Visualizing the unit ball of the AGY norm

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Introduction

Goal & Set Up

- By translating triangles and gluing them along certain edges in $\mathbb{R}^2$, we can get some surfaces, called translation surface.
- We wish to understand the Avila-Gouezel-Yoccoz (AGY) norm defined on a vector space attached to a translation surface.
- Specifically speaking, these translation surfaces have some singularities and therefore induce a relative homology space. We want to study the norm on that space.
- The goal of this project is to visualize two-dimensional slices of the unit ball under this norm.

We built a program to derive and visualize connections between points in the translation surface, known as **Saddle Connections**, which will help us visualize the unit ball.

1. Saddle Connections & Preliminaries

Translation Surface

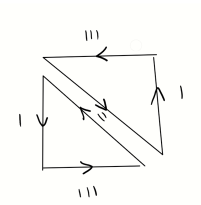
Definition. *A **translation surface** is the space obtained by identifying pairwise by translations the sides of a collection of plane polygons.*

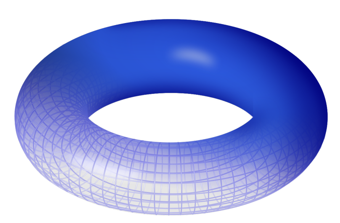
Since each polygon can be subdivided into triangles, we can say translation surfaces are built out of triangles.

- The identifications follow some basic properties:
  - every edge is identified with exactly one other (parallel) edge
  - Every edge has a fixed orientation (with respect to the triangle, edges must be oriented counter clockwise)
  - Only edges with opposite orientation can be identified.
- Any two triangles will be regarded as the same if there is a translation between them.
- The vertices in this surface are called **singularities**.
  - *Note*: Not every vertex is a singularity, but here we treat every vertex as one. In reality, to be a singularity, the sum of the angles around the vertex must be $> 2\pi$.

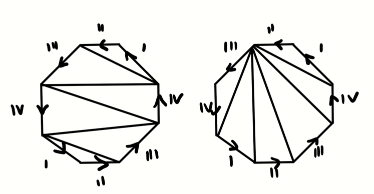
Tori are simple examples of translation surfaces. See the picture below.

Example 1 (A translation surface built by triangles)**.** *Image credit to Wikipedia.*





Example 2 (A translation surface built by an octagon)**.** *Image credit to Wikipedia.*



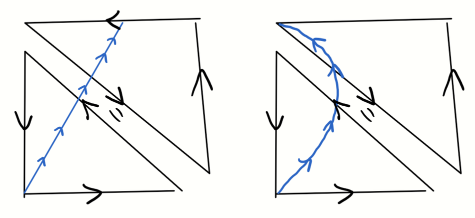
On the left, the octagon is triangulated and the translation surface turns out to be a genus-two surface.

Saddle Connections

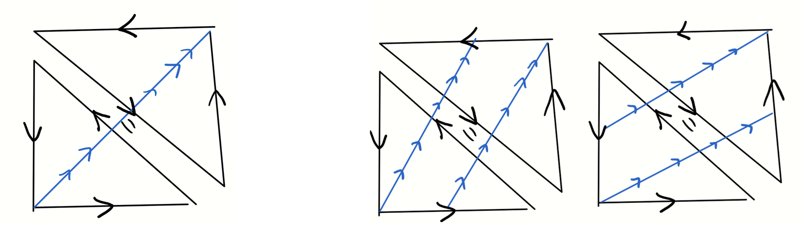
Definition. *A **Saddle Connection** is a straight line inside the triangles joining one singularity of a triangle to another.*

While a saddle connection can go through as many triangles as it wants, it cannot go through other singularities.
For example, the edges of each triangle are trivial saddle connections.

Example 3 (Saddle Connection v.s Non Saddle Connection)**.** *The line segment on the left is a saddle connection, while on the right it's not, because it is not a straight line.*



Example 4 (Saddle Connections in Tori)**.**



These are different saddle connections of a torus. In fact, we have infinitely many saddle connections in torus.

Methods and Results

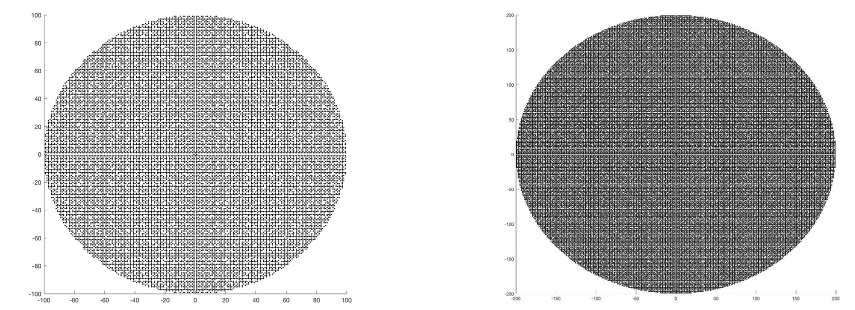
Algorithm:

- Given $L \in \mathbb{R}_{>0}$ and S a surface built out of triangles, we want to find all the saddle connections with the length smaller than L.
- A saddle connection can be represented as a vector emanating from a singularity and ending at a singularity.
- We first find the saddle connections originating from a singularity $p \in S$ going through some triangle of which p is a vertex.
- We loop this through all such possible triangles and apply this approach to each singularity.
- We then plot each of these saddle connections as vectors in $\mathbb{R}^2$.
- Starting at a singularity $p \in S$ of a triangle T we initialize a "window" as the two edges of T adjacent to p as vectors emanating away from p.
- The window gives bounds for where a possible saddle connection may be.
- We move into the next triangle extending the window vectors until they hit an edge or vertex of the triangle.
- If the vectors of the window hit the same edge there is no saddle connection found yet and we go to the next triangle.
- If the vectors of the window hit different edges we have found a saddle connection, there are now two different windows as we continue forward through the surface.
- Repeat until all saddle connections shorter than L have been found.

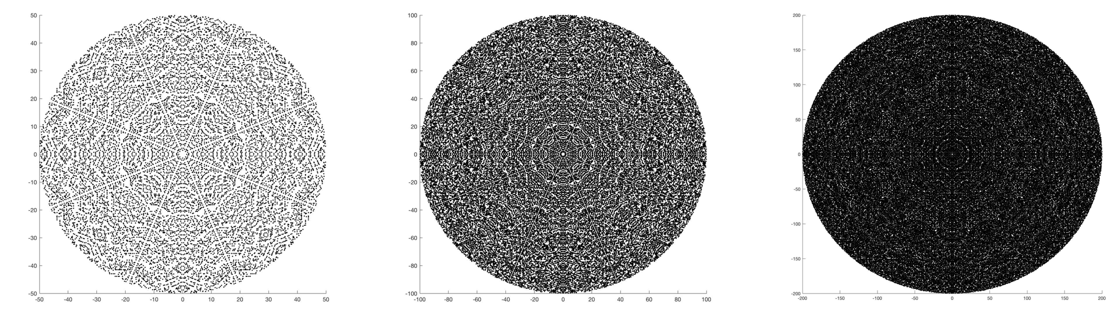
Some Pictures:

Using the above algorithm, we got some great pictures about saddle connections of torus and translation surface built out of an octagon.

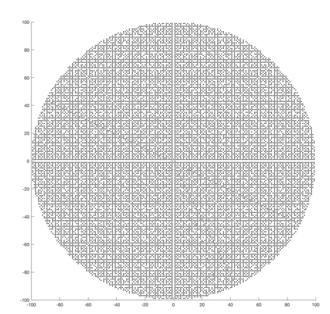
Example 5 (Saddle Connections in Tori)**.** *Here are what we have for the torus. When $L = 100, 200$, we got the pictures from left to right. It can be verified that the saddle connections should be exactly all the co-prime lattice points in $\mathbb{R}^2$. From pictures, it can be seen there are a lot of symmetries, since the base object torus is full of symmetries.*



Example 6 (Saddle Connections in the Genus-two Surface)**.** *This genus two surface is constructed by gluing opposite sides of a regular octagon together, the pictures of saddle connections for lengths 50, 100 and 200 are pictured bellow. We see that these pictures have an 8-fold symmetry coming from the octagon as expected. In addition they have quite a lot of unexpected structure which implies there is something more interesting going on in this case and more to explore.*



Example 7 (Saddle Connections in the L-shape)**.** *We have found interestingly that the picture of saddle connections for the L-shape looks exactly the same as for the Torus. This makes sense as one can tessellate the plane (with overlaps) with L-shapes and hence the possible saddle connections are exactly the lattice $\mathbb{Z}^2$. Thus we get actual saddle connections being pairs of coprime integers, exactly the same as the Torus.*



Future Directions

Having these saddle connections, we can actually build the relative homology space of these translation surfaces. With the information in hand, we can compute the **AGY norm** and study it.

1.1 AGY Norm

The AGY Norm is a norm defined on the space of linear functionals $\mathbb{R}^n \to \mathbb{R}$ where n is the number of edges in the triangulation and saddle connections are thought of as a formal linear combination of the edges. Think of (v_x, v_y) as the explicit coordinates of the saddle connection. In particular for a such a linear functional φ the AGY norm is given by

$$||\varphi|| = \sup_{\text{saddle connections } v} \frac{\varphi(v)}{|(v_x, v_y)|}$$

Where $|\cdot|$ is thought of a way to measure the length of the saddle connection, we will use the euclidean norm to begin.
The next step is thus to plot 2D slices of this norm.

References

[1] Alex Wright. *Visualizing the Unit Ball of the AGY Norm.* 2018.

[2] Lab of Geometry at Michigan. LoG(M) Poster Template. *University of Michigan Department of Mathematics.* 2018.