On The Classification of Bivariate Symmetric Ideals

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Introduction

Let S_∞ denote the infinite symmetric group. Let $k[X] := k[x_1, x_2, \dots,]$, for k some algebraically closed, characteristic 0 field. Similarly, denote $k[X, Y] := k[x_1, x_2, \dots, y_1, y_2, \dots,]$. S_∞ has a natural action on both $k[X]$ and $k[X, Y]$, via permuting the formal variables. In the $k[X, Y]$ case, S_∞ permutes the variables x_i and y_i simultaneously. A set S is S_∞ stable if $S_\infty \circlearrowright S = S$.

Goal

Classify certain S_∞-stable ideals of $k[X, Y]$. In particular, we would like to classify the S_∞-stable analogue of prime ideals, or c-prime radical ideals.

c-Prime Ideals

First, we define the S_∞-stable analogue of a prime ideal more explicitly. Let $I \subset k[X, Y]$ be an S_∞-stable ideal.

Definition. *For subsets $A, B \subset k[X, Y]$, we say I is a **Prime Ideal** if $AB \subset I$ implies that either $A \subset I$ or $B \subset I$.*

Equivalently,

I is prime if and only if for any $a, b \in k[X, Y]$, $ab \in I$ implies that $a \in I$ or $b \in I$.

Definition. *For S_∞-stable subsets $A, B \subset k[X, Y]$, we say I is a **c-prime Ideal** if $AB \subset I$ implies that either $A \subset I$ or $B \subset I$.*

Equivalently,

I is c-prime if and only if for any given $a, b \in k[X, Y]$ where $a(\sigma \cdot b) \in I$ for all $\sigma \in S_\infty$, $a \in I$ or $b \in I$.

Restricting specifically to S_∞-stable ideals, we find that being c-prime is a weaker condition than being prime. In particular, being c-prime does not imply that the ideal is radical, while it is known that all prime ideals are radical.

Example 1. *The ideal $I = (x_1^2, x_2^2, \dots)$ is c-prime, but not radical. I is clearly not radical, since $x_1 \in \sqrt{I}$, but $x_1 \notin I$. We sketch our proof for I being c-prime.*
Let $f = x_1^{m_1} + x_2^{m_2} + x_3^{m_3}$, $g = x_2^{m_4} + x_3^{m_5} + x_4^{m_6}$.

$(x_1^{m_1} + x_2^{m_2} + x_3^{m_3})(\sigma \cdot (x_2^{m_4} + x_3^{m_5} + x_4^{m_6})) \in I$ $\xrightarrow{\text{choice of } \sigma}$ $(x_1^{m_1} + x_2^{m_2} + x_3^{m_3})(x_4^{m_4} + x_5^{m_5} + x_6^{m_6}) \in I$ $\xrightarrow{I \text{ is generated by squares}}$ $2 | m_i$ $\xrightarrow{\text{each } x_i \text{ is a power of a square}}$ $f, g \in I$

c-Irreducible Sets

Recall that there exists a bijection between prime ideals and their corresponding irreducible sets. We would like a similar bijection; to do this, we define an S_∞-stable analogue of an irreducible set. Let $X \subset (\mathbb{A}^2)^\infty$ be a closed set under the Zariski Topology.

Definition. *X is **irreducible** if it cannot be written as the union of two proper closed subsets.*

Any algebraically closed set can be written as a finite union of irreducible sets.

Definition. *X is **c-irreducible** if it cannot be written as the union of two proper S_∞-stable closed subsets.*

Any S_∞-stable algebraically closed set can be written as a finite union of c-irreducible sets.

