a_{n-1}u^{n-1} + \cdots + a_0v^{n-1}) \end{aligned}$$

Now, notice that in a UFD, every nonzero non-unit has a unique factorization into primes. I claim that v is unit. If not, there exists a prime p that divides v . By above, it would also divide u^n , and thus u . This is a contradiction, as we took u/v to be in its reduced form.

Problem 4. Prove that an intersection of normal rings (with fraction field K) is normal. Use Theorem 1 to deduce that if R is Noetherian and normal, then the ring $\bigcap_{ht1P \in \text{Spec } R} R_P$ is normal. [In fact: for Noetherian rings, R is normal if and only if $R = \bigcap_{ht1P \in \text{Spec } R} R_P$; See Hochster's Dec 4 lecture.]

Solution: Let $\{R_i\}_{i \in I}$ be a collection of normal rings with fraction field K . Recall that for an extension of rings $R \subset S$, if there exists an R -module map $S \rightarrow R$ which restricts to the identity on R , then if S is normal then R is normal (this is from the worksheet on going down and integral extensions).

$\bigcap_{i \in I} R_i \rightarrow R_i$ is certainly an injection for each R_i , and for each R_i the projection map to the intersection clearly restricts to the identity. Since each R_i is normal, it follows that $\bigcap_{i \in I} R_i \rightarrow R_i$.

If R is Noetherian and normal, notice that R_P is normal for all $P \in \text{Spec}(R)$, in particular just the ones of height 1 are normal. Thus, since each R_P is normal, $\bigcap_{\text{ht} 1 P \in \text{Spec} R} R_P$ is normal.

Problem 5. Let R be a Noetherian normal domain with fraction field K . Define a group homomorphism $K^\times \rightarrow \text{Div} R$ so that the divisor class group of R is the cokernel.

Solution: The map $\varphi : a/b \mapsto \sum_{i=1}^t \nu_{P_i}(f/1)[P_i]$ for P_i the minimal primes of (a) and ν_{P_i} the valuation on R_{P_i} is a map from $K^\times \rightarrow \text{Div}(R)$. Notice that all elements of form quotiented over $\text{Div}(R)$ yield the class group. Since $\text{coker}(\varphi) \cong \text{Div}(R)/\text{im}(\varphi)$, it follows that $\text{coker}(\varphi) \cong \text{CL}(R)$.

Problem 6. Let R be a normal Noetherian domain. Let \mathfrak{q} be \mathfrak{p} -primary where \mathfrak{p} is height one.

- (a) Show that $\mathfrak{q} = \mathfrak{p}^{(n)}$ for some n , where by definition, $\mathfrak{p}^{(n)} = \mathfrak{p}^n R_{\mathfrak{p}} \cap R$.
 [Hint: Recall that \mathfrak{q} is \mathfrak{p} -primary if and only if $\sqrt{\mathfrak{q}} = \mathfrak{p}$ and $\mathfrak{q}R_{\mathfrak{p}} \cap R = \mathfrak{q}$. You will also need Thm 1.]
- (b) Show $\text{div}(f) = n_1[\mathfrak{p}_1] + \dots + n_t[\mathfrak{p}_t]$ where $\langle f \rangle = \mathfrak{p}_1^{(n_1)} \cap \dots \cap \mathfrak{p}_t^{(n_t)}$ is a primary decomp.

Solution:

- (a) Theorem 1 tells us that $R_{\mathfrak{p}}$ is a DVR, and in particular, a local PID. Thus, $p = (t)$ for some t , implying that $q = (t^n)$. Then, notice that

$$q = qR_{\mathfrak{p}} \cap R = (t^n) \cap R = p^n R_{\mathfrak{p}} \cap R = p^{(n)}$$

- (b) We know that the radicals of all elements in the primary decomposition are precisely the minimal primes of (f) . Thus, all we need to verify is that $\nu_{\mathfrak{p}_i}(f/1) = n_i$. From our worksheet on valuation rings, the method used to construct a discrete valuation from a DVR immediately implies this fact, since for $p = (t)$ (the maximal ideal), $p^{(n_i)} = q = (t^{n_i})$.

Problem 7. DEDKIND DOMAINS. A **Dedekind domain** is a normal Noetherian ring of dimension 1. Show that the normalization of any finite integral extension of \mathbb{Z} is a Dedekind domain. Such a ring is called a **number ring**. [You may assume that the normalization of a finitely generated \mathbb{Z} -algebra is Noetherian. This is a non-trivial fact.]

Solution: The ring is a normalization, so it is clearly normal. $\dim(\mathbb{Z}) = 1$, and since dimension is preserved over integral extensions, the dimension of the integral extension is still 1. Furthermore, finiteness of the extension (via the assumption) preserves Noetherianity, implying we have a normal Noetherian ring of dimension 1 or a Dedekind domain.

Problem 8. Let R be a Dedekind domain. Let \mathfrak{q} and \mathfrak{q}' be non-zero primary ideals with distinct radicals.

- (a) Show $\mathfrak{q} \cap \mathfrak{q}' = \mathfrak{q}\mathfrak{q}'$. [Hint: Observe $\mathfrak{q} + \mathfrak{q}' = R$.]
- (b) Explain why the primary decomposition of any ideal in R is *unique*.
- (c) Show $\mathfrak{p}^n \neq \mathfrak{p}^{n+1}$ for all $n \in \mathbb{N}$ and all non-zero $\mathfrak{p} \in \text{Spec } R$. [Hint: NAK!]
- (d) Let \mathfrak{q} be \mathfrak{p} -primary. Show that $\mathfrak{q} = \mathfrak{p}^n$ for some n . [Hint: Recall that \mathfrak{q} is \mathfrak{p} -primary if and only if $\sqrt{\mathfrak{q}} = \mathfrak{p}$ and $\mathfrak{q}R_{\mathfrak{p}} \cap R = \mathfrak{q}$. You will also need Theorem 1.]
- (e) Prove ideals in a Dedekind domain factor uniquely as a product of prime ideals.

Solution:

- (a) It is clear that $\mathfrak{q}\mathfrak{q}' \subset \mathfrak{q}, \mathfrak{q}'$, so $\mathfrak{q}\mathfrak{q}' \subset \mathfrak{q} \cap \mathfrak{q}'$. We now verify that $\mathfrak{q} \cap \mathfrak{q}' \subset \mathfrak{q}\mathfrak{q}'$.
Choose $x_1 \in \mathfrak{q}$. Since $\mathfrak{q} + \mathfrak{q}' = R$, we can choose $x_2 \in \mathfrak{q}'$ such that $x_1 + x_2 = 1$. Next, choose $x \in \mathfrak{q} \cap \mathfrak{q}'$. Notice that $x(x_1 + x_2) = x$, but $xx_1 \in \mathfrak{q}\mathfrak{q}', xx_2 \in \mathfrak{q}'\mathfrak{q}$, so $x = x(x_1 + x_2) \in \mathfrak{q}\mathfrak{q}'$.
- (b) In 1 dimensional rings, all primes over an ideal are minimal primes. Thus, any primary decomposition is immediately a minimal primary decomposition, and is thus unique.
- (c) Notice that in 1 dimensional rings, all primes are defacto maximal primes. Furthermore, observe that \mathfrak{p}^n is an R -module that is finitely generated over a local ring. Thus, if $\mathfrak{p}^n = \mathfrak{p}^{n+1}$, it follows that $\mathfrak{p} \cdot \mathfrak{p}^n = \mathfrak{p}^n$, and we could cite Nakayama's Lemma to tell us that $\mathfrak{p}^n = 0$, a contradiction to the fact that R is a domain.
- (d) We already know that $\mathfrak{q} = \mathfrak{p}^{(n)} = \mathfrak{p}^n R_{\mathfrak{p}} \cap R$. Well, \mathfrak{p}^n is clearly \mathfrak{p} -primary, and we know \mathfrak{p} -primary ideals to be of the form $\mathfrak{p}^n R_{\mathfrak{p}} \cap R = \mathfrak{p}^{(n)}$. It follows that $\mathfrak{q} = \mathfrak{p}^{(n)} = \mathfrak{p}^n$.
- (e) Let I be an ideal of R . Part (b) gives us a unique primary decomposition, and from the (c) and (d) we know it is of the form $I = \mathfrak{p}_1^{n_1} \cap \dots \cap \mathfrak{p}_t^{n_t}$, for $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ prime ideals. By part (a), we know that

$$I = \mathfrak{p}_1^{n_1} \cap \dots \cap \mathfrak{p}_t^{n_t} = \mathfrak{p}_1^{n_1} \mathfrak{p}_2^{n_2} \cdots \mathfrak{p}_t^{n_t}$$

(which, again, is unique is part (b)).