Again, c-irreducibility is a weaker condition than irreducibility. We have a criterion for relating the two below:

Lemma 1. *Suppose $X \subset (\mathbb{A}^2)^\infty$ is S_∞-stable and closed. Let $X = \bigcup X_i$ denote its decomposition into irreducible sets. If the S_∞ orbit of an irreducible component is dense in X, then X is c-irreducible.*

Relating c-Prime Ideals and c-Irreducible Sets

Theorem 1

From classical algebraic geometry, we have the following bijection:

$$\{\text{Prime Ideals of } k[x_1, \dots, x_n]\} \longleftrightarrow \{\text{Irreducible Sets in } \mathbb{A}^n\}$$

When only considering S_∞-stable ideals within the infinite variable polynomial ring (with any number of collections of infinitely many variables), the following bijection holds:

$$\{c\text{-Prime Radical Ideals of } k[X_1, \dots, X_n]\} \longleftrightarrow \{c\text{-Irreducible Sets in } (\mathbb{A}^n)^\infty\}$$

Where each X_i corresponds to an infinite collection of variables $x_{1,i}, x_{2,i}, \dots$ for each i, and S_∞ acts on each X_i simultaneously. (We are studying the case where $n = 2$, though our cofinal set can be extended to the general case.)

Just as the first bijection establishes a geometric picture of prime ideals, the second bijection establishes a similar picture for c-prime radical ideals. This reduces our problem of studying c-prime radical ideals to one about studying c-irreducible sets. We give examples of c-irreducible sets below.

Example 2. *Define*

$$X := \{(a_1, a_2, \dots, b_1, b_2, \dots) \in \left(\mathbb{A}^2\right)^\infty \mid [a_1, a_2, \dots], [b_1, b_2, \dots] \text{ are linearly dependent}\}$$

Recall that the two vectors are linearly dependent if only if $\det \begin{bmatrix} a_i & b_i \\ a_j & b_j \end{bmatrix} = 0 \ \forall i, j \in \mathbb{N}$. This is equivalent to saying that each sequence satisfies the polynomial $x_i y_j - x_j y_i$ for all i, j. Thus, the S_∞ orbit of $x_1 y_2 - x_2 y_1$ has zero set X, so X is closed. Furthermore, $x_1 y_2 - x_2 y_1$ is prime in $k[X, Y]$, implying that its corresponding zero set is irreducible. Irreducibility implies c-irreducibility for S_∞-stable sets, so X is c-irreducible.

Example 3. *Define*

$$Z_{n,f} = \{(a_\bullet, b_\bullet) \in \left(\mathbb{A}^2\right)^\infty \mid f(a_\bullet, b_\bullet) \text{ takes} \leq n \text{ values}\}$$

For some fixed $f \in k[x, y]$ non-constant. We sketch a proof of why $Z_{n,f}$ is closed below.

$(a_\bullet, b_\bullet) \in Z_{n,f}$ $\xrightarrow{f(a_1,b_1),\dots,f(a_n,b_n) \text{ distinct}}$ $\exists i \leq n$ such that $f(a_{n+1}, b_{n+1}) = f(a_i, b_i)$ $\xrightarrow{V(\Delta_n)=\{a_\bullet | \leq n \text{ distinct values}\}}$ $V(\Delta[f(x_i, y_i) \mid i \leq n+1]) = Z_{n,f}$ $\longrightarrow$ $Z_{n,f}$ is closed

This shows that $Z_{n,f}$ is closed, so we now only need to sketch a proof of irreducibility. Choose $A_1, A_2, \dots, A_n \subset \mathbb{N}$ such that $\bigsqcup_{i=1}^n A_i = [\infty]$. Using these, define

$$X_A := \{(a_1, a_2, \dots, b_1, b_2, \dots,) \mid f(a_i, b_i) \text{ is constant on each } i \in A_1, \dots, i \in A_n\}$$

X_A is the set of sequences where the n distinguished values are taken in specific indices in the sequence. $X_A \subset Z_{n,f}$, and X_A is closed and irreducible.

X_A closed, irreducible $\xrightarrow{\text{Choose all } A=A_1\sqcup\cdots\sqcup A_n=[\infty]}$ $Z_{n,f} = \bigcup_A X_A$ $\xrightarrow{\text{Choose } A \text{ such that each } |A_i| = \infty}$ $(X_A)_{S_\infty}$ is dense $\xrightarrow{\text{lemma 1}}$ $Z_{n,f}$ is c-irreducible

Constructing The Cofinal Set

Theorem 2

The set

$$Z_{n,d} = \{(a_\bullet, b_\bullet) \in \left(\mathbb{A}^2\right)^\infty \mid \exists f \in k[x, y] \text{ of degree} \leq d \text{ such that } f(a_\bullet, b_\bullet) \text{ takes} \leq n \text{ distinct values}\}$$

is closed. Furthermore, for any S_∞-stable closed set $X \subset (\mathbb{A}^2)^\infty$, $\exists n, d \in \mathbb{N}$ such that $X \subset Z_{n,d}$ (i.e. $Z_{n,d}$ is cofinal over S_∞-stable closed sets).