Problem 9. Let $R = K[x, y, z, w]/\langle xy - zw \rangle$.

- (a) Prove that R is a three dimensional domain.
- (b) Prove that R is not a UFD.
- (c) Prove that $\langle x, z \rangle, \langle x, w \rangle, \langle y, z \rangle, \langle y, w \rangle$ are all height one prime ideals.
- (d) Show that $R_{\langle x, z \rangle} \cong K[z, y, \frac{1}{y}, w]_{\langle z \rangle} \cong K[z, y, w]_{\langle z \rangle}$.
- (e) Compute the divisor of $\bar{z} \in R$. [Hint: First find its minimal primes.]
- (f) Show that all primes in (c) are equal in $Cl(R)$ up to sign.
- (g) * Prove that $Cl(R)$ is isomorphic to \mathbb{Z} .

Solution:

- (a) We know that $K[x, y, z, w]$ is a 4 dimensional domain, and that $xy - zw$ is prime of height 1. It follows that any chain $0 \subset (xy - zw) \subset P_2 \subset P_3 \subset P_4$ can be lifted to a chain $0 \subset P_2/(xy - zw) \subset P_3/(xy - zw) \subset P_4/(xy - zw)$ of length 3, and since this chain cannot have length ≥ 4 , this is a three dimensional domain.
- (b) This is not UFD because $\bar{x} \cdot \bar{y} = \overline{xy} = \overline{zw} = \bar{z} \cdot \bar{w}$.
- (c) It is clear that these are all prime, so we just need to verify that they are 1 dimensional. Notice that (x) is not prime, since $xy = zw \in (x)$, but $z, w \notin (x)$. Similar logic is used to show that $(y), (z), (w)$ are all not prime. It follows that each of these ideals, which are height 2 over $K[x, y, z, w]$, are height 1 in the quotient, as any ideal contained in it is either principally generated by x, y, z, w , and it not prime, or is generated by a product of the generators, which is certainly not prime.
- (d) Localizing by (z) inverts y , so it is clear that $K[z, y, \frac{1}{y}, w]_{\langle z \rangle} \cong K[z, y, w]_{\langle z \rangle}$. Recall that quotients commute with localization.

$$R_{\langle x, z \rangle} \cong K(y, w)[x, z]/K(y, w)[x, z](xy - zw)$$

Notice that in here $xy = zw$, where both y, w are inverted. Thus, we have that $x/w = z/y$, and in particular, $x = zw/y$. This implies that x can be written in terms of z, y, w , and can be ignored. From this, we have that $R_{\langle x, z \rangle} \cong K[z, y, \frac{1}{y}, w]_{\langle z \rangle}$, so we can conclude.

- (e) We know that $(x, z), (y, z)$ are both minimal primes, as they contain (z) and are height 1. Furthermore, we can't take powers, since $z \notin (x, z)^2, (y, z)^2$. Thus, it follows that the minimal primary decomposition of (z) is $(x, z) \cap (y, z)$, so the divisor of (z) is $[(x, z)] + [(y, z)]$.
- (f) This follows immediately from the previous part, as the divisor of any ideal generated by an indeterminate is precisely the sum of two of the ideals in (c). Modding out by these thus gives the desired result.

Problem 10. Let V be a valuation ring with fraction field K . Prove that V is normal. [Hint: For $\lambda \in K$, consider an equation of integral dependence over V and the possible values of ν for its terms.]

Solution: Suppose there exists $\lambda \notin V$ satisfying an equation of integral dependence $\lambda^n + r_{n-1}\lambda^{n-1} + \cdots + r_0 = 0$, where each $r_i \in V$. Well, since $\lambda \notin V$, we know that $\lambda^{-1} \in V$, and thus $\lambda^{1-n} \in V$, since V is a valuation ring. Multiplying our equation by λ^{1-n} yields

$$\lambda + r_{n-1}\lambda^{-1} + \cdots + r_0\lambda^{1-n} = 0 \Rightarrow \lambda = -(r_{n-1}\lambda^{-1} + \cdots + r_0\lambda^{1-n})$$

Thus, λ is equal to a sum of elements all contained in V by construction, so $\lambda \in V$, a contradiction.

Problem 11. POLYNOMIAL RINGS OVER NORMAL RINGS ARE NORMAL.

1. Let R be a normal domain. Prove that $R[x]$ is a Normal domain. [Hint: Use the fact that $K[x]$ is a UFD to reduce to considering elements of $K[x]$ integral over $R[x]$.]
2. Prove that a directed union of normal domains is normal.
3. Prove that a polynomial ring over any normal ring in any number (even infinite) of variables is normal.

Solution:

- (a) R is the intersection of DVRs $V \subset \text{Frac}(R)$, implying that $R[x]$ is the intersection of the rings $V[x] \subset \text{Frac}(R)[x]$. Since V is a UFD, $V[x]$ is a UFD for each V , so $R[x]$ is normal.
- (b) An element x in the fraction field integral over the union will be in the fraction field of one of these domains, and likewise will be integral over one of them. But some domain D in the family will contain both, and since D is normal, x is integral over D and in the fraction field of D , and so must be in D .
- (c) If $R[x_1]$ is normal, $R[x_1][x_2] = R[x_1, x_2]$ is normal by the first part. Induction tells us that any polynomial ring in finitely many variables over a normal ring is normal, and part (b) tells us we can extend this to infinitely many variables.

Math 614: Commutative Algebra

IBL Lab Notebook

Class 25: December 5th

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Problem 1. Show that the subset $\mathfrak{C}^I(R)$ of Cauchy sequences in $R^\mathbb{N}$ is a subring.

Solution: Clearly the sequence of all 0s and all 1, converge, so we have a multiplicative and additive identity. Now, choose $r_\bullet, s_\bullet \in \mathfrak{C}^I(R)$. We first verify that $r_\bullet + s_\bullet \in \mathfrak{C}^I(R)$. Fix $t \in \mathbb{N}$. There exists d_1 such that $\forall n, m \geq d_1, r_n - r_m \in I^t$. Furthermore, $\exists d_2$ such that $\forall k, \ell \geq d_2, s_k - s_\ell \in I^t$. Let $D = \max(d_1, d_2)$. Then $\forall n, m \geq D, r_n - r_m, s_n - s_m \in I^t$, so $(r_n - r_m) + (s_n - s_m) \in I^t$. Grouping terms, we find that $(r_n + s_n) - (r_m + s_m) \in I^t$, so it follows that we have additive closure. If Furthermore, if $\exists d$ such that $\forall n, m \geq d, r_n - r_m \in I^t$, then $-(r_n - r_m) = -r_n - (-r_m) \in I^t$, so we have additive inverses as well.

It is sufficient then to verify multiplicative closure. Take r_\bullet, s_\bullet as before, with t fixed and d_1, d_2, D as before. We want to show that $s_m r_m - s_n r_n \in I^t$. Well, $s_m r_m - s_n r_n = s_m(r_m - r_n) - r_n(s_m - s_n)$. Notice that $r_m - r_n, s_m - s_n \in I^t$, so $s_m(r_m - r_n), r_n(s_m - s_n) \in I^t$, so $s_m(r_m - r_n) - r_n(s_m - s_n) = s_m r_m - s_n r_n \in I^t$ as desired, giving us multiplicative closure.

Problem 2. Show that the map $R \rightarrow \mathfrak{C}^I(R)$ sending r to the constant sequence (r) is a ring homomorphism, making $\mathfrak{C}^I(R)$ into an R -algebra.

Solution: It is clear that $rs \mapsto (rs)_\bullet = r_\bullet s_\bullet$ and that $r + s \mapsto (r + s)_\bullet = r_\bullet + s_\bullet$. Furthermore, the all 1s sequence is the multiplicative identity, as the all 0s sequence is the additive identity, so identities are preserved. Thus, this is a ring homomorphism.

Problem 3. Let $\mathfrak{C}_I^0(R)$ be the set of sequences $(r_i)_{i \in \mathbb{N}}$ that converge to zero, meaning that for all n there exists m such that for all $i \geq m, r_i \in I^n$. Prove that $\mathfrak{C}_I^0(R)$ is an ideal of $\mathfrak{C}^I(R)$.

Solution: It is clear that the all 0s sequence is in $\mathfrak{C}_I^0(R)$, since $0 \in I^t$ for any t . Suppose that $r_\bullet, s_\bullet \in \mathfrak{C}_I^0(R)$. We fix $t \in \mathbb{N}$ for the rest of the problem. There exists d_1, d_2 such that $\forall i \geq d_1, j \geq d_2, r_i, s_j \in I^t$. Well, for $D = \max(d_1, d_2)$, we have that $\forall i \geq D, r_i, s_i \in I^t$, so $r_i + s_i \in I^t$, so we have additive closure. In addition, if $r_i \in I^t$, then $-r_i \in I^t$, so we have additive inverses as well.

Now, suppose instead that $r_\bullet \in \mathfrak{C}^I(R)$, and $s_\bullet \in \mathfrak{C}_I^0(R)$. Choose i sufficiently large enough that $s_i \in I^t$. Notice that if $s_i \in I^t$, then $r_i s_i \in I^t$, since I^t is an ideal. It follows that $r_\bullet s_\bullet \in \mathfrak{C}_I^0(R)$, so $\mathfrak{C}_I^0(R)$ is an ideal of $\mathfrak{C}^I(R)$.

Problem 4. Observe that a subsequence of a Cauchy sequence is Cauchy, and differs from the original by a sequence converging to zero.

Solution: Suppose $r_\bullet \in \mathfrak{C}_I(R)$. For fixed t , there exists a d such that $n, m \geq d$ implies that $r_n - r_m \in I^t$. Notice that the choice of n, m is independent of sequencing, i.e., for any subsequence of r_{i_\bullet} of r_\bullet , for any indices of the subsequence $i_j \geq d$, the same statement holds. Thus, any subsequence of a Cauchy sequence is Cauchy.

Furthermore, the difference between $r_{i_n} - r_{i_m}$ and $r_n - r_m$ for $i_n = n, i_m = m$ is clearly zero, so it follows that the difference between a sequence and any subsequence is a sequence that converges to zero.

Problem 5. The elements of \widehat{R}^I are equivalence classes of Cauchy sequences that differ by Cauchy sequences converging to zero. We call such an equivalence class a **limit of a Cauchy sequence** in R . Compare this to the construction of the real numbers as the collection of limits of Cauchy sequences of rational numbers.

Solution: Recall that \mathbb{R} is constructed via taking the Cauchy sequence completion with respect to \mathbb{Q} . The only ideals of \mathbb{Q} are 0 and \mathbb{Q} , and completing with respect to \mathbb{Q} is a trivial action. Thus, completing \mathbb{Q} in the new sense with respect to the zero ideal yields \mathbb{R} .

Problem 6. Discuss a natural map $R \rightarrow \widehat{R}^I$. What is its kernel?

Solution: This is the map $R \rightarrow \mathfrak{C}_I(R) \rightarrow \mathfrak{C}_I(R)/\mathfrak{C}_I^0(R)$, where the first map was defined in problem 2, and the second map is the standard projection map. The only elements that are zero in the image are sequences where terms of sufficiently high index live in I^t for any t . Since the image consists of only constant sequences, index is irrelevant, so the kernel consists of elements that live in I^t for any t . Concretely, $\ker = \bigcap_{i \in \mathbb{N}} I^i$.

Problem 7. Suppose (R, m) is Noetherian and local. Prove that $R \rightarrow \widehat{R}^m$ is injective.

Solution: It follows that $\bigcap_{i \in \mathbb{N}} m^i = 0$ via Krull's intersection theorem, so the map $R \rightarrow \widehat{R}^m$ has trivial kernel.

Problem 8. Let $R = \mathbb{Z}$ and $I = \langle p \rangle$. Show that

$$\left(\sum_{i=0}^n p^i \right)_{n \in \mathbb{N}} = (1, 1 + p, 1 + p + p^2, 1 + p + p^2 + p^3, \dots)$$

is a Cauchy sequence for the p -adic topology on \mathbb{Z} . Show also that it represents the inverse of $1 - p$ in the completion $\widehat{\mathbb{Z}}^p$ (also denoted $\widehat{\mathbb{Z}}_p$.)

Solution: Let r_i denote the i th element in this sequence. Notice that (for $i < j$) $r_i - r_j = p_i + \cdots + p_j = p_i(1 + \cdots + p^{j-i}) \in (p^i) = I^i$. Thus, this is a Cauchy sequence. Notice that $\frac{1}{1-p} = \sum_{i=1}^{\infty} p^i$; the fact that this Cauchy sequence represents the inverse of this follows, as $p^n(1-p) = p^n - p^{n+1}$, reducing the infinite sequence to 1 via a telescoping argument, and reducing a finite sum $\sum_{i=1}^n p^i$ via a similar telescoping argument to $1 - p^n$.

Problem 9. Let $R = K[x, y]$ and $I = \langle x, y \rangle$.

(a) Show that

$$\left(\sum_{i=0}^n r_i x^i y^{i+2} \right)_{n \in \mathbb{N}} = (r_0 y^2, r_0 y^2 + r_1 x y^3, r_0 y^2 + r_1 x y^3 + r_2 x^2 y^4, \dots)$$

is a Cauchy sequence in the I -adic topology.

(b) Write a Cauchy sequence different from the one in (a) but which has the same limit (that is, represents the same element of \widehat{R}^I , or equivalently, differs by a sequence converging to zero).

(c) Describe a natural ring map³ $K[[x, y]] \rightarrow \widehat{R}^I$ taking a power series $\sum_{i=0}^{\infty} r_{ij} x^i y^j$ to an equivalence class of Cauchy sequences. Can you show it's injective? an isomorphism?

Solution:

(a) Notice that the difference between the i th and j th terms of the sequence (for $i \leq j$) is of the form $r_i x^i y^{i+2} + \cdots + r_j x^j y^{j+2}$. By factoring out terms, we see that

$$r_i x^i y^{i+2} + \cdots + r_j x^j y^{j+2} = (x^i y^i) r_i y^2 + \cdots + r_j x^{j-i} y^{j+2-i} \in (x^i, y^i) = (x, y)^i = I^i$$

It follows that this is Cauchy.

(b) Because being Cauchy is a criterion for terms of sufficiently large index, we can change the first term of the sequence and still get a sequence that has the same limit. Thus, just take the sequence exactly as above, but make the first 5 terms 1. This is a different sequence, but one that has the same limit.

(c) We take the natural map φ which sends $\sum_{i,j=0}^{\infty} r_{ij} x^i y^j \mapsto \left(\sum_{i,j=0}^n r_{ij} x^i y^j \right)_{n \in \mathbb{N}}$. We first verify that this is injective. Choose $\sum_{i,j=1}^{\infty} r_{ij} x^i y^j \in \ker(\varphi)$. It follows that for sufficiently large N , $\sum_{i,j=1}^n r_{ij} x^i y^j = 0$ for each $n \geq N$. Well, since these terms are all linearly independent (as $r_i \in K$, and all power series monomials have differing degrees in x, y by construction) it follows that the whole sequence must be the zero sequence. Thus, φ is injective.

³think: successive truncations

To verify this is surjective, choose $(p_i(x, y))_{i \in \mathbb{N}} \in \widehat{R}^I$. Notice that $\sum_{i=1}^{\infty} p_i(x, y)$ maps to the class of $(p_i(x, y))_{i \in \mathbb{N}}$. Though the image of $\sum_{i=1}^{\infty} p_i(x, y)$ is not exactly $(p_i(x, y))_{i \in \mathbb{N}}$, they have the same limit, and thus are members of the same class in the completion (i.e. their difference is a sequence which converges to 0).

Problem 10. Explain how to think of an element of \widehat{R}^I (non-uniquely) as an infinite power series $\sum_{n=0}^{\infty} x_n$ where $x_n \in I^n$.

Solution: Under this isomorphism, a sequence $(x_n)_{n \in \mathbb{N}} \in \widehat{R}^I$ can be thought of as a power series $\sum_{n=0}^{\infty} x_n$. This is not unique, as you can reparenthesize the x_i terms in the power series to get a "new" power series.

Problem 11. Assume the Theorem. Let $R = \mathbb{R}[x, y]/\langle y^2 - x^2 - x^3 \rangle$. Let $m = \langle x, y \rangle$.

- (a) Prove that R is a domain.
- (b) Show that \widehat{R}^m is not a domain. [Hint: find a power series representing $\sqrt{1+x}$.]
- (c) Draw a picture of the curve in \mathbb{R}^2 with coordinate ring R .
- (d) There is a sense in which $\text{Spec } \widehat{R}^m$ can be considered a very small neighborhood around $m \in \text{Spec } R$. How is this reflected in your picture?

Solution:

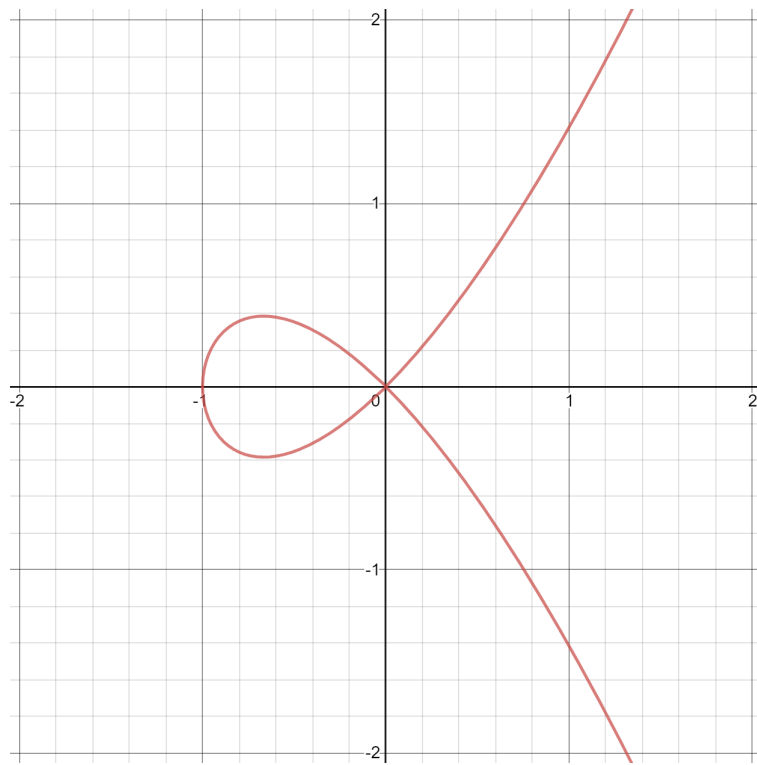
- (a) By Eisenstein's criterion on y , we know that $y^2 - x^2 - x^3$ is an irreducible polynomial. Thus, it generates a prime ideal, and the quotient by that ideal is a domain.

- (b) First, observe that

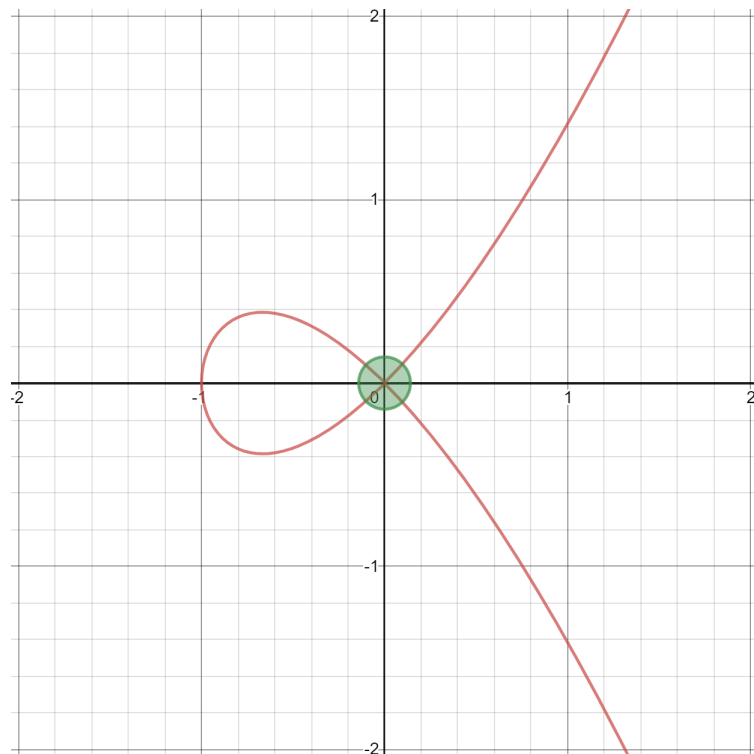
$$\sqrt{1+x} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n n!} x^n \in \widehat{R}^m$$

(This is simply the Taylor expansion, which makes sense over \mathbb{R} .) It follows that since in \widehat{R}^m , $y^2 - x^2 - x^3 = 0$, we have that $y^2 - (x\sqrt{1+x})^2 = 0$. A sum of squares can be factored as $(y + x\sqrt{1+x})(y - x\sqrt{1+x}) = 0$, implying that a product of two nonzero things is zero, implying that there are zero divisors, and that \widehat{R}^m is a domain.

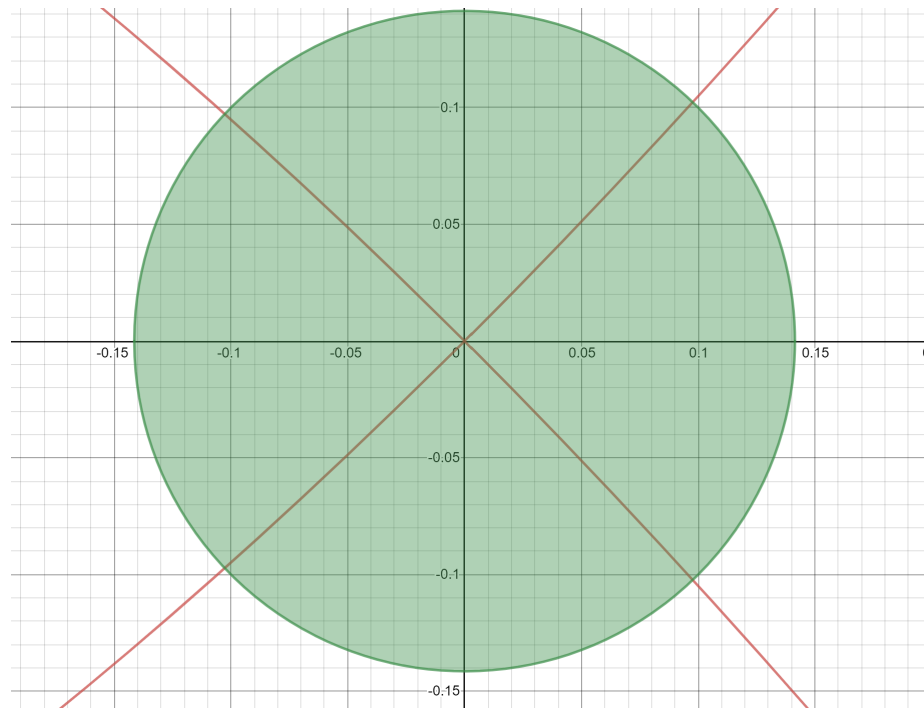
- (c) The graph of this curve looks like this in \mathbb{R}^2 :



- (d) The maximal ideal (x, y) via the Nullstellensatz corresponds to the point $(0, 0)$, meaning that our completion is centered around the center point as shown here:



Zooming in, we see the picture looks more like this:



Which geometrically shows how the whole picture is "smooth" and "connected" (implying that algebraically we have an irreducible object), but in this piece it is not "smooth" (implying our space is reducible into multiple parts).

Problem 12. Let $(r_n)_{n \in \mathbb{N}} \in \mathfrak{C}_I(R)$ be a Cauchy sequence. Fix $t \in \mathbb{N}$.

- Explain why the sequence of residues $r_n \pmod{I^t}$ is eventually constant as $n \rightarrow \infty$.
- Use (a) to show there is a surjective ring homomorphism $\mathfrak{C}_I \twoheadrightarrow R/I^t$.
- Use (b) to show there is a surjective ring homomorphism $\widehat{R}^I \twoheadrightarrow R/I^t$.
- Show that for any $t' \leq t$, the map for t' is the composition $\widehat{R}^I \twoheadrightarrow R/I^t \twoheadrightarrow R/I^{t'}$.

Solution:

- Fix $t \in \mathbb{N}$ for the rest of the problem. For $i, j \geq d$ (sufficiently large), we have that $r_i - r_j \in I^t$. This implies that $\overline{r_i - r_j} = 0$ in the quotient R/I^t , implying that $\overline{r_i} = \overline{r_j}$. Since this holds $\forall i, j$ sufficiently large, we can conclude that the sequence is eventually constant in the quotient.

- (b) The map $r_\bullet \rightarrow \bar{r}_i$ for $i \gg 0$ is well defined, by the previous part. It is also clearly surjective, as for any element in $\bar{r} \in R/I^t$, the constant sequence $(\bar{r}, \bar{r}, \dots)$ is a possible preimage.
- (c) Let's call the map above φ . Suppose we choose $r_\bullet \in \mathfrak{C}_I^0(R)$. For $i \gg 0$, $r_i \in I^t$, so $\bar{r}_i = 0$ in R/I^t . It follows that $\varphi(\mathfrak{C}_I^0(R)) = 0$, implying that by the universal property of quotients, φ factors through $\mathfrak{C}_I(R)/\mathfrak{C}_I^0(R) = \widehat{R}^I$ as follows:

$$\begin{array}{ccc} \mathfrak{C}_I(R) & \xrightarrow{\pi} & \widehat{R}^I \\ & \searrow \varphi & \downarrow \bar{\varphi} \\ & & R/I^t \end{array}$$

We now verify that this is surjective. Suppose for $\bar{r} \in R/I^t$, $\exists a_\bullet$ such that $\varphi(a_\bullet) = \bar{r}$. It follows by commutivity that $\bar{\varphi}(\pi(a_\bullet)) = \varphi(a_\bullet) = \bar{r}$, implying that $\bar{\varphi}$ is still surjective.

- (d) Choose $r_\bullet \in \widehat{R}^I$. Suppose that this maps to $\bar{r}_i \in R/I^t$ and $\tilde{r}_i \in R/I^{t'}$. We just need to verify that $\bar{r}_i \mapsto \tilde{r}_i$ under $R/I^t \twoheadrightarrow R/I^{t'}$. Since $t' \leq t$, it is clear that the natural projection map $\pi' : R \rightarrow R/I^{t'}$ factors through R/I^t , as $\pi(I^t) = 0$. By similar logic to the above, we know that since π' is surjective, the map $R/I^t \rightarrow R/I^{t'}$ is also surjective. Thus, we have maps $\widehat{R}^I \twoheadrightarrow R/I^t$, $\widehat{R}^I \twoheadrightarrow R/I^{t'}$, and $R/I^t \twoheadrightarrow R/I^{t'}$, and it is clear by construction that the limit $\tilde{r}_i \in R/I^{t'}$ must map to the limit $\bar{r}_i \in R/I^t$ under the projection map $R/I^t \twoheadrightarrow R/I^{t'}$.

Math 614: Commutative Algebra

IBL Lab Notebook

Class 26: December 10th

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Collaborators: Aniruddh Agrawal and James Yang

Problem 1. Let $\mathfrak{C}_I(M)$ denote the set of Cauchy sequences of M .

- Prove that $\mathfrak{C}_I(M)$ is a module over the ring $\mathfrak{C}_I(R)$.
- Prove that the set $\mathfrak{C}_I^0(M)$ of sequences converging to zero forms an $\mathfrak{C}_I(R)$ submodule of $\mathfrak{C}_I(M)$.
- Prove that the abelian group $\mathfrak{C}_I(M)/\mathfrak{C}_I^0(M)$ has a structure of a \widehat{R}^I module. We denote this module \widehat{M}^I , and call it the completion of M at I .

Solution: Fix $t \in \mathbb{N}$ for the rest of the problem. Whenever we say indices are "sufficiently" large, we are just supposing that $i, j \geq d$, where d is defined as in the definition of being Cauchy.

- From similar logic to the last worksheet, we can see that $\mathfrak{C}_I(M)$ is an abelian group. We just need to show that it is compatible with an $\mathfrak{C}_I(R)$ action. Choose $m_\bullet \in \mathfrak{C}_I(M)$, and $r_\bullet \in \mathfrak{C}_I(R)$. For i, j sufficiently large, we know that $r_i - r_j \in I^t$ and $m_i - m_j \in I^t M$. We want to verify that $r_i m_i - r_j m_j \in I^t M$, well,

$$r_i m_i - r_j m_j = r_i(m_i - m_j) - r_j m_j + r_i m_j = r_i(m_i - m_j) + (r_i - r_j)m_j \in I^t M$$

Distributively follows from the fact that M is an R -module.

- It is clear that $\mathfrak{C}_I^0(M) \subset \mathfrak{C}_I(M)$, so we just need to verify that $\mathfrak{C}_I(R) \cdot \mathfrak{C}_I^0(M) = \mathfrak{C}_I^0(M)$. Choose $m_\bullet \in \mathfrak{C}_I^0(M)$, and $r_\bullet \in \mathfrak{C}_I(R)$. For i, j sufficiently large, we know that $r_i - r_j \in I^t$, and $m_i, m_j \in I^t M$. Well, notice that $r_i m_i, r_j m_j \in I^t M$. It follows that $r_\bullet m_\bullet \in \mathfrak{C}_I^0(M)$.
- Since $\mathfrak{C}_I(M), \mathfrak{C}_I^0(M)$ are $\mathfrak{C}_I(R)$, modules, it follows that their quotient is as well. This tells us that $\mathfrak{C}_I(M)/\mathfrak{C}_I^0(M)$ has the structure of an abelian group. We just need to show that there is a compatible $\mathfrak{C}_I(R)/\mathfrak{C}_I^0(R)$ action. It is sufficient to verify that acting by an element of $\mathfrak{C}_I^0(R)$ is trivial in the quotient. That is, we want to show that $\mathfrak{C}_I^0(R) \cdot \widehat{M}^I \subset \mathfrak{C}_I^0(M)$.

Well, choose $m_\bullet \in \widehat{M}^I$ and $r_\bullet \in \mathfrak{C}_I^0(R)$. For sufficiently large i , notice that $r_i \in I^t$. It follows that $r_i m_i \in I^t M$ for any $m \in M$. Thus, $r_i m_i \in I^t M$, implying that $r_\bullet m_\bullet \in \mathfrak{C}_I^0(M)$. Thus, there is a natural \widehat{R}^I action on \widehat{M}^I , making it a \widehat{R}^I module.

Problem 2. Consider the inverse limit system maps of sets

$$X_1 \xleftarrow{\nu_{12}} X_2 \xleftarrow{\nu_{23}} X_3 \xleftarrow{\nu_{34}} \dots$$

and let ν_{ij} be the resulting composition from X_j to X_i . Define the inverse limit $\varprojlim X_n$ to be the subset of the direct product $\prod_{i=1}^{\infty} X_i$ consisting of all sequences x_{\bullet} such that $\nu_{ij}(x_j) = x_i$ for all i, j .

- Prove that if the X_i are abelian groups and the ν are group homomorphisms, then $\varprojlim X_n$ has a naturally induced structure as an abelian group.
- Prove that there are natural group homomorphisms $\pi_n : \varprojlim X_n \rightarrow X_n$ for all n such that $\pi_m = \nu_{mn}\pi_n$ for all $n \geq m$.
- Verify that (a) and (b) also hold in the category of rings, as well as the category of R -modules.
- Assume that (in the setting of (a)) G is an abelian group such that
 - There exists a group homomorphism $\varphi_n : G \rightarrow X_n$ for all n .
 - $\varphi_m = \nu_{mn}\varphi_n$ for all $n \geq m$.

Prove that there exists a unique group homomorphism $G \rightarrow \varprojlim X_n$ making all the relative diagrams commute.

- Think through (d) in the category of sets, rings, and R -modules.

Solution:

- To make part (c) easier, we verify this holds in the category of associative R -algebras (with identity). Algebras are modules, and rings are just associative \mathbb{Z} -algebras, and abelian groups are just \mathbb{Z} -modules. It follows that treating this case generalizes to all the cases in part (c).

Suppose each X_i is an associative R -algebra with identity. We verify that $\varprojlim X_n$ has an Algebra structure. First, observe that any algebra homomorphism sends $0 \rightarrow 0$, $1 \rightarrow 1$, so the sequences $0_{\bullet}, 1_{\bullet} \in \varprojlim X_n$, as $\nu_{ij}(0) = 0, \nu_{ij}(1) = 1$ for any i, j . Now, Choose $x_{\bullet}, y_{\bullet} \in \varprojlim X_n$. We first check that $x_{\bullet} + y_{\bullet} \in \varprojlim$.

Notice that $\nu_{ij}(x_j + y_j) = \nu_{ij}(x_j) + \nu_{ij}(y_j) = x_i + y_i$ for any i, j , implying additive closure. Furthermore, notice that $\nu_{ij}(x_j y_j) = \nu_{ij}(x_j)\nu_{ij}(y_j) = x_i y_i$, so by similar logic $x_{\bullet} y_{\bullet} \in \varprojlim X_n$, implying multiplicative closure. Finally, we verify that there is a compatible R action.

Choose $r \in R$, $x_{\bullet} \in \varprojlim X_n$. We want to show that $rx_{\bullet} \in \varprojlim X_n$. Well, Notice that for any i, j , $\nu_{ij}(rx_j) = r\nu_{ij}(x_j) = rx_i$, so we can conclude.

- the natural map is defined as $\pi_n(x_{\bullet}) = x_n$. By the definitions of addition and multiplication and R -actions above, it is clear that π_n is an R -algebra homomorphism.

$$\pi_n(x_{\bullet} + y_{\bullet}) = x_n + y_n = \pi_n(x_{\bullet}) + \pi_n(y_{\bullet})$$

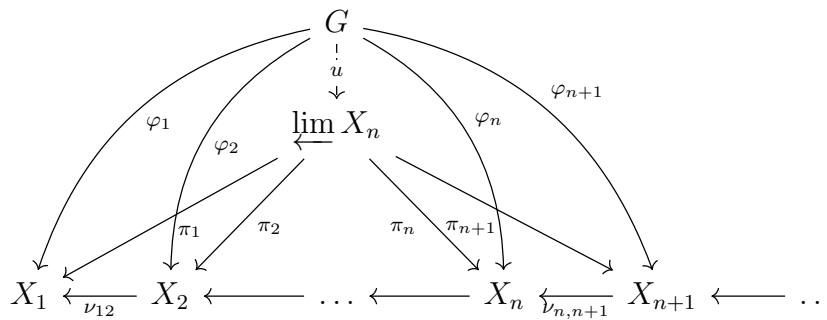
$$\begin{aligned}\pi_n(x_\bullet y_\bullet) &= x_n y_n = \pi_n(x_\bullet) \pi_n(y_\bullet) \\ \pi_n(rx_\bullet) &= rx_n = r \pi_n(x_\bullet)\end{aligned}$$

We verify the identities below.

$$\begin{aligned}\pi_m(x_\bullet + y_\bullet) &= x_m + y_m = \nu_{mn}(x_n) + \nu_{mn}(y_n) = \nu_{mn}(x_n + y_n) = \nu_{mn}(\pi_n(x_\bullet + y_\bullet)) \\ \pi_m(x_\bullet y_\bullet) &= x_m y_m = \nu_{mn}(x_n) \nu_{mn}(y_n) = \nu_{mn}(x_n y_n) = \nu_{mn}(\pi_n(x_\bullet y_\bullet)) \\ \pi_m(rx_\bullet) &= rx_m = \nu_{mn}(rx_n) = \nu_{mn}(\pi_n(rx_\bullet))\end{aligned}$$

- (c) Follows immediately from the work we did above.
- (d) As before, we prove this in the context of associative R -algebras with unity, so (e) falls out immediately, from the same logic as the previous time we did this. Notice that in the category of sets, the morphisms are just set maps. Thus, any proof we do here will hold in the category of sets, provided we forget the additional structure of all the maps involved.

Putting this in a diagram gives us the following:



Given we have commutativity with the ν_{ij} maps, it is sufficient to verify that $\exists u : G \rightarrow \varprojlim X_n$ such that $\pi_n \circ u = \varphi_n$ for each n . Well, consider the map

$$u(g) \mapsto (\varphi_n(g))_{n \in \mathbb{N}}$$

It is clear that this would be an R -algebra homomorphism. Furthermore,

$$\pi_n \circ u(g) = \pi_n((\varphi_n(g))_{n \in \mathbb{N}}) = \varphi_n(g)$$

Thus, the diagram commutes, and we have existence. We now verify uniqueness. Suppose that \exists another map that fits in the diagram, denoted ψ . Suppose for $g \in G$, $\psi(g) = r_\bullet \in \varprojlim X_n$. Well, notice that $\pi_n \circ \psi(g) = \varphi_n(g)$, so since $\pi_n \circ \psi(g) = r_n$, we have that $\varphi_n(g) = r_n$, so $r_\bullet = (\varphi_n(g))_{n \in \mathbb{N}}$, implying that $\psi(g) = u(g)$. Since this holds for all g , we have uniqueness.

We now go further, and show that if any object G satisfies the universal property of inverse limits, then $G \cong \varprojlim X_n$. G satisfies this, then we have the following commutative diagram:

Each dotted line represents the unique map that must exist, as both G and $\varprojlim X_n$ satisfy the universal property. Notice that $\text{Id} : \varprojlim X_n \rightarrow \varprojlim X_n$ satisfies the universal property, so by uniqueness that must be the map in the diagram. This implies that $u \circ u' = \text{Id}$. A similar argument (where G is replaced with $\varprojlim X_n$ and vice versa, and with their corresponding maps switched) gives us that $u' \circ u = \text{Id}$ via another uniqueness argument. It follows that $G \cong \varprojlim X_n$.

(e) From how we did the previous part, every case follows.

Problem 3. Let $x_\bullet \in \mathfrak{C}_I(M)$ be a Cauchy sequence. Fix $t \in \mathbb{N}$.

- (a) Explain why the sequence of residues x_n modulo $I^t M$ is eventually constant as $n \rightarrow \infty$.
- (b) Use (a) to show that there is a surjective group homomorphism $\mathfrak{C}_I(M) \rightarrow M/I^t M$.
- (c) Use (b) to show that there is a surjective group homomorphism $\widehat{M}^I \rightarrow M/I^t M$.
- (d) Show that for any $t' \leq t$, the map for t' is the composition $\widehat{M}^I \rightarrow M/I^t M \rightarrow M/I^{t'} M$.
- (e) Find a ring homomorphism $\widehat{R}^I \rightarrow \varprojlim R/I^n$.
- (f) Find a group homomorphism $\widehat{M}^I \rightarrow \varprojlim M/I^n M$, and show that it is \widehat{R}^I -linear.
- (g) Prove the maps in (e) and (f) are isomorphisms. [Hint: Given an element in \widehat{R}^I , arbitrarily lift elements of R/I^n to a sequence in R . Show that it is Cauchy and well defined up to \mathfrak{C}_I^0 .]

Solution:

- (a) For sufficient large i, j , $x_i - x_j \in I^t M$. It follows that $\overline{x_i - x_j} = 0$ in the quotient $M/I^t M$, implying that $\overline{x_i} = \overline{x_j}$. Thus, for sufficiently large indices, the sequences of residues is eventually constant modulo $I^t M$.

- (b) The map $x_\bullet \mapsto \bar{x}_i$ for this constant point at $i \gg 0$ is clearly a surjective group homomorphism, from similar logic to problem 12 in the previous worksheet.
- (c) Through the universal property of quotients, the map in the pervious part factors through \widehat{M}^I , through similar logic to problem 12 in the previous worksheet.
- (d) This holds through similar logic to problem 12 in the previous worksheet.
- (e) We have maps $\varphi_n : \widehat{R}^I \rightarrow R/I^n$ for each n from the previous worksheet. Problem 12(d) from the previous worksheet shows us that these maps are compatible with the maps $\nu_{mn} : R/I^n \rightarrow R/I^m$. Thus, it follows that $\varphi_m = \nu_{mn}\varphi_n$ for all $n \geq m$, implying there is a unique homomorphism $\widehat{R}^I \rightarrow \varprojlim R/I^n$ by 2(d) and 2(e).
- (f) We have maps $\varphi_n : \widehat{M}^I \rightarrow M/I^n M$ for each n from the previous part. Part (d) shows us that these maps are compatible with the maps $\nu_{mn} : M/I^n M \rightarrow M/I^m M$. Thus, it follows that $\varphi_m = \nu_{mn}\varphi_n$ for all $n \geq m$, implying there is a unique \widehat{R}^I -module homomorphism $\widehat{M}^I \rightarrow \varprojlim M/I^n M$ by 2(d) and 2(e).
- (g) To verify that these maps are isomorphisms, it is sufficient to verify that both \widehat{M}^I and \widehat{R}^I satisfy the universal property of inverse limits, as we've shown that any objects that satisfy these properties are isomorphic to the inverse limit of the system.

We treat the \widehat{M}^I case, as the \widehat{R}^I follows immediately, since \widehat{R}^I is an \widehat{R}^I -algebra, and thus a \widehat{R}^I -module, over itself compatible with the more general case of \widehat{M}^I .

Suppose we have a module N such with maps $\psi_n : N \rightarrow R/I^n$ that are compatible with the ν maps. We want to show that there exists a map $u : N \rightarrow \widehat{M}^I$ that makes the appropriate maps commute. It is sufficient to check that $\varphi_n \circ u = \psi_n$, well, the map

$$u(x) = (\psi_i(x))_{i \in \mathbb{N}}$$

Still works. To show this, we just need to verify that $\varphi_n((\psi_i(x))_{i \in \mathbb{N}})$ and $\psi_n(x)$ belong to the same class in $M/I^n M$. Well, this follows because the sequences which converge to zero are precisely those contained in $\bigcap I^j M$. We've chosen φ_n such that we map any sequence to its point of convergence modulo n . Thus, it follows that, up to some sequence tending to 0, φ_n picks the n th term of a sequence. Thus, $\varphi_n((\psi_i(x))_{i \in \mathbb{N}})$ and $\psi_n(x)$ belong to the same class in $M/I^n M$.

Problem 4. Let $S = R[x_1, \dots, x_n]$ be a polynomial ring over R , and let $I = (x_1, \dots, x_n) \subset S$.

- (a) Explain why an element of S/I^n is uniquely represented by a polynomial of degree less than n in the x_i .
- (b) Use (a) to describe the sequences in $\prod_{n \in \mathbb{N}} S/I^n$ that represent an element of the inverse limit $\varprojlim S/I^n$?
- (c) Prove that $\varprojlim S/I^n \cong R[[x_1, \dots, x_n]]$.

Solution:

(a) Choosing a polynomial $f \in S$ with arbitrary degree terms, we see that quotienting by $(x_1, \dots, x_n)^n$ kills all monomials with degree $\geq n$. In other words, if we have a monomial $x_1^{m_1} \dots, x_n^{m_n}$, where $\sum m_i \geq n$, then $x_1^{m_1} \dots, x_n^{m_n} \in I^n$. This implies that for \bar{f} to be a nonzero object in S/I^n , all monomials must be of degree $< n$, implying f , its representative, itself must be a polynomial of degree $< n$, since all of its monomial components have degree $< n$.

(b) We first characterize the map $S/I^n \rightarrow S/I^{n-1}$. If $f \in S/I^n$ is a degree $n - 1$ polynomial, the map simply kills the $n - 1$ th degree term of the polynomial. Otherwise, the map is the identity.

Using this, we now can characterize elements of $\varprojlim S/I^n$. The first term of the sequence must be a monomial of degree < 1 , so of degree 0 and thus a constant. The next term must be a degree < 2 polynomial such that the degree 0 term matches the first term of the sequence. This is because the map $S/I^2 \rightarrow S/I$ must be compatible with the terms in the sequence. Continuing onwards, we see that the n th term must be a polynomial f of degree $n - 1$, where the i th term for $i < n$ must be the sum of all terms in f of degree $i - 1$ or less.

(c) We take the map $\varprojlim S/I^n \rightarrow R[[x_1, \dots, x_n]]$ where a sequence $f_\bullet \mapsto \sum_{m \in \mathbb{N}} f_{m+1} - f_m$. Notice that $f_{m+1} - f_m$ is precisely the a term of degree m or is 0. This is clearly a ring homomorphism, so we now just check that there exists an inverse. Well, consider the map

$$\sum_{\sum m_i = \ell, \ell \in \mathbb{N}} x_1^{m_1} \dots x_n^{m_n} \mapsto \left(\sum_{\sum m_i = \ell, \ell < j} x_1^{m_1} \dots x_n^{m_n} \right)_{j \in \mathbb{N}}$$

This is well defined, as the n th term is a polynomial of degree at most $n - 1$, where the i th term for $i < n$ is the sum of all the terms of the $n - 1$ degree polynomial of degree $i - 1$ or less. It is also clear that this is a ring homomorphism, so we just need to verify inverses. Notice that

$$\left(\sum_{\sum m_i = \ell, \ell < j} x_1^{m_1} \dots x_n^{m_n} \right)_{j \in \mathbb{N}} \mapsto \sum_{j \in \mathbb{N}} \sum_{\sum m_i = \ell, \ell < j+1} x_1^{m_1} \dots x_n^{m_n} - \sum_{\sum m_i = \ell, \ell < j} x_1^{m_1} \dots x_n^{m_n}$$

From our observation above, we see that this difference just cuts out the degree j term, implying that

$$\sum_{j \in \mathbb{N}} \left(\sum_{\sum m_i = \ell, \ell < j+1} x_1^{m_1} \dots x_n^{m_n} - \sum_{\sum m_i = \ell, \ell < j} x_1^{m_1} \dots x_n^{m_n} \right) = \sum_{\sum m_i = \ell, \ell \in \mathbb{N}} x_1^{m_1} \dots x_n^{m_n}$$

This verifies that these two are inverses, and completes the proof.

Problem 5. Prove that if R is a Noetherian ring, then \widehat{R}^I is Noetherian. [Hint: Map the

polynomial ring $S = R[x_1, \dots, x_n]$ surjectively to R such that the x_i map to generators of I . Construct a surjection for the inverse limits $R[[x_1, \dots, x_n]] \rightarrow \widehat{R}^I$.]

Solution: We follow the hint. Let $I = (f_1, \dots, f_n)$, and define the surjection $R[x_1, \dots, x_n] \rightarrow R$ by $x_i \mapsto f_i$. We show in 7(c) that completion takes surjections to surjections, so we cite that to verify that $R[[x_1, \dots, x_n]] \rightarrow \widehat{R}^I$ is surjective. Since we know $R[[x_1, \dots, x_n]]$ is Noetherian, so is \widehat{R}^I .

Problem 6. let $R = \frac{K[x_1, \dots, x_n]}{(g_1, \dots, g_t)}$. Prove that completing at (x_1, \dots, x_n) produces $\frac{K[[x_1, \dots, x_n]]}{(g_1, \dots, g_t)}$.

Solution: This follows via taking the same isomorphism in 4(c), as we know that $\widehat{R}^I \cong \varprojlim R/I^n$. More concretely we can define the map $R \rightarrow K[[x_1, \dots, x_n]]/(g_1, \dots, g_t)$ via the isomorphism

$$(\overline{f_i}) \mapsto \sum_{j \in \mathbb{N}} \overline{f_{j+1}} - \overline{f_j}$$

This is still well defined, as is the inverse

$$\sum_{\sum m_i = \ell, \ell \in \mathbb{N}} \overline{x_1^{m_1} \dots x_n^{m_n}} \mapsto \left(\sum_{\sum m_i = \ell, \ell < j} \overline{x_1^{m_1} \dots x_n^{m_n}} \right)_{j \in \mathbb{N}}$$

Since the quotient we are taking is identical on either side of the maps.

Problem 7. The completion functor.

- (a) Prove that if $M \rightarrow N$ is an R -module homomorphism, then there is a natural induced \widehat{R}^I -module homomorphism $\widehat{M}^I \rightarrow \widehat{N}^I$.
- (b) Prove that completion defines a functor from R -modules to \widehat{R}^I modules.
- (c) Prove that the completion functor is right exact.

Solution:

- (a) Let $\varphi : M \rightarrow N$ be some R -module homomorphism. The map $m_\bullet \mapsto (\varphi(m))_\bullet$ is still a \widehat{R}^I homomorphism which sends \widehat{M}^I to \widehat{N}^I . To see why, notice that this is definitely a map $\mathfrak{C}_I(M) \rightarrow \mathfrak{C}_I(N)$, and since $\varphi(I^n M) \subset I^n N$, it follows that it sends 0-convergent series to 0-convergent series, and extends to a map \widehat{M}^I to \widehat{N}^I .

(b) We know this sends objects to objects from Problem 1. For morphisms $\varphi : L \rightarrow M, \psi : M \rightarrow N$, notice that Their composition under the completion functor is of the form $\ell_{\bullet} \mapsto (\varphi(\ell))_{\bullet} \mapsto (\psi \circ \varphi(\ell))_{\bullet}$, which is the same as composing then completing, so it follows that completion is a (covariant) functor.

(c) First, we verify that completing in general (even with respect to two different ideals) is surjective. In other words, for a surjective R -module homomorphism $M \rightarrow N$, I claim that $\widehat{M}^I \rightarrow \widehat{N}^I$ is surjective. Notice that each element of \widehat{N}^I can be represented as partial sums of a series s_{\bullet} , for $s_i \in I^n N$. Since our surjective morphism maps $I^n M \rightarrow I^n N$, we can find $r_{\bullet} \in \widehat{M}^I$ where $r_n \in I^n M$ and $r_n \mapsto s_n$ for all n , yielding surjectivity.

Thus, the completion functor takes surjections to surjections.

So here I actually cannot prove that completion is right exact in general, so I googled around and found out it isn't true in full generality. A counter example is found on the Stack Project, Tag: **05JF**.

Problem 8. Let $N \subset M$ be Noetherian modules over the Noetherian ring R .

- (a) Assuming the Artin-Rees Lemma, show that there is a constant c such that for all $n \geq c$, $I^n M \cap N \subset I^{n-c} N$.
- (b) Show that if a sequence of elements in N is an I -adic Cauchy sequence in M , (respectively, converges to 0 in M), then it is an I -adic Cauchy sequence in N (respectively, converges to 0 in N).
- (c) Prove that $\widehat{N}^I \rightarrow \widehat{M}^I$ is injective.
- (d) Conclude that the completion functor is exact on finitely generated R -modules.

Solution:

- (a) $I^n M \cap N = I^{n-c}(I^c M \cap N) \subset I^{n-c}(M \cap N) = I^{n-c} N$.
- (b) Fix $t \in \mathbb{N}$ and choose $i, j \gg 0$ sufficiently large. If $m_{\bullet} \in \mathfrak{C}_I(M)$, then $m_i - m_j \in I^t M$. If we presuppose that each $m_\ell \in N$, then it follows that $m_i - m_j \in I^t M \cap N \subset I^{t-c} N$ for some c . Thus, $m_{\bullet} \in \mathfrak{C}_I(N)$. Similarly, if $m_{\bullet} \in \mathfrak{C}_I^0(M)$, but $m_\ell \in N \forall \ell$, it follows that $m_i \in I^t M \cap N \subset I^{t-c} N$ for some c , so $m_{\bullet} \in \mathfrak{C}_I^0(N)$.
- (c) We've shown that if a Cauchy sequence in N converges to 0 in $\mathfrak{C}_I^0(M)$, then it is in $\mathfrak{C}_I^0(N)$. In other words, if $n_{\bullet} \in \ker(\widehat{N}^I \rightarrow \widehat{M}^I)$, then $n_{\bullet} = 0 \in \widehat{N}^I$, as we quotient by 0-convergent subsequences. It follows that $\widehat{N}^I \rightarrow \widehat{M}^I$ has trivial kernel, and is injective.
- (d) Since we've shown above that the completion functor (in the case of finitely generated modules) takes surjections to surjections and injections to injections, we

just need to verify that it preserves exactness in the middle. To do this, consider the exact sequence $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$. It is sufficient to verify that $\text{im}(\widehat{N}^I \rightarrow \widehat{M}^I) \supset \ker(\widehat{M}^I \rightarrow \widehat{L}^I)$, as the other case is immediate. Define these maps ϕ, ψ respectively.

Choose $m_\bullet \in \ker(\psi)$. We can assume that $m_{t+1} - m_t \in I^t M$ by previous results, so the image of $m_t \in L \cong M/N$ converge to 0, implying $m_t \in I^t(M/N)$, so $m_t \in I^t M + N$. Supposing that $m_t = x_t + y_t$ for each t , it follows that the sequence y_t is contained in N , and converges in M , so $y_{t+1} - y_t \in I^t M \cap N = I^{t-c} N$ by Artin-Rees, so it converges in N . It follows that $m_\bullet \in \text{im}(\varphi)$.

Problem 9. (a) Show that there is a natural map of R -modules $M \rightarrow \widehat{M}^I$ whose kernel is $\bigcap_{n \in \mathbb{N}} I^n M$.

(b) Show that there is a natural map of \widehat{R}^I -modules $\widehat{R}^I \otimes_R M \rightarrow \widehat{M}^I$.

(c) Show that if R and M are Noetherian, then $\widehat{R}^I \otimes_R M \cong \widehat{M}^I$ as \widehat{R}^I -modules.

Solution:

(a) This was already proven in the case of rings in problem 6 in the last worksheet, and the case of modules is identical.

(b) First, consider the map $\widehat{R}^I \times M \rightarrow \widehat{M}^I$ where $((r_i)_{i \in \mathbb{N}}, m) \mapsto (r_i m)_{i \in \mathbb{N}}$. This is clearly well defined and a \widehat{R}^I -module homomorphism. It is also bilinear.

From the universal property of tensor products, this extends uniquely to a map $\widehat{R}^I \otimes_R M \rightarrow \widehat{M}^I$.

(c) First, we prove this in the case where M is a finitely generated free module. Notice that completion clearly commutes with direct sums. Thus,

$$\widehat{R}^I \otimes R^{\oplus m} \cong \bigoplus_{i=1}^n \widehat{R}^I \otimes R \cong \bigoplus_{i=1}^n \widehat{R}^I \cong \widehat{R^{\oplus m}}^I$$

We now tackle the general case. Notice that Noetherian modules are always finitely generated, and over Noetherian rings are finitely presented. Thus, we can take a presentation $R^m \rightarrow R^n \rightarrow M \rightarrow 0$. Tensoring is right exact, and completion is exact, yielding the following exact sequences with maps between them:

$$\begin{array}{ccccccc} \widehat{R}^I \otimes R^{\oplus m} & \longrightarrow & \widehat{R}^I \otimes R^{\oplus n} & \longrightarrow & \widehat{R}^I \otimes M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \widehat{R^{\oplus m}}^I & \longrightarrow & \widehat{R^{\oplus n}}^I & \longrightarrow & \widehat{M}^I & \longrightarrow & 0 \end{array}$$

Since the first two are isomorphisms, it follows that the third is as well.

Problem 10. (a) Prove that an R -module F is flat if and only if whenever $N \subset M$ are finitely generated R -modules, the map $F \otimes_R N \rightarrow F \otimes_R M$ is injective.

(b) Deduce that \widehat{R}^I is a flat R -algebra when R is Noetherian.

Solution:

(a) Tensoring is always right exact, so it is sufficient to verify that the functor $F \otimes -$ sends injective maps to injective maps. If that happens, then $F \otimes -$ is an exact functor, implying that F is a flat module. The converse is similar.

(b) Over Noetherian rings, we know that the functor $\widehat{(-)}^I$ is exact. In other words, we know that for injections $M \rightarrow N$, $\widehat{M}^I \rightarrow \widehat{N}^I$ is also injective. From Problem 9(c), we know that $\widehat{R}^I \otimes_R M \rightarrow \widehat{M}^I$ and $\widehat{R}^I \otimes_R N \rightarrow \widehat{N}^I$, implying that the map $\widehat{R}^I \otimes M \rightarrow \widehat{R}^I \otimes N$ is injective. It follows from the previous part that \widehat{R}^I is flat.