Closure

To prove closure, we define the following (purported) projective variety:

$$\overline{Z}_{n,d} = \left\{(a_\bullet, b_\bullet, \alpha_0, \dots, \alpha_d) \,\middle|\, f_\alpha(a_\bullet, b_\bullet) \text{ takes} \leq n \text{ distinct values}\right\} \subset \left(\mathbb{A}^2\right)^\infty \times \mathbb{P}^d$$

Where f_α is a degree $\leq d$ polynomial with coefficients $\alpha_0, \dots, \alpha_d$. Observe that

$$\pi : \left(\mathbb{A}^2\right)^\infty \times \mathbb{P}^d \longrightarrow \left(\mathbb{A}^2\right)^\infty$$
$$\overline{Z}_{n,d} \longmapsto Z_{n,d}$$

Since this projection map is proper, we can conclude $Z_{n,d}$ is closed via the closure of $\overline{Z}_{n,d}$, which follows similarly to the closure proof of example 3.

Cofinality

Let X be an affine variety over $(\mathbb{A}^2)^\infty$. It suffices to treat the case where X is the zero set of the S_∞ orbit of a fixed non-zero polynomial f. Since polynomials have finitely many terms, we can assume that this polynomial has maximal terms x_m, y_n (with respect to index), and thus $f \in k[x_1, \dots, x_m, y_1, \dots, y_n]$. Choose $(a_1, \dots, a_m, \dots, b_1, \dots, b_n, \dots) \in V(f)$. Separating the terms a_1, b_1 and $a_2, \dots, a_m, b_2, \dots, b_n$, we know that there exists some polynomial F_1 such that $F_1(a_1, b_1) = C$, for $C \in k$ written in terms of $a_2, \dots, a_m, b_2, \dots, b_n$. Note here that $F_1 \in k[x, y]$. (Here, we ignore the case where $F_1 = 0$, but this case can be resolved via a shifting argument).
Now note that for $i > \max(m, n)$, $(a_i, a_2, \dots, a_m, b_i, b_2, \dots, b_m)$ also satisfies f. This implies that $F_1(a_i, b_i) = C$, for the same C as before. Thus, for any $(a_\bullet, b_\bullet) \in V(f)$, we have that

$$F_1(a_i, b_i) = \begin{cases} C & i = 1 \\ C & i > \max(m, n) \\ \text{other values} & \text{otherwise} \end{cases}$$

Thus, $F_1(a_i, b_i)$ takes on at most $\max(m, n)$ distinct values, so there exists N, D such that $X \subset Z_{N,D}$.

Future Directions

Question 1 Does there exist a classification of c-prime ideals? (i.e. without the radical condition).

Question 2 Even more generally, if we have a finite dimensional variety (or scheme) $\mathcal{X}$, how do we characterize the S_∞-stable closed sets of $(\mathcal{X})^\infty$? The case where $\mathcal{X} = \mathbb{A}$ was completely solved in [1], and the case where $\mathcal{X}$ is a (finite-dimensional) affine variety is nearly solved here. We conjecture that the projective case is nearly identical to the affine case, though the map used to prove closure in Theorem 2 would no longer be proper. I, as well as A. Snowden and R. Nagpal are working on a follow-up paper treating the most general case, which is currently in preparation.

References

[1] A. Snowden, R. Nagpal, *On Symmetric Ideals*, In preparation.

[2] J. Draisma *Noetherianity up to Symmetry*, In: Combinatorial Algebraic Geometry. Lecture Notes in Mathematics, **2108** (2014). Springer, Cham.

[3] W. Fulton *Algebraic Curves* (2008